

Variation of Log Canonical Thresholds in Linear Systems

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Dedicated to the memory of Prof. Șerban Basarab

We investigate the variation of log canonical thresholds in (graded) linear systems. For toric log Fano varieties, we give a sharp lower bound for log canonical thresholds of the anticanonical members in terms of the global minimal log discrepancy.

1 Introduction

An interesting question in the classification theory of algebraic varieties is the conjecture of Alexeev and Borisov brothers: (singular) Fano varieties X of fixed dimension d and with minimal log discrepancy $\text{mld}(X) \geq \epsilon > 0$, belong to a bounded family. This conjecture is known if $d = 2$ [1] or X is a toric variety [8]. To a Fano variety X we associate the α -invariant

$$\gamma(X) = \inf \left\{ \text{lct} \left(X; \frac{D_n}{n} \right); n \geq 1, D_n \in |-nK_X| \right\}.$$

It is known that $\gamma(X) \cdot \sqrt[d]{(-K_X)^d} \leq d$, so the anticanonical volume is bounded above if the α -invariant is bounded away from zero. It follows that the above conjecture reduces

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to two local statements: (i) a lower bound $\gamma(X) \geq \gamma(d, \epsilon) > 0$ and (ii) an upper bound $r \leq r(d, \epsilon)$ for the smallest integer $r \geq 1$ such that rK_X is Cartier. The first aim of this paper is to propose a sharp lower bound $\gamma(d, \epsilon)$, and to establish it in the toric case.

Theorem 1.1. Let (X, B) be a toric log Fano variety, with $\dim X = d$ and $\text{mld}(X, B) \geq \frac{1}{q}$, for some integer $q \geq 1$. Then

$$\gamma(X, B) = \inf \left\{ \text{lct} \left(X, B; \frac{D_n}{n} \right); n \geq 1, D_n \in |-nK_X - nB| \right\} \geq \frac{q}{u_{d+1,q}},$$

where $(u_{p,q})_{p,q \geq 1}$ is the sequence of integers defined recursively by $u_{1,q} = q$, $u_{p+1,q} = u_{p,q}(1 + u_{p,q})$. \square

Theorem 1.1 is sharp (see Example 6.3). An interesting feature is that the type of coefficients of B do not matter. We expect the same bound holds in the non-toric case. This is easy to see in dimension one, but unclear in dimension two.

The α -invariant can be localized, and defined for polarized log varieties $(X, B; H)$. One defines

$$\gamma_P(X, B; |nH|) = \inf_{D \in |nH|} \text{lct}_P(X, B; D),$$

$\gamma_P(X, B; H) = \inf_{n \geq 1} n \cdot \gamma_P(X, B; |nH|)$ and $\gamma(X, B; H) = \inf_{P \in X} \gamma_P(X, B; H)$. The second aim of the paper is to study the variation of $\text{lct}_P(X, B; D)$ in the variables P, D , as suggested in [3]. The variation of $\text{lct}(X, B; D)$ in the variable D is implicit in the work of Viehweg [20, Sections 5.3, 8.2], where it is assumed that $B = 0$ and X has Kawamata log terminal singularities, and $\text{lct}(X, 0; D)$ is replaced by $e(X, 0; D)$, the largest positive integer e such that $\frac{1}{e} < \text{lct}(X, 0; D)$. We extend Viehweg's results to the setting of families of log varieties (X, B) . We show that log canonical thresholds are lower semi-continuous in flat families. In particular, $\text{lct}_P(X, B; D)$ is lower semi-continuous in two variables $(P, D) \in X \times |nH|$. By projection on to the second factor, we obtain extensions to the log category of the semi-continuity results of Varchenko [19] and Demailly and Kollár [11]. By projection on to the first factor, we obtain that $P \mapsto \gamma_P(X, B; |nH|)$ is lower semi-continuous and takes only finitely many values. It would be interesting to find out if the same holds for the asymptotic version $P \mapsto \gamma_P(X, B; H)$.

We outline the structure of this paper. In Section 2, we introduce families of log varieties, a logarithmic version of Viehweg's moduli functor (see [13] for a presentation of Viehweg's and Kollár's moduli functors). We define the boundary of a family only on

the smooth locus of the family, not on the whole total space (compare with [2]). The main result is the openness of the locus where the fibers have log canonical singularities (Theorem 2.8). It follows that log canonical thresholds are semi-continuous in flat families (Corollary 2.10).

In Section 3, we study the variation of log canonical thresholds $\text{lct}_P(X, B; D)$ in P and D , where D moves in a linear system. The results follow from those of Section 1 applied to the universal divisor of the linear system. We also generalize the product theorem of Viehweg to distinct factors (Theorem 3.7). To an \mathbb{N} -graded convex family of linear systems Λ_\bullet on a given log variety (X, B) , we associate the local α -invariant at x as follows:

$$\gamma_x(X, B; \Lambda_\bullet) = \inf \left\{ \text{lct}_x \left(X, B; \frac{D_n}{n} \right); n \geq 1, D_n \in \Lambda_n \right\}.$$

It is very interesting to study the variation in x of this functional (see Question 3.8), but we can only say little in general. We compute this invariant if X is a curve, recall some known results, and express it in terms of width. We can say more in the toric case. In Section 4, we compute the local α -invariant in the generic point of the invariant primes of a toric variety, and show that their minimum is exactly the global α -invariant (Theorem 4.4). In particular, we can compute combinatorially the α -invariant of a line bundle on a toric variety. With some extra assumptions, this was independently obtained in [10, Theorem 3.4], with analytic methods.

The main result of Section 4 is Theorem 5.6, a logarithmic effective version of a diophantine approximation result of Hensley [14, Lemma 2]. The special case $q = 1$ was solved by Averkov [6, Theorem 1.1], and our proof is inspired from his. We also give sharp versions of the original results of Hensley [14].

In Section 6, we give the sharp lower bound for the α -invariant of a toric log Fano (Theorem 6.1), a result essentially equivalent to Theorem 5.6. In particular, we obtain an upper bound for the anti-log canonical volume (Corollary 6.2). Theorem 6.4 extends to the log category and simplifies the proof of the finiteness of d -dimensional ϵ -log canonical toric Fano varieties [8]. We end Section 6 with the simplest examples of toric log Fano varieties, where we can explicitly compute both the minimal log discrepancy and the α -invariant, and see their relation with diophantine approximation.

2 Families of Log Varieties

Throughout this paper, we fix a base field k , of characteristic zero. By *scheme* we mean a k -scheme of finite type.

2.1 Relative effective Cartier divisors

Let $\pi: X \rightarrow S$ be a flat morphism of schemes, and D an effective Cartier divisor on X . Recall that D is called *relative over S* if one of the following equivalent conditions holds:

- (a) D is flat over S .
- (b) for every point $x \in X$ and a local equation f for D at x , f does not divide zero in $\mathcal{O}_{X_{\pi(x)}, X}$.
- (c) for every $s \in S$, $\text{Supp} D$ contains no associated prime of X_s .

If the fibers of π are reduced, (c) means that $\text{Supp} D$ contains no irreducible component of a fiber of π . If $S' \rightarrow S$ is a morphism of schemes, the pullback of D is a well defined effective Cartier divisor on $X \times_S S'$, relative over S' .

Let $\pi: X \rightarrow S$ be a flat morphism of schemes. An *effective \mathbb{Q} -Cartier divisor on X relative over S* is a formal product xD , where $x \geq 0$ is a rational number and D is an effective Cartier divisor on X relative over S . If $rx \in \mathbb{Z}$, we can write it as $\frac{1}{r}D'$, where $D' = rxD$ is an effective Cartier divisor on X relative over S .

We will use the following special case of [13, Proposition 3.5].

Lemma 2.1. Let $f: X \rightarrow S$ be a flat morphism of schemes, whose fibers satisfy Serre's property (S_2) . Let $w: U \subseteq X$ be an open subset such that its complement Z satisfies $\text{codim}(Z_s, X_s) \geq 2$ for every $s \in S$. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Then $\mathcal{H}_Z^i(\mathcal{L}) = 0$ ($i = 0, 1$), that is $\mathcal{L} \xrightarrow{\sim} w_*(\mathcal{L}|_U)$. \square

2.2 Log varieties, log canonical thresholds

Let $(X/k, B)$ be a log variety. Log discrepancies in geometric valuations of X , and minimal log discrepancies in scheme points, or closed subsets of X , are well defined by the existence of log resolutions (see e.g., [3, 5]). Denote by $(X, B)_{lc}$ the largest open locus in X where (X, B) has log canonical singularities, and $(X, B)_{-\infty}$ its complement. For an effective \mathbb{Q} -divisor D on X and a scheme point $x \in (X, B)_{lc}$, the *log canonical threshold at x of D with respect to (X, B)* is defined as

$$\text{lct}_x(X, B; D) = \sup \{t \geq 0; \text{mld}_x(X, B + tD) \geq 0\}.$$

It is $+\infty$ if $x \notin \text{Supp} D$, 0 if $x \in \text{Supp} D$ and $\text{mld}_x(X, B) = 0$, and a positive real number if $x \in \text{Supp} D$ and $\text{mld}_x(X, B) > 0$. It is rational if so are B, D near x . The reciprocal

$$\mu_x(X, B; D) = 1/\text{lct}_x(X, B; D)$$

is called the *Arnold multiplicity* at x of D with respect to (X, B) . It is 0 if $x \notin \text{Supp} D$, $+\infty$ if $x \in \text{Supp} D$ and $\text{mld}_x(X, B) = 0$, and a positive real number if $x \in \text{Supp} D$ and $\text{mld}_x(X, B) > 0$.

2.3 Families of log varieties

A *family of log varieties* $(X/S, B)$ consists of the following data:

- (a) a flat morphism of schemes $\pi: X \rightarrow S$, such that X_s is normal for every $s \in S$;
- (b) an effective \mathbb{Q} -Cartier divisor B defined on $U = U_\pi$ and relative over S , where $w: U \subseteq X$ is the open locus where the morphism π is smooth,

satisfying the following property: there exists an integer $r \geq 1$ such that rB is Cartier and the \mathcal{O}_X -module $w_*((\Omega_{U/S}^{\text{top}})^{\otimes r}(rB))$ is locally free. The smallest r with this property is called the *index* of the family.

Here $\Omega_{U/S}^{\text{top}}$ is the top exterior product of $\Omega_{U/S}^1$, corresponding to the locally constant dimension of the fibers. Property (a) implies that $Z = X \setminus U$ is the union of the singular locus of X_s , after all $s \in S$. In particular, $\text{codim}(Z_s, X_s) \geq 2$ for every $s \in S$. Recall that normality of a scheme is defined locally, so it does not imply irreducibility. The fibers of π are normal if and only if the geometric fibers $X_{\overline{k(s)}}$ ($s \in S$) are normal, where $\overline{k(s)}$ is an algebraic closure of $k(s)$.

Consider a family of log varieties of index r . For $l \in r\mathbb{Z}$, lB is a Cartier divisor on U , and we can define

$$\omega^{[l]} = w_* \left((\Omega_{U/S}^{\text{top}})^{\otimes l} \otimes \mathcal{O}_U(lB) \right).$$

The \mathcal{O}_X -module $\omega^{[l]}$ is coherent (EGA IV, Proposition 5.11.1). By assumption, $\omega^{[r]}$ is an invertible \mathcal{O}_X -module.

Lemma 2.2. $(\omega^{[r]})^{\otimes l} \xrightarrow{\sim} \omega^{[l]}$ for every $l \in r\mathbb{Z}$. □

Proof. We have natural multiplication maps $\omega^{[l]} \otimes \omega^{[l']} \rightarrow \omega^{[l+l']}$. By Lemma 2.1, $\omega^{[0]} = \mathcal{O}_X$. Therefore suffices to show that $\omega^{[r]} \otimes \omega^{[l]} \rightarrow \omega^{[r+l]}$ is an isomorphism. Indeed, denote $\mathcal{F} = \omega^{[r]} \otimes \omega^{[l]}$. Our homomorphism factors as

$$\mathcal{F} \rightarrow w_*(\mathcal{F}|_U) \xrightarrow{\sim} \omega^{[r+l]}.$$

We have $\mathcal{H}_Z^i(\mathcal{F}) = 0$ ($i = 0, 1$), since $\omega^{[l]}$ satisfies this property by Lemma 2.1, and $\omega^{[r]}$ is locally trivial. Therefore the first map is also an isomorphism. ■

Lemma 2.3. Let $(X/S, B)$ be a family of log varieties of index r . Let $g: S' \rightarrow S$ be a morphism of schemes. Consider the induced base change diagram

$$\begin{array}{ccc} X & \xleftarrow{G} & X' \\ \pi \downarrow & & \downarrow \pi' \\ S & \xleftarrow{g} & S' \end{array}$$

Then $U_{\pi'} = G^{-1}(U_{\pi})$, $(X'/S', (G|_{U_{\pi'}})^*B)$ is a family of log varieties over S' , and we have natural isomorphisms

$$G^*(\omega^{[l]}) \xrightarrow{\sim} \omega'^{[l]} \quad (l \in r\mathbb{Z}). \quad \square$$

Proof. We have $G^*(\Omega_{X/S}^1) \xrightarrow{\sim} \Omega_{X'/S'}^1$. Therefore, $U_{\pi'} = G^{-1}(U_{\pi})$ and we have a base change diagram

$$\begin{array}{ccc} U & \xleftarrow{G|_U} & U' \\ w \downarrow & & \downarrow w' \\ X & \xleftarrow{G} & X' \end{array}$$

For $l \in r\mathbb{Z}$, we obtain a natural homomorphism

$$G^*(\omega^{[l]}) = G^*w_*((\Omega_{U/S}^{\text{top}})^{\otimes l}(LB)) \rightarrow w'_*((G|_U)^*(\Omega_{U/S}^{\text{top}})^{\otimes l}(LB)) \xrightarrow{\sim} w'_*(\Omega_{U'/S'}^{\text{top}})^{\otimes l}(LB') = \omega'^{[l]}.$$

If we denote this homomorphism by $\mathcal{F}' \rightarrow \mathcal{G}'$, it factors as $\mathcal{F}' \rightarrow w'_*(\mathcal{F}'|_{U'}) \xrightarrow{\sim} \mathcal{G}'$. Since $\omega^{[l]}$ is locally trivial, so is \mathcal{F}' . By Lemma 2.1, $\mathcal{H}_{Z'}^i(\mathcal{F}') = 0$ ($i = 0, 1$). Therefore, $\mathcal{F}' \rightarrow w'_*(\mathcal{F}'|_{U'})$ is also an isomorphism. ■

In particular, for every $s \in S$, the fiber X_s is a normal variety defined over $k(s)$, rB_s is an effective Cartier divisor on $U_s = X_s \setminus \text{Sing}(X_s)$, and if $r\bar{B}_s$ is the effective Weil divisor which is the closure of rB_s in X_s , we have a base change diagram

$$\begin{array}{ccc} U_s & \hookrightarrow & U \\ w_s \downarrow & & \downarrow w \\ X_s & \hookrightarrow & X \end{array}$$

and an isomorphism

$$\omega^{[r]}|_{X_s} \xrightarrow{\sim} w_{s*} \left(\left(\Omega_{U_s/k}^{\text{top}} \right)^{\otimes r} (rB_s) \right) = \mathcal{O}_{X_s} (rK_{X_s} + r\bar{B}_s).$$

Therefore $rK_{X_s} + r\bar{B}_s$ is Cartier, so that (X_s, \bar{B}_s) is a log variety. We think of (X_s, \bar{B}_s) ($s \in S$) as an algebraic family of log varieties parametrized by S . The boundary coefficients may vary. To simplify notation, we denote \bar{B}_s by B_s .

Examples 2.4.

- (1) Let $\pi: X \rightarrow S$ be a smooth morphism of schemes. Then $(X/S, B)$ is a family of log varieties over S if and only if $B = \frac{1}{r}D$, for some integer $r \geq 1$ and an effective Cartier divisor D on X which is relative over S . The fibers are the log varieties $(X_s/k(s), \frac{1}{r}D|_{X_s})$ ($s \in S$).
- (2) Let $(F/k, B_F)$ be a log variety. Let S be a scheme. Denote by $B_F \times S$ the \mathbb{Q} -Cartier divisor $p_1^*(B_{F^0})$, defined on $F^0 \times S$, where F^0 is the smooth locus of F . Then $(F \times S/S, B_F \times S)$ is the trivial family of log varieties over S , with constant fiber (F, B_F) .
- (3) Let $(X/S, B)$ be a family of log varieties. Let D be an effective Cartier divisor on X , relative over S . Then $(X/S, B + tD)$ is a family of log divisors for every rational $t \geq 0$.
- (4) A family of log curves consists of a smooth morphism $\pi: X \rightarrow S$ of relative dimension one, endowed with an effective \mathbb{Q} -Cartier divisor B , relative over S . If r is the index of the family, then $rB = D$ is an effective Cartier divisor on X which is finite flat over S . \square

Lemma 2.5. Let $\pi: X \rightarrow S$ be a flat morphism of schemes with normal fibers, and S smooth over k . Then $(X/S, B)$ is a family of log varieties if and only if $(X/k, B)$ is a log variety and the boundary supports no irreducible components of fibers of π . Moreover, they have the same index r , and $\pi^* \omega_{S/k}^{\otimes r} \otimes \omega_{(X/S, B)}^{[r]} \simeq \omega_{(X/k, B)}^{[r]}$. \square

Proof. Since S and the fibers are normal, so is X . Let $U = U_\pi$ be the smooth locus of π . It follows that $\text{codim}(X \setminus U, X) \geq 2$. In particular, lB is Cartier on U if and only if $l\bar{B}$ is a Weil divisor on X . We have a short exact sequence

$$0 \rightarrow \pi^* \Omega_{S/k}^1 \rightarrow \Omega_{U/k}^1 \rightarrow \Omega_{U/S}^1 \rightarrow 0.$$

It induces an isomorphism $\pi^* \Omega_{S/k}^{\text{top}} \otimes \Omega_{U/S}^{\text{top}} \simeq \Omega_{U/k}^{\text{top}}$. If lB is Cartier on U , we obtain an isomorphism

$$\pi^* \omega_{S/k}^{\otimes l} \otimes \omega_{(X/S, B)}^{[l]} \simeq \omega_{(X/k, B)}^{[l]}.$$

Therefore, $\omega_{(X/S, B)}^{[l]}$ is locally free if and only if $\omega_{(X/k, B)}^{[l]}$ is locally free, and the first claim follows. The second follows from the first. ■

Lemma 2.6. Let (X, B) be a log variety. Let $f: X \rightarrow S$ be a morphism, with S a reduced scheme. Then there exists an open subset $\emptyset \neq V \subseteq S$ such that $(X, B)|_{f^{-1}V} \rightarrow V$ is a family of log varieties. □

Proof. We may shrink S to an open subset, so that S is smooth, and f is flat with normal fibers. Let $U \subseteq X$ be the smooth locus of f . Let $r \geq 1$ such that $rK_X + rB$ is Cartier. We may further shrink S so that the effective Cartier divisor $rB|_U$ becomes flat over S . By Lemma 2.5, $(X/S, B|_U)$ is a family of log varieties. ■

Lemma 2.7. Let $\pi: (X, B) \rightarrow S$ be a family of log varieties. Let S be regular at s , and choose a regular system of parameters $(h_i)_i$ for $\mathcal{O}_{S, s}$. The following are equivalent for $x \in X_s$:

- (a) the fiber (X_s, B_s) has log canonical singularities near x ;
- (b) the log variety $(X, B + \pi^* \Sigma_s)$ has log canonical singularities near x , where $\Sigma_s = \sum_i \text{div}(h_i)$. □

Proof. Let $S' = \text{div}(h_j)$. It is defined locally near s , but we may shrink S to a neighborhood of s . Let $X' = \pi^*(S')$, $\pi': X' \rightarrow S'$ the induced morphism, and $B' = B|_{U_{\pi'}}$. The base change data $(X', B') \rightarrow S' \ni s, (h_i|_{S'})_{i \neq j}$ satisfy the same properties. Now $(X, B + X')$ is a log variety with lc center X' , a normal Cartier divisor in X . Therefore, the different is zero, and the codimension one adjunction formula is

$$(K_X + B + X')|_{X'} = K_{X'} + B'.$$

If we denote $\Sigma'_s = \sum_{i \neq j} \text{div}(h_i)$, we obtain $(K_X + B + \pi^* \Sigma_s)|_{X'} = K_{X'} + B' + \pi'^* \Sigma'_s$. By Inversion of Adjunction [15], $(X, B + \pi^* \Sigma_s)$ has log canonical singularities near x if and only if $(X', B' + \pi'^* \Sigma'_s)$ has log canonical singularities near x . Iterating this argument proves the equivalence. ■

Theorem 2.8. Let $\pi: (X, B) \rightarrow S$ be a family of log varieties. Then the set of points $x \in X$ such that $(X_{\pi(x)}, B_{\pi(x)})$ has log canonical singularities at x , is open in X . □

Proof. We have to show that the complement $Z(\pi) = \bigcup_{s \in S} (X_s, B_s)_{-\infty}$ is closed in X . For this, it suffices to construct a non-empty open subset $V \subseteq S$ such that the restriction $Z(\pi)|_{\pi^{-1}(V)}$ is closed. Indeed, replacing S by $S' = S \setminus V$ and π by the induced base change family, we have

$$Z(\pi) = (Z(\pi)|_{\pi^{-1}(V)}) \cup Z(\pi|_{S'}).$$

By Noetherian induction, $Z(\pi|_{S'})$ is closed. Therefore $Z(\pi)$ is closed.

It remains to prove the claim. For this, we may base change with the reduced structure on S and then restrict to the regular locus. Therefore, S is regular. By Lemma 2.5, (X, B) is a log variety. We show that we have an inclusion $(X, B)_{-\infty} \subseteq Z(\pi)$, which is an equality over some non-empty open subset of S .

For the inclusion, let $x \in X \setminus Z(\pi)$, so that $(X_{\pi(x)}, B_{\pi(x)})$ has log canonical singularities at x . By Lemma 2.7, $(X, B + \pi^* \Sigma_S)$ has log canonical singularities at x . Therefore, (X, B) has log canonical singularities at x , that is $x \notin (X, B)_{-\infty}$.

By Hironaka, there exists a desingularization $\mu: X' \rightarrow X$ and a normal crossing divisor $\sum_i E_i$ which supports $B' = \mu^*(K_X + B) - K_{X'}$. After shrinking S to an open subset, the morphism $(X', \sum_i E_i) \rightarrow S$ becomes log smooth: $X' \rightarrow S$ is smooth and $\sum_i E_i$ is a relative normal crossings divisor. In this case, we show that the inclusion is an equality. Assuming (X, B) has log canonical singularities at x , we have to show that (X_s, B_s) has log canonical singularities at x , where $s = \pi(x)$. We may shrink X to a neighborhood of x , and suppose (X, B) has log canonical singularities. That is the coefficients of B' are at most 1. By an iteration of adjunction using a local system of parameters at s (see Lemma 2.7 and its proof), we obtain

$$\mu_s^*(K_{X_s} + B_s) = K_{X'_s} + B'_s,$$

where $B'_s = \sum_i b_i E_i|_{X'_s}$. Since $(X', \sum_i E_i) \rightarrow S$ is log smooth, a prime divisor on X'_s is contained in at most one E_i . Therefore, the coefficients of B'_s are some of the b_i 's, so at most 1. Therefore, (X_s, B_s) has log canonical singularities at x . ■

Proposition 2.9. Let $\pi: (X, B) \rightarrow S$ be a family of log varieties whose fibers have at most log canonical singularities. Let D be an effective Cartier divisor on X , relative over S . Then the function $X \ni x \mapsto \text{lct}_x(X_{\pi(x)}, B_{\pi(x)}; D_{\pi(x)})$ is lower semi-continuous and takes only finitely many values. □

Proof. Fix $t \geq 0$. Then $\text{lct}_x(X_{\pi(x)}, B_{\pi(x)}; D_{\pi(x)}) \geq t$ if and only if $(X_{\pi(x)}, B_{\pi(x)} + tD_{\pi(x)})$ has log canonical singularities at x . Since $(X, B + tD|_U) \rightarrow S$ is a family of log varieties, the locus of such x is open by Theorem 2.8. Therefore the function is lower semi-continuous.

To show that the function takes only finitely many values, it suffices to prove this holds over some open non-empty subset of S (by Noetherian induction). Then we may assume S is reduced and regular. Let $X' \rightarrow X$ be a desingularization, endowed with a normal crossing divisor $\sum_i E_i$ which supports both $B' = \mu^*(K_X + B) - K_{X'}$ and $D' = \mu^*D$. We finally shrink S to an open subset such that $(X', \sum_i E_i) \rightarrow S$ becomes log smooth. We have

$$\mu_s^*(K_{X_s} + B_s + tD_s) = K_{X'_s} + B'_s + tD'_s,$$

and the coefficients of $B'_s + tD'_s$ are some of the $b'_i + td'_i$. Therefore each $\text{lct}_x(X_s, B_s; D_s)$ is either $+\infty$, or the largest $t \geq 0$ such that $b'_i + td'_i \leq 1$ for certain i . They belong to a finite set. ■

Corollary 2.10. Let $\pi: (X, B) \rightarrow S$ be a family of log varieties. Let $Z \subseteq X$ be a closed subset, such that $Z \rightarrow S$ is proper surjective, and (X_s, B_s) has log canonical singularities near Z_s for every $s \in S$. Let D be an effective Cartier divisor on X , relative over S . Then the function $S \ni s \mapsto \text{lct}_{Z_s}(X_s, B_s; D_s)$ is lower semi-continuous and takes only finitely many values. □

Proof. By Theorem 2.8, we may shrink X near Z , so that the fibers of π have log canonical singularities. Denote $\gamma(x) = \text{lct}_x(X_{\pi(x)}, B_{\pi(x)}; D_{\pi(x)})$. Then $\text{lct}_{Z_s}(X_s, B_s; D_s) = \min_{x \in Z_s} \gamma(x)$, and

$$\{s \in S; \text{lct}_{Z_s}(X_s, B_s; D_s) < t\} = \pi(Z \cap \{x \in X; \gamma(x) < t\}).$$

So it follows from Proposition 2.9. ■

3 Lct-Variation in a Linear System

Let $(X/k, B)$ be a log variety. We assume k is algebraically closed. Let Λ be a non-empty, finite dimensional linear system on X . That is \mathcal{L} is an invertible \mathcal{O}_X -module, $V \subseteq \Gamma(X, \mathcal{L})$ is a non-zero finite dimensional k -vector subspace, and Λ is the family of divisors of zeros of sections in V . Using a basis of V , we may identify Λ with \mathbb{P}_k^n , where n is the dimension of Λ . Inside $X \times \Lambda$ we have the universal divisor H , given by $\sum_{i=0}^n s_i(x)\lambda_i = 0$, where s_0, \dots, s_n is basis of V over k .

Proposition 3.1. The function $(X, B)_{lc} \times \Lambda \rightarrow [0, \infty]$, $(P, D) \mapsto \text{lct}_P(X, B; D)$ is lower semi-continuous and takes only finitely many values. □

Proof. Denote $\mathcal{X} = X \times X \times \Lambda$, $S = X \times \Lambda$, and $\pi = p_{23}: \mathcal{X} \rightarrow S$ the projection on the last two factors. Let σ be the section of π which is the diagonal on X and the identity on Λ . Let $\mathcal{B} = p_1^* B$ and $\mathcal{H} = p_{13}^* H$. Then $\pi: (\mathcal{X}, \mathcal{B}) \rightarrow S$ is a family of log varieties, σ is a section of π , and \mathcal{H} is an effective Cartier divisor relative over S . For $s = (P, D) \in S$, the fiber $(\mathcal{X}_s, \mathcal{B}_s + t\mathcal{H}_s)$ is isomorphic to $(X, B + tD)$ and $\sigma(s) \in \mathcal{X}_s$ corresponds to $P \in X$. If we restrict the family to $(X, B)_{lc} \times \Lambda$, it becomes log canonical, and $\text{lct}_{\sigma(s)}(\mathcal{X}_s, \mathcal{B}_s; \mathcal{H}_s) = \text{lct}_P(X, B; D)$. We conclude by Proposition 2.9. ■

Example 3.2. $\text{lct}_P(X, B; D) < +\infty$ if and only if $(P, D) \in H$. □

Example 3.3. Endow the affine line \mathbb{A}_k^1 with a boundary $B = \sum_P b_P P$, where $b_P \in [0, 1]$, and only finitely many are non-zero. Let $f_0, \dots, f_n \in k[t]$ be polynomials, linearly independent over k . They induce a linear system $\Lambda = \left\{ D_\lambda = \text{div}\left(\sum_i \lambda_i f_i\right); \lambda \in \mathbb{P}^n \right\}$. Denote by ∂ the canonical derivation of $k[t]$. Then $\text{lct}_P(\mathbb{A}^1, B; D_\lambda) < t$ if and only if $\sum_i \frac{\partial^m f_i(P)}{m!} \lambda_i = 0$ for every integer $1 \leq m \leq \frac{1-b_P}{t}$. □

Theorem 3.4. Let $x \in (X, B)_{lc}$ be a scheme point. The function $\text{lct}_x(X, B; \cdot): \Lambda \rightarrow [0, +\infty]$ is lower semi-continuous and takes only finitely many values. □

Proof. We have $\text{lct}_x(X, B; D) = \max_{P \in \bar{x}} \text{lct}_P(X, B; D)$. By Proposition 3.1, it takes only finitely many values. For $t > 0$, we have

$$\{D \in \Lambda; \text{lct}_x(X, B; D) < t\} = \cap_{P \in \bar{x}} \{D \in \Lambda; \text{lct}_P(X, B; D) < t\}.$$

Each term on the right-hand side is closed, by Proposition 3.1. Therefore the left-hand side is also closed. ■

In particular, $\text{lct}_x(X, B; \cdot)$ attains its maximum (respectively, minimum) on a dense open (respectively, special closed) subset of Λ . Define

$$\gamma_x(X, B; \Lambda) = \min_{D \in \Lambda} \text{lct}_x(X, B; D).$$

Denote $\mu_x(X, B; \Lambda) = 1/\gamma_x(X, B; \Lambda)$, so that $\mu_x(X, B; \Lambda) = \max_{D \in \Lambda} \mu_x(X, B; D)$.

Theorem 3.5. The function $(X, B)_{lc} \rightarrow [0, +\infty]$, $P \mapsto \gamma_P(X, B; \Lambda)$ is lower semi-continuous and takes only finitely many values. □

Proof. By Proposition 3.1, the values belong to a finite set. And $\{P \in (X, B)_{lc}; \gamma_P(X, B; \Lambda) < t\}$ is the projection on the first factor of $\{(P, D) \in (X, B)_{lc} \times$

$\Lambda; \gamma_P(X, B; D) < t\}$. The latter is closed by Proposition 3.1, and since Λ is proper over k , it follows that our level set is closed. ■

Theorem 3.6. Let $Z \subseteq X$ be a closed subset such that Z/k is proper and (X, B) has log canonical singularities near Z . The function $\Lambda \rightarrow [0, \infty], D \mapsto \text{lct}_Z(X, B; D)$ is lower semi-continuous and takes only finitely many values. □

Proof. Denote $\mathcal{X} = X \times_k \Lambda$, $S = \Lambda$, and $\pi: \mathcal{X} \rightarrow S$ the second projection. Let $\mathcal{B} = p_1^* B$ and $\mathcal{H} \subset X \times \Lambda$ the universal divisor. Then $\pi: (\mathcal{X}, \mathcal{B}) \rightarrow S$ is a family of log varieties, \mathcal{H} is an effective Cartier divisor relative over S , $\mathcal{Z} = Z \times S$ is a closed subset of X which is proper over S . For $s = [D] \in S$, the fiber $(\mathcal{X}_s, \mathcal{B}_s + t\mathcal{H}_s)$ is isomorphic to $(X, B + tD)$ and $\text{lct}_{\mathcal{Z}_s}(\mathcal{X}_s, \mathcal{B}_s; \mathcal{H}_s) = \text{lct}_Z(X, B; D)$. We conclude by Corollary 2.10. ■

In particular, $\text{lct}_Z(X, B; \cdot)$ attains the maximal (respectively, minimal) value on a dense open (respectively, special closed) subset of Λ . Define

$$\gamma_Z(X, B; \Lambda) = \min_{D \in \Lambda} \text{lct}_Z(X, B; D).$$

If X/k is proper, denote $\gamma_X(X, B; \Lambda)$ by $\gamma(X, B; \Lambda)$. Define similarly $\mu_Z(X, B; \Lambda)$ and $\mu(X, B; \Lambda)$.

Theorem 3.7. Let $(X_i/k, B_i)$ be finitely many proper log varieties, with log canonical singularities. Let $|L_i|$ be non-empty complete linear systems on X_i . Let $X = \prod_i X_i$, $B = \sum_i p_i^*(B_i)$, $L = \sum_i p_i^*(L_i)$. Then the product log variety (X, B) has log canonical singularities, the complete linear system $|L|$ is non-empty, and $\gamma(X, B; |L|) = \min_i \gamma(X_i, B_i; |L_i|)$. □

Proof. By induction, suffices to consider only two factors.

Let $D_i \in |L_i|$, for $i = 1, 2$. Let $t_i \geq 0$ be maximal such that $(X_i, B_i + t_i D_i)$ has log canonical singularities. Set $t = \min(t_1, t_2)$ and $D = p_1^* D_1 + p_2^* D_2$. Then $D \in |L|$ and t is maximal such that $(X, B + tD)$ has log canonical singularities. Therefore, $\gamma \leq \text{lct}(X, B; D) = t$. Taking minimum after all members, we obtain $\gamma(X, B; |L|) \leq \min_i \gamma(X_i, B_i; |L_i|)$.

Suppose by contradiction that $\gamma(X, B; |L|) < \min_i \gamma(X_i, B_i; |L_i|)$. That is, there exists $D \in |L|$ and $t \leq \min_i \gamma(X_i, B_i; |L_i|)$ such that $(X, B + tD)$ does not have log canonical singularities. Denote $Z = (X, B + tD)_{-\infty}$. It is a proper subset of X . If we show that $Z = \pi_i^{-1} \pi_i(Z)$ for every i , it follows that $Z = \emptyset$, a contradiction.

We prove the claim for the second projection $\pi: X \rightarrow X_2$. Choose a closed point $P \in X_2$ and suppose $X_P \not\subseteq Z$. We will show that $X_P \cap Z = \emptyset$. Indeed, by Hironaka's flattening, there exists a desingularization $g: X'_2 \rightarrow X_2$ such that the induced Cartier divisor D' on $X' = X_1 \times X'_2$ admits a decomposition $D' = D'' + \pi^*(D_2)$, where D'' is an effective Cartier divisor on X' relative over X'_2 , and D_2 is an effective Cartier divisor on X'_2 with normal crossing support.

Choose a point $Q \in g^{-1}(P)$. Since $D''|_{X'_Q} \in |L_1|$ and $t \leq \gamma(X_1, B_1; |L_1|)$, the log variety $(X_1, B_1 + tD''|_{X_1 \times Q})$ has log canonical singularities. This is the fiber at Q of the family of log varieties $(X'/X'_2, p_1^*(B_1) + tD'')$, so by Lemma 2.7, the log variety $(X', p_1^*(B_1) + tD'' + \pi'^*\Sigma_Q)$ has log canonical singularities near X'_Q , where Σ_Q is any local divisor cut out by a regular system of parameters of $\mathcal{O}_{X'_2, Q}$.

On the other hand, $(X, B + tD)$ has log canonical singularities at the generic point of X_P . Therefore $(X', B' + tD')$ has log canonical singularities at the generic point of X'_Q . Let $g^*(K_{X_2} + B_2) = K_{X'_2} + B'_2$. It follows that $(X'_2, B'_2 + tD_2)$ has log canonical singularities. Therefore, $B'_2 + tD'_2 \leq \Sigma_Q$ for some choice of local parameters at Q . We deduce that $(X', p_1^*(B_1) + tD'' + \pi'^*(B'_2 + tD'_2))$ has log canonical singularities near X'_Q . That is $(X', B' + tD')$ has log canonical singularities near X'_Q . This holds for every $Q \in g^{-1}(P)$, so we deduce that $(X', B' + tD')$ has log canonical singularities over an open neighborhood of $g^{-1}(P)$. Since $(X', B' + tD') \rightarrow (X, B + tD)$ is log crepant, it follows that $(X, B + tD)$ has log canonical singularities near X_P , that is $Z \cap X_P = \emptyset$. ■

3.1 Graded case

Let L be a \mathbb{Q} -Cartier divisor on X such that $|nL| \neq \emptyset$ for some $n \geq 1$. Let $x \in (X, B)_{lc}$ be a scheme point. The α -invariant at x of L with respect to (X, B) is defined as

$$\gamma_x(X, B; L) = \inf \left\{ \text{lct}_x \left(X, B; \frac{D_n}{n} \right); n \geq 1, D_n \in |nL| \right\}.$$

Question 3.8. Is $(X, B)_{lc} \ni x \mapsto \gamma_x(X, B; L)$ is lower semi-continuous, with finitely many rational values? □

Let $m \geq 1$ such that mL is Cartier and $|mL| \neq \emptyset$. Then $\gamma_x(X, B; L) = \inf_{m|n} n\gamma_x(X, B; |nL|)$.

Lemma 3.9. If $\dim X = 1$, $\deg L > 0$ and $x \in X$ is a closed point, then $\gamma_x(X, B; L) = \frac{1-b_x}{\deg L}$. □

Proof. We may suppose $b_x < 1$. Let mL be Cartier and $|mL| \neq \emptyset$. Let g be the genus of C . Recall that any complete linear system of degree g is non-empty. Let $m|n$ and

$n \deg L > g$. Then $|nL - (n \deg L - g)x| \neq \emptyset$. Therefore

$$n \frac{\deg L}{1 - b_x} - \frac{g}{1 - b_x} \leq \mu_x(X, B; |nL|) \leq n \frac{\deg L}{1 - b_x}.$$

Dividing by n and letting $n \rightarrow \infty$, we obtain the claim. \blacksquare

Proposition 3.10 ([16, Theorem 6.7.1]). Suppose X^d is proper and L is a nef and big \mathbb{Q} -divisor. Then $\gamma_P(X, B; L) \cdot \sqrt[d]{(L^d)} \leq d$ for every $P \in (X, B)_{lc}$. \square

Proof. We may scale L and suppose it is Cartier. Let $0 < c < \sqrt[d]{(L^d)}$ be a rational number. Since $h^0(nL) = (L^d) \frac{n^d}{d!} + O(n^{d-1})$ and $\binom{nc+d}{d} = c^d \frac{n^d}{d!} + O(n^{d-1})$, there exists an integer $n \geq 1$ such that $nc \in \mathbb{Z}$ and $h^0(nL) > \binom{nc+d}{d}$.

Let $Q \in X \setminus (\text{Sing} X \cup \text{Supp} B)$. The evaluation map $\Gamma(X, nL) \rightarrow \mathcal{O}_Q / \mathfrak{m}_Q^{nc+1}$ has non-trivial kernel, by dimension count. Therefore there exists $D \in |nL|$ such that $\text{mult}_Q(D) > nc$. Let $(X, B + \gamma D)$ be maximally log canonical at Q . Let v be the valuation induced by the exceptional divisor of the blow-up of $Q \in X$. Then $0 \leq a(v; X, B + \gamma D) = d - \gamma \text{mult}_v(D) < d - \gamma nc$. Therefore $\gamma < \frac{d}{nc}$. We conclude $\gamma_Q(X, B; |nL|) < \frac{d}{nc}$.

Since the points Q are dense in X , Theorem 3.5 gives $\gamma_P(X, B; |nL|) < \frac{d}{nc}$ for every $P \in (X, B)_{lc}$. Then $\gamma_P(X, B; L) \leq n \gamma_P(X, B; |nL|) < \frac{d}{c}$ for every $P \in (X, B)_{lc}$. Letting c converge to $\sqrt[d]{(L^d)}$, we obtain the claim. \blacksquare

If X/k is proper and (X, B) has log canonical singularities, define the α -invariant of L with respect to (X, B) as

$$\gamma(X, B; L) = \inf \left\{ \text{lct} \left(X, B; \frac{D_n}{n} \right); n \geq 1, D_n \in |nL| \right\}.$$

Question 3.11. Does $\text{mld}(X, B) > 0$ imply $\gamma(X, B; L) > 0$? \square

For example, $\gamma(X, B; L) = \frac{\text{mld}(X, B)}{\deg L}$ if X is a curve and $L \not\sim_{\mathbb{Q}} 0$ (by Lemma 3.9). If X is smooth, $B = 0$, and A is a very ample divisor on X , the argument of [20, Corollary 5.11] shows that $\gamma(X, 0; L) \geq \frac{1}{(L \cdot A^{d-1})}$. Theorem 3.7 gives the following corollary.

Corollary 3.12. Let $(X_i/k, B_i)$ be finitely many proper log varieties, with log canonical singularities. Let L_i be \mathbb{Q} -Cartier divisors such that $|mL_i| \neq \emptyset$ for some $m \geq 1$. Let $X = \prod_i X_i$, $B = \sum_i p_i^*(B_i)$, $L = \sum_i p_i^*(L_i)$. Then the product log variety (X, B) has log canonical singularities and

$$\gamma(X, B; L) = \min_i \gamma(X_i, B_i; L_i). \quad \square$$

3.2 The α -invariant in terms of width

Let X be a normal variety. Let Λ be a non-empty, finite dimensional linear system on X . For a prime divisor $E \subset X$, define the *width of Λ at E* as

$$w_E(\Lambda) = \sup \{ \text{mult}_E(D) ; D \in \Lambda \}.$$

It is a non-negative integer, zero if and only if E is not a fixed component of Λ and $\phi_\Lambda(E) = \phi_\Lambda(X)$. If $f: X' \rightarrow X$ is a proper modification and $E \subset X'$ is a prime divisor, define the width of Λ at E as $w_E(f^*\Lambda)$. It depends only on the valuation of X defined by E (called *geometric valuation of X*).

Let L be a \mathbb{Q} -Cartier divisor such that mL is Cartier and $|mL| \neq \emptyset$ for some $m \geq 1$. Let E be a geometric valuation of X . Define the *width of L at E* as

$$w_E(L) = \sup \left\{ \frac{w_E(|nL|)}{n} ; m \mid n \right\}.$$

If X is proper, $w_E(L) = 0$ for every geometric valuation E of X if and only if $L \sim_{\mathbb{Q}} 0$. If X is projective of dimension d , A is a very ample divisor on X , and E is a prime divisor on X , then $w_E(L) \leq (L \cdot A^{d-1})$. It follows that if X is proper, then $w_E(L)$ is a non-negative real number, for every geometric valuation E of X . By definition, the following formulas hold:

– Let (X, B) be a log variety, Λ a non-empty finite dimensional linear system on X . Let E be a geometric valuation of $(X, B)_{lc}$. Then

$$\gamma_E(X, B; \Lambda) = \begin{cases} +\infty, & w_E(\Lambda) = 0 \\ \frac{a(E; X, B)}{w_E(\Lambda)}, & w_E(\Lambda) > 0 \end{cases}$$

– Let (X, B) be a proper log variety, with log canonical singularities. Let L be a \mathbb{Q} -Cartier divisor such that mL is Cartier and $|mL| \neq \emptyset$ for some $m \geq 1$. Then $\gamma(X, B; L)$ is the infimum of $\gamma_E(X, B; L)$ after all geometric valuations E of X . Equivalently,

$$\gamma(X, B; L) = \begin{cases} +\infty, & L \sim_{\mathbb{Q}} 0 \\ \inf_{w_E(L) > 0} \frac{a(E; X, B)}{w_E(L)}, & L \not\sim_{\mathbb{Q}} 0 \end{cases}$$

4 The α -Invariant on Toric Varieties

Let $(X/k, B)$ be a *toric log variety* with log canonical singularities. That is X/k is a toric variety, B is an effective \mathbb{Q} -divisor which is torus invariant, $K_X + B$ is \mathbb{Q} -Cartier and

(X, B) has at most log canonical singularities. Due to the existence of log resolutions in the toric category, the latter condition is equivalent to $B = \sum_i b_i E_i$, where E_i are the torus invariant prime divisors of X and $b_i \in [0, 1] \cap \mathbb{Q}$. We have $\alpha(E_i; X, B) = 1 - b_i$. Throughout this section, we also assume that X/k is proper.

Let $L = \sum_i l_i E_i$ be a torus invariant \mathbb{Q} -Cartier divisor. Recall that $\Gamma(X, L) = \bigoplus_{m \in M \cap \square_L} k \cdot \chi^m$, where $\square_L = \cap_i \{m \in M_{\mathbb{R}}; \langle m, e_i \rangle + l_i \geq 0\}$ and $\{e_i\} = \Delta_X(1)$.

Let $V \subseteq \Gamma(X, L)$ be a non-zero, torus invariant k -vector subspace. There exists a finite set $A \subseteq M \cap \square_L$ such that $V = \bigoplus_{m \in A} k \cdot \chi^m$. Let $\Lambda = \{(f) + L; f \in V \setminus \{0\}\}$ be the corresponding linear system.

Lemma 4.1. $w_{E_i}(\Lambda)$ is attained by a torus invariant member, computed by the formula:

$$w_{E_i}(\Lambda) = \max_{m \in A} \langle m, e_i \rangle + l_i. \quad \square$$

Proof. Let $t \geq 0$. The set $\{f \in V; \text{mult}_{E_i}((f) + L) \geq t\}$ is a torus invariant vector subspace of V . So it is non-zero if and only if it contains χ^m for some $m \in A$. It follows that the maximal (also minimal) value among $\text{mult}_{E_i}(D)$ ($D \in \Lambda$) is attained within the subset $\text{mult}_{E_i}((\chi^m) + L) = \langle m, e_i \rangle + l_i$ ($m \in A$). ■

We have $w_{E_i}(\Lambda) = 0$ if and only if A is contained in the hyperplane $\langle \cdot, e_i \rangle + l_i = 0$. It follows that we can compute $\gamma_x(X, B; \Lambda)$ for every torus invariant codimension one point $x \in X$. Indeed, x is the generic point of some E_i , and $\gamma_{E_i}(X, B; \Lambda)$ is $+\infty$ if $w_{E_i}(\Lambda) = 0$, and $\frac{1-b_i}{\max_{m \in A} \langle m, e_i \rangle + l_i}$ otherwise. Proposition 4.2 states that only these valuations determine $\gamma(X, B; \Lambda)$.

Proposition 4.2. $\gamma(X, B; \Lambda) = \min_i \gamma_{E_i}(X, B; \Lambda)$. In particular, $\gamma(X, B; \Lambda)$ is attained by a torus invariant member of Λ . ■

Proof. The inequality \leq is clear. For the converse, let $t \leq \min_i \gamma_{E_i}(X, B; \Lambda)$ and $D \in \Lambda$. We have to show that $(X, B + tD)$ has log canonical singularities.

Let $\mu: X' \rightarrow X$ be a toric birational modification which is an isomorphism in codimension one, and such that X' is \mathbb{Q} -factorial. The toric varieties X, X' have the same invariant prime divisors, and therefore we may pullback our data to X' . Therefore we may suppose X is \mathbb{Q} -factorial.

The conclusion is local on X , so we may shrink X to a torus invariant affine open neighborhood U of a fixed point. Thus $U = T_N \text{emb}(\sigma)$, where σ is a simplicial cone in $N_{\mathbb{R}}$ which generates $N_{\mathbb{R}}$. Let N' be the lattice generated by the primitive vectors $e_i \in N$ which generate the extremal rays of σ . The inclusion $N' \subseteq N$ induces a

toric morphism $\tau: U' \rightarrow U$ which is finite, and étale in codimension one. We have $(U', U' \setminus T) \simeq (\mathbb{A}_k^d, \sum_{j=1}^d H_j)$, where H_i are the standard hyperplanes of the affine space. We have $\tau^*(K_X + B + tD|_U) = K_{\mathbb{A}_k^d} + \sum_{j=1}^d b_j H_j + tD'$. We may suppose $[\tau^*L] = 0$, and therefore $(\mathbb{A}_k^d, \sum_{j=1}^d b_j H_j + tD' = \sum_{j=1}^d b_j H_j + t\{D'\} + t[D'])$ has log canonical singularities by Lemma 4.3. Therefore $(X, B + tD)$ has log canonical singularities on U . ■

Lemma 4.3. Let H_i be the standard hyperplanes of the affine space \mathbb{A}_k^d . Let $b_i \in [0, 1]$, $t > 0$, and $0 \neq P \in k[z_1, \dots, z_d]$ such that $\deg_{z_i}(P) \leq \frac{1-b_i}{t}$ for every $1 \leq i \leq d$. Let D be the divisor of zeros of P . Then $(\mathbb{A}_k^d, \sum_{i=1}^d b_i H_i + tD)$ has log canonical singularities. □

Proof. Denote $w_i = \deg_{z_i}(P)$. Consider the product of log varieties $\prod_{i=1}^d (\mathbb{P}^1, b_i \cdot 0)$. We have $\gamma(\mathbb{P}^1, b_i \cdot 0; |\mathcal{O}(w_i)|) = \frac{1-b_i}{w_i} \geq t$. By Theorem 3.7, $\gamma(\prod_{i=1}^d (\mathbb{P}^1, b_i \cdot 0); |\mathcal{O}(w_1, \dots, w_d)|) \geq t$.

Now P defines a divisor $D' \in |\mathcal{O}(w_1, \dots, w_d)|$. Therefore $(\prod_i \mathbb{P}^1, \boxtimes_i b_i \cdot 0 + tD')$ has log canonical singularities. After restricting to the complement of $\boxtimes_i \infty$, we obtain that $(\mathbb{A}_k^d, \sum_{i=1}^d b_i H_i + tD)$ has log canonical singularities. ■

Suppose now that $|nL| \neq \emptyset$ for some $n \geq 1$, that is $\square_L \neq \emptyset$. The width of L at E_i is computed by the formula

$$w_{E_i}(L) = \max_{m \in \square_L} \langle m, e_i \rangle + l_i.$$

Let $r \geq 1$ be the smallest integer such that the extremal points of $r\square_L$ belong to the lattice M (i.e., $\Gamma(rL) \otimes \Gamma(nL) \rightarrow \Gamma((r+n)L)$ is surjective for $n \gg 0$). Then $w_{E_i}(L) = \frac{w_{E_i}(rL)}{r}$.

The width of L at E_i is zero if and only if \square_L is contained in the hyperplane $\langle \cdot, e_i \rangle + l_i = 0$. In this case, $\gamma_{E_i}(X, B; L) = +\infty$. If $w_{E_i}(L) > 0$, then

$$\gamma_{E_i}(X, B; L) = \frac{1 - b_i}{w_{E_i}(L)}.$$

Proposition 4.2 for the complete linear systems $|nL|$ gives the following theorem.

Theorem 4.4. $\gamma(X, B; L) = \min_i \gamma_{E_i}(X, B; L)$. In particular, $\gamma(X, B; L)$ is attained by some invariant member $(\chi^m) + L$ ($m \in M_{\mathbb{Q}} \cap \square_L$). □

Recall that the *stable fixed multiplicity* of L in E_i is defined as

$$f_{E_i}(L) = \inf \left\{ \frac{\text{mult}_{E_i}(D_n)}{n}; n \geq 1, D_n \in |nL| \right\}.$$

In our toric setting, it has the following combinatorial formula:

$$f_{E_i}(L) = \min_{m \in \square_L} \langle m, e_i \rangle + l_i.$$

It is zero if and only if E_i is not fixed by $|nL|$ for some $n \geq 1$. More precisely, $rf_{E_i}(L) = \text{mult}_{E_i} \text{Fix}(|rL|)$, where r is defined above.

Recall that the *width* of a convex set $\square \subset M_{\mathbb{R}}$ along a direction $e \in N_{\mathbb{R}} \setminus 0$ is defined as $w(\square; e) = \sup\{\langle m', e \rangle - \langle m, e \rangle; m', m \in \square\}$. We obtain the identity:

$$w(\square_L; e_i) = w_{E_i}(L) - f_{E_i}(L).$$

Corollary 4.5. Let (X, B) be a proper toric log variety with log canonical singularities. Let L be a torus invariant \mathbb{Q} -Cartier divisor such that the linear system $|nL|$ is mobile for some $n \geq 1$. Then

$$\gamma(X, B; L) = \sup\{t \geq 0; t(\square_L - \square_L) \subseteq \square_{-K_X - B}\}. \quad \square$$

Proof. We have $t(\square_L - \square_L) \subseteq \square_{-K_X - B}$ if and only if $t(\square_L - m) \subseteq \square_{-K_X - B}$ for every $m \in \square_L$. Since $-K_X - B = \sum_i (1 - b_i)E_i$, this is equivalent to $t\langle m' - m, e_i \rangle + 1 - b_i \geq 0$ for every invariant prime divisor $E_i \subset X$, and $m, m' \in \square_L$. Equivalently, $t \cdot w(\square_L; e_i) \leq 1 - b_i$ for every i . By assumption, $f_{E_i}(L) = 0$ for every i . That is $w(\square_L; e_i) = w_{E_i}(L)$. The condition becomes $t \cdot w_{E_i}(L) \leq 1 - b_i$ for every i , that is $t \leq \gamma(X, B; L)$, by Theorem 4.4. ■

Lemma 4.6. Let (X, B) be a proper toric log variety, with log canonical singularities. Let L be a torus invariant \mathbb{Q} -Cartier divisor such that $|nL| \neq \emptyset$ for some $n \geq 1$.

- (a) $X \ni P \mapsto \gamma_P(X, B; L)$ is lower semi-continuous, and takes only finitely many values.
- (b) Let $P \in X \setminus T$ be a closed point outside the torus. Let O be the generic point of the unique torus orbit which contains P . Then $\gamma_P(X, B; L) = \gamma_O(X, B; L)$. ■

Proof. (a) Fix $t > 0$. Then $\{P \in X; \gamma_P(X, B; L) < t\} = \cup_{n \geq 1} Z(t, n)$, where $Z(t, n) = \cup_{D_n \in |nL|} (X, B + \frac{t}{n}D_n)_{-\infty}$. Each $Z(t, n)$ is torus invariant. And is closed by Theorem 2.8 applied to the universal divisor of $|nL|$. Since X contains only finitely many closed torus invariant subsets, $Z(t, n)$ belong to a finite set. Therefore, $\cup_n Z(t, n)$ is closed in X . We conclude that $P \mapsto \gamma_P(X, B; L)$ is lower semi-continuous. The function is constant on the torus orbits, and since X has only finitely many orbits, we conclude that the function takes only finitely many values.

(b) The inequality \leq is clear. For the converse, let $t \leq \gamma_O(X, B; L)$. Then $Z(t, n)$ is a closed torus invariant subset of X which does not contain the generic point of the orbit O . Therefore $Z(t, n)$ is disjoint from O . Therefore $t \leq \gamma_P(X, B; L)$. ■

5 Hensley-Type Diophantine Approximation

The main result of this section is Theorem 5.6, the combinatorial statement behind Theorem 6.1. This technically looking statement may be reformulated in terms of approximating a point on the real torus by its multiples (see the examples of Section 5, especially Lemma 6.8). Theorem 5.6 provides an effective way to show that if the origin is the unique interior lattice point of a certain polytope, then it cannot be too close to the boundary of the polytope. The interested reader should consult [6–9, 14, 17] for more on this topic.

5.1 Geometry of numbers [12]

Let $V \simeq \mathbb{R}^d$ be a finite dimensional \mathbb{R} -vector space. Let V^* be the dual vector space. The *dual* of a non-empty convex set $\square \subseteq V$ is defined as

$$\square^* = \{v^* \in V^*; \langle v^*, v \rangle + 1 \geq 0 \ \forall v \in \square\}.$$

It is closed, convex subset of V^* , containing the origin. The Duality Theorem states that if \square is compact convex and contains the origin in its interior, then $(\square^*)^* = \square$.

Let $\square \subset V$ be a compact convex set. Let $P \in \square$ be a point. Denote

$$\gamma(P \in \square) = \sup \{t \geq 0; P + t(\square - \square) \subseteq \square\}.$$

It is a well-defined non-negative real number, zero if and only if P does not belong to the relative interior of \square . We can think of $\gamma(\cdot \in \square)$ as a distance function to the boundary of \square .

Suppose $\dim \square = \dim V$ and P is an interior point of \square . For each $v \in V \setminus 0$, there exist $l^+, l^- > 0$ such that $P + l^+v, P - l^-v \in \partial \square$. The supremum of the ratio l^+/l^- , after directions $v \in V \setminus 0$, is called the *coefficient of asymmetry of \square about P* , denoted $c(P \in \square)$. We have

$$\gamma(P \in \square) = \frac{1}{1 + c(P \in \square)}.$$

By definition, $c(P \in \square) \geq 1$. Therefore $\gamma(P \in \square) \leq \frac{1}{2}$, and equality holds if and only if \square is symmetric about P .

If 0 is an interior point of a compact convex set \square , then so is $0 \in \square^*$, and $\gamma(0 \in \square) = \gamma(0 \in \square^*)$.

Suppose \square is a compact polytope and P is an interior point. To compute $c(P \in \square)$ it suffices to consider the directions v such that $P + \mathbb{R}v$ contains some vertex of \square . In particular, the supremum in the definition of $c(P \in \square)$ is a maximum. And

if both P and \square are rational with respect to some lattice Λ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$, then $c(P \in \square) \in \mathbb{Q}$. Fixing a line which attains the maximum and passes through a vertex of \square , we can apply Caratheodory's Theorem to the other boundary point, and obtain the following statement (called the *simplex trick*): there exists a simplex S (with $\dim S \leq \dim V$), with vertices among those of \square , containing P in its relative interior, and with $c(P \in \square) \leq c(P \in S)$.

If S is a simplex and P has barycentric coordinates $(\gamma_i)_i$ with respect to the vertices of S , then $\gamma(P \in S) = \min_i \gamma_i$.

We will use an asymmetric version of Minkowski's first theorem (cf. [12, Theorem 2, p. 52]).

Theorem 5.1. Let $\square \subset \mathbb{R}^d$ be a compact convex set, containing the origin in the interior. Then

$$|\mathbb{Z}^d \cap \text{int}(\square)| \geq \gamma(0 \in \square)^d \text{vol}_{\mathbb{Z}^d}(\square). \quad \square$$

Proof. Denote $\gamma = \gamma(0 \in \square)$. Let $k \geq 0$ be the largest integer such that $\text{vol}_{\mathbb{Z}^d}(\gamma\square) > k$. By Van der Corput's Theorem (see [12, Theorem 1, p. 47]) applied to $(\gamma - \epsilon)\square$ ($0 < \epsilon \ll 1$), there exist pairwise distinct elements v_1, \dots, v_{k+1} in the interior of $\gamma\square$, such that $v_i - v_j \in \mathbb{Z}^d$ for all i, j . Since $\gamma\square - \gamma\square \subseteq \square$, it follows that $(v_i - v_1)_{i=1}^{k+1}$ are $k+1$ vectors in $\mathbb{Z}^d \cap \text{int}(\square)$. Therefore, $|\mathbb{Z}^d \cap \text{int}(\square)| \geq k+1$. This proves that $|\mathbb{Z}^d \cap \text{int}(\square)| \geq \text{vol}_{\mathbb{Z}^d}(\gamma\square) = \gamma^d \text{vol}_{\mathbb{Z}^d}(\square)$. ■

The original Minkowski's first theorem asserts that if \square is symmetric about the origin, that is $\gamma(0 \in \square) = \frac{1}{2}$, then $\{0\} \subsetneq \mathbb{Z}^d \cap \text{int}(\square)$ if $\text{vol}_{\mathbb{Z}^d}(\square) > 2^d$.

5.2 Diophantine approximation

For positive integers p, q , define integers $u_{p,q}$ recursively as follows: $u_{1,q} = q$, $u_{p+1,q} = u_{p,q}(1 + u_{p,q})$. The following properties hold:

- $\sum_{i=1}^p \frac{1}{1+u_{i,q}} = \frac{1}{q} - \frac{1}{u_{p+1,q}}$, $\prod_{i=1}^p (1 + u_{i,q}) = \frac{u_{p+1,q}}{q}$.
- $\prod_{i=1}^p \frac{1}{1+u_{i,q}} = 1 - q \sum_{i=1}^p \frac{1}{1+u_{i,q}}$.
- $q \mid u_{p,q}$ and $\gcd(1 + u_{p,q}, 1 + u_{p',q}) = 1$ for $p \neq p'$.

The sequence $(1 + u_{p,1})_{p \geq 1} = (2, 3, 7, 43, \dots)$ is called the Sylvester sequence in the literature, and $t_{p,q} = 1 + u_{p,q}$ was also considered in [17]. And $u_{p,q}$ can be expressed as a polynomial in q , with leading term $q^{2^{p-1}}$.

Lemma 5.2. [18] Let d be a positive integer. Let $x_1 \geq \dots \geq x_d > 0$ and $y_1 \geq \dots \geq y_d > 0$ be real numbers such that $\prod_{i=1}^l x_i \geq \prod_{i=1}^l y_i$ for every $1 \leq l \leq d$. Then $\sum_{i=1}^d x_i \geq \sum_{i=1}^d y_i$, and equality holds if and only if $x_i = y_i$ for every i . \square

Lemma 5.3 (cf. [18]). Let d, q be positive integers. Let $x_1 \geq \dots \geq x_d > 0$ be real numbers such that $\prod_{i=1}^l x_i \leq 1 - q \sum_{i=1}^l x_i$ for every $1 \leq l \leq d$. Then $\sum_{i=1}^d x_i \leq \sum_{i=1}^d \frac{1}{1+u_{i,q}}$, and equality holds if and only if $x_i = \frac{1}{1+u_{i,q}}$ for every i . \square

Proof. Denote $y_i = \frac{1}{1+u_{i,q}}$. We show by induction on d that $\sum_{i=1}^d x_i \geq \sum_{i=1}^d y_i$ implies $y_i = x_i$ for all i .

Let $d = 1$. Then $x_1 \leq 1 - qx_1$ is equivalent to $x_1 \leq y_1$.

Let $d > 1$. We have $\prod_{i=1}^d x_i \leq 1 - q \sum_{i=1}^d x_i \leq 1 - q \sum_{i=1}^d y_i = \prod_{i=1}^d y_i$. Therefore, we may define l to be maximal with the property $\prod_{i \geq l} x_i \leq \prod_{i \geq l} y_i$. We distinguish two cases:

Case $l = d$. Then $x_d \leq y_d$. Then $\sum_{i=1}^{d-1} x_i \geq \sum_{i=1}^{d-1} y_i$. By induction, $y_i = x_i$ for $i < d$. It follows that $y_d = x_d$ too.

Case $l < d$. By the maximality of l , it follows that $\prod_{i=l}^k x_i \leq \prod_{i=l}^k y_i$ for every $l \leq k \leq d$, and the inequality is strict for $k < d$. Lemma 5.2 gives $\sum_{i=l}^d x_i < \sum_{i=l}^d y_i$. But then $\sum_{i=1}^{l-1} x_i < \sum_{i=1}^{l-1} y_i$. By induction, this is a contradiction. \blacksquare

Lemma 5.4. For indeterminates T_1, \dots, T_d , the following formula holds:

$$\det \begin{pmatrix} 1+T_1 & 1 & & 1 \\ 1 & 1+T_2 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+T_d \end{pmatrix} = \left(1 + \sum_{i=1}^d \frac{1}{T_i}\right) \prod_{i=1}^d T_i.$$

\square

Proof. We use induction on d . The case $d = 1$ is clear. Let $d \geq 2$. The determinant is of the form $c_1 T_1 + c_0$, where c_0, c_1 are polynomials in T_2, \dots, T_d . The constant term is obtained by setting $T_1 = 0$, and we compute $c_0 = \prod_{i=2}^d T_i$. The other term is the difference

$$\det \begin{pmatrix} 1+T_1 & 1 & & 1 \\ 1 & 1+T_2 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+T_d \end{pmatrix} - \det \begin{pmatrix} T_1 & 1 & & 1 \\ 1 & 1+T_2 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+T_d \end{pmatrix}.$$

We compute the determinants using the formula by permutations σ of $\{1, \dots, d\}$. If $\sigma(1) \neq 1$, the corresponding difference is zero. Therefore, $c_1 = (1 + T_1) \det_{d-1} - T_1 \det_{d-1} = \det_{d-1}$. By induction, $c_1 = \sum_{i=2}^d \prod_{j \neq i, 1} T_j + \prod_{i=2}^d T_i$. Then the determinant is $\sum_{i=2}^d \prod_{j \neq i} T_j + \prod_{i=1}^d T_i + \prod_{j \neq 1} T_j$, so the desired identity holds for d . ■

Lemma 5.5. Let $x_1, \dots, x_d > 0$ and $c_1, \dots, c_d \geq 1$ such that $1 - \prod_{i=1}^d x_i < \sum_{i=1}^d c_i x_i < 1$. Then there exists $z \in \mathbb{N}^d \setminus 0$ such that $\frac{z_j}{1 + \sum_i c_i z_i} < x_j$ for all j . □

Proof. Consider the convex set $U = \{z \in \mathbb{R}^d; \|Az\|_\infty < 1\}$, where A is the $d \times d$ matrix

$$\begin{pmatrix} c_1 - \frac{1}{x_1} & c_2 & & c_d \\ c_1 & c_2 - \frac{1}{x_2} & & c_d \\ \vdots & & \ddots & \vdots \\ c_1 & \dots & c_{d-1} & c_d - \frac{1}{x_d} \end{pmatrix}$$

and the norm $\|\cdot\|_\infty$ is the maximum absolute value of the components. By Lemma 5.4, we compute

$$\det A = (-1)^d \frac{1 - \sum_i c_i x_i}{\prod_i x_i}.$$

By assumption, $0 < |\det A| < 1$. Then $\text{vol}_{\mathbb{Z}^d}(U) = \frac{2^d}{|\det A|} > 2^d$. The convex body U is symmetric about the origin. By Minkowski's first theorem, there exists $0 \neq z \in \mathbb{Z}^d \cap U$. That is $z \in \mathbb{Z}^d \setminus 0$ and $|\sum_i c_i z_i - \frac{z_j}{x_j}| < 1$ for every j . We may suppose $\sum_i c_i z_i \geq 0$, after possibly replacing z by $-z$.

The inequality $\sum_i c_i z_i - \frac{z_j}{x_j} < 1$ gives $\frac{z_j}{x_j} > -1$, that is $z_j > -x_j$. Since $c_i \geq 1$, we obtain $\sum_{i=1}^d x_i < 1$. In particular, $x_j < 1$. Therefore $z_j > -1$, that is $z_j \geq 0$.

The other inequality $-1 < \sum_i c_i z_i - \frac{z_j}{x_j}$ is equivalent to $\frac{z_j}{1 + \sum_i c_i z_i} < x_j$. ■

The following statement is the effective version of [14, Lemma 2.4]. The case $q = 1$ was obtained in [6, Theorem 1.1].

Theorem 5.6. Let q be a positive integer, let $1 \leq c_1, \dots, c_d \leq q$. Let $x_1 \geq \dots \geq x_d > 0$ such that $\sum_{i=1}^d x_i \geq \sum_{i=1}^d \frac{q}{1+u_{i,q}}$. Suppose $x_i \neq \frac{q}{1+u_{i,q}}$ for some i . Then there exists $z \in \mathbb{N}^d \setminus 0$ such that $\frac{c_j z_j}{1 + \sum_i c_i z_i} < x_j$ for every j . □

Proof. We may suppose $\sum_{i=1}^d x_i < 1$. By Lemma 5.3 for $(\frac{x_i}{q})_i$, there exists $1 \leq l \leq d$ such that $\prod_{i=1}^l x_i > q^l (1 - \sum_{i=1}^l x_i)$. In particular, $\prod_{i=1}^l x_i > (\prod_{i=1}^l c_i) (1 - \sum_{i=1}^l x_i)$.

By Lemma 5.5 for $(\frac{x_i}{c_i})_i$, there exists $z \in \mathbb{N}^l \setminus 0$ such that $\frac{c_j z_j}{1 + \sum_{i=1}^l c_i z_i} < x_j$ for all $1 \leq j \leq l$. Set $z_j = 0$ for $j > l$. Then $z \in \mathbb{N}^d \setminus 0$ satisfies the claim. ■

5.3 Upper bound for coefficient of asymmetry

The case $q = 1$ of the following result was obtained in [7, Theorem 2.1].

Theorem 5.7. Let q be a positive integer. Let N be a d -dimensional lattice and S a simplex with vertices in N such that $N \cap \text{int}(\frac{1}{q}S) = \{0\}$. Then $\gamma(0 \in S) \geq \frac{q}{u_{d+1,q}}$, and equality holds if and only if there exists an isomorphism $N \simeq \mathbb{Z}^d$ which maps S to the convex hull of e_0, \dots, e_d , where e_1, \dots, e_d is the standard basis of \mathbb{Z}^d and $e_0 = -\sum_{i=1}^d \frac{u_{d+1,q}}{1+u_{i,q}} e_i$. □

Proof. Let v_0, \dots, v_d be the vertices of S . Let $0 = \sum_{i=0}^d \gamma_i v_i$ be the barycentric coordinates of the origin. Suppose $\gamma_0 = \min_i \gamma_i$.

Suppose by contradiction that $\gamma(0 \in S) < \frac{q}{u_{d+1,q}}$. That is $\gamma_0 < \frac{q}{u_{d+1,q}}$. Then $\sum_{i=1}^d \gamma_i > \sum_{i=1}^d \frac{q}{1+u_{i,q}}$. By Theorem 5.6, there exists $z \in \mathbb{N}^d \setminus 0$ such that $\frac{q z_i}{1 + \sum_i q z_i} < \gamma_i$ ($1 \leq i \leq d$). Set $z_0 = 0$ and denote $|z| = \sum_i z_i$. We have

$$-q \sum_i z_i v_i = \sum_i ((1 + q|z|) \gamma_i - q z_i) v_i.$$

On the right-hand side, the coefficients of v_i are positive, and add up to 1. Therefore $-q \sum_i z_i v_i \in \text{int}(S)$. That is $-\sum_i z_i v_i \in N \cap \text{int}(\frac{1}{q}S) \setminus 0$, a contradiction.

We conclude that $\gamma_0 \geq \frac{q}{u_{d+1,q}}$. Suppose now that $\gamma_0 = \frac{q}{u_{d+1,q}}$. The above arguments and Theorem 5.6 give $\gamma_i = \frac{q}{1+u_{i,q}}$ for $1 \leq i \leq d$. We obtain the barycentric coordinates

$$0 = \frac{q}{u_{d+1,q}} v_0 + \sum_{i=1}^d \frac{q}{1+u_{i,q}} v_i.$$

In particular, $\sum_{i=1}^d \frac{q}{1+u_{i,q}} (v_i - v_0) = -v_0 \in N$. Since $(1 + u_{i,q})_i$ are pairwise relatively prime, we obtain $\frac{q}{1+u_{i,q}} (v_i - v_0) \in N$ for every i . Since $q \mid u_{i,q}$, we deduce $v_i - v_0 = (1 + u_{i,q}) w_i$ for some $w_i \in N$. The vectors $w_1, \dots, w_d \in N$ are linearly independent, and $v_0 = -q \sum_{i=1}^d w_i$.

The inequality $\sum_{i=1}^d \frac{1}{1+u_{i,q}} < \frac{1}{q}$ implies $v_0 + \sum_{i=1}^d (0, 1] w_i \subset \text{int}(\frac{1}{q}S)$. Let $x_i \in (0, 1]$ such that $\sum_{i=1}^d x_i w_i \in N$. Then $v_0 + \sum_{i=1}^d x_i w_i \in N \cap \text{int}(\frac{1}{q}S)$. Therefore $v_0 + \sum_{i=1}^d x_i w_i = 0$. Therefore $x_i = 1$ for all i . We conclude that w_1, \dots, w_d is a basis for the lattice N .

But $v_i = (1 + u_{i,q}) w_i - \sum_{j=1}^d q w_j = \sum_j a_{ij} w_j$ ($1 \leq i \leq d$). By Lemma 5.4, we compute $\det(a_{ij}) = 1$. Therefore v_1, \dots, v_d is a basis of N . This induces an isomorphism $N \simeq \mathbb{Z}^d$ such that v_i ($1 \leq i \leq d$) correspond to the standard basis e_i ($1 \leq i \leq d$), and v_0 corresponds to $-\sum_{i=1}^d \frac{u_{d+1,q}}{1+u_{i,q}} e_i$. ■

The following is the sharp version of [14, Corollary 3.2].

Theorem 5.8. Let $\square \subset \mathbb{R}^d$ be a compact polytope with vertices in \mathbb{Z}^d , of dimension d . Suppose $\mathbb{Z}^d \cap \text{int}\square$ has cardinality $q \geq 1$. Then for every $P \in \mathbb{Z}^d \cap \text{int}\square$ we have $\gamma(P \in \square) \geq \frac{q}{u_{d+1,q}}$. \square

Proof. Fix $P \in \square$. We may replace \square by $\square - P$, so that $P = 0$. Suppose by contradiction that $\gamma(0 \in \square) < \frac{q}{u_{d+1,q}}$. By the simplex trick, there exists a simplex S with vertices among those of \square , which contains 0 in its relative interior, and $\gamma(0 \in \square) \geq \gamma(0 \in S)$.

Let $\dim S = d' \leq d$. Then $\gamma(0 \in S) \leq \gamma(0 \in \square) < \frac{q}{u_{d+1,q}} \leq \frac{q}{u_{d'+1,q}}$. By Theorem 5.7, there exists $0 \neq e \in \mathbb{Z}^d \cap \text{relint}(\frac{1}{q}S)$. Then $0, e, 2e, \dots, qe$ are $q+1$ distinct lattice points in the relative interior of S . They must be contained in $\mathbb{Z}^d \cap \text{int}\square$. Contradiction! \blacksquare

As in [14, Theorem 3.6, Corollary 3.7], we obtain the following corollary.

Corollary 5.9. Let \mathbb{Z}^d, \square, q as above. Then $\text{vol}_{\mathbb{Z}^d}(\square) \leq q(\frac{u_{d+1,q}}{q})^d$ and $\mathbb{Z}^d \cap \square$ has cardinality at most $d!q(\frac{u_{d+1,q}}{q})^d$. \square

5.4 Errata to [4]

The upper bound $n \leq c_d q^d$ in [4, Theorem 1.1] is not correct. In Step 1 of the proof, the constant γ depends not only on $d-1$, but on q as well. Since $\Lambda \simeq \mathbb{Z}^{d-1}$, S has vertices in $\frac{1}{q}\Lambda$ and $\Lambda \cap \text{int}(S) = \{0\}$, Lemma 5.10 gives $\gamma \geq \frac{q}{u_{d,q}}$. Step 2 of the proof gives $j \leq d!q^{d-1}\gamma^{-d+1}$. Since $ja \in \mathbb{Z}$, we obtain a correct effective upper bound for Theorem 1.1

$$n \leq d!u_{d,q}^{d-1}q.$$

This bound is probably not sharp.

Lemma 5.10. Let S be a simplex with vertices in $\frac{1}{q}\mathbb{Z}^d$, and such that $\{0\} = \mathbb{Z}^d \cap \text{int}(S)$. Then $\gamma(0 \in S) \geq \frac{q}{u_{d+1,q}}$. \square

Proof. The simplex $S' = qS$ has vertices in \mathbb{Z}^d and $\{0\} = \mathbb{Z}^d \cap \text{int}(\frac{1}{q}S')$. By Theorem 5.7, $\gamma(0 \in S') \geq \frac{q}{u_{d+1,q}}$. But $\gamma(0 \in S') = \gamma(0 \in S)$. \blacksquare

6 Toric Log Fano Varieties

Let (X, B) be a toric log variety with log canonical singularities. So $X = T_N \text{emb}(\Delta)$ is a toric variety and $B = \sum_i (1 - a_i)E_i$ with $0 \leq a_i \leq 1$. Since $K_X + \sum_i E_i = 0$, we obtain

$$K_X + B = \sum_i -a_i E_i.$$

Let $\sigma \in \Delta(\text{top})$. There exists $\psi_\sigma \in M_{\mathbb{Q}}$ such that $(\chi^{\psi_\sigma}) + K_X + B$ is zero on the open subset U_σ of X . That is $\langle \psi_\sigma, e_i \rangle = a_i$ for every $e_i \in \sigma(1)$.

6.1 Minimal log discrepancies

Each $e \in N^{\text{prim}} \cap \sigma$ defines a toric valuation E_e over U_σ , with log discrepancy computed by the formula

$$a(E_e; X, B) = \langle \psi_\sigma, e \rangle.$$

Since log resolutions exist in the toric category, we obtain

$$\text{mld}(U_\sigma; X, B) = \min \{ \langle \psi_\sigma, e \rangle; 0 \neq e \in N \cap \sigma \}.$$

The moment polytope associated with the \mathbb{Q} -divisor $-K_X - B$ is

$$\square_{-K_X - B} = \{ m \in M_{\mathbb{R}}; \langle m, e_i \rangle + a_i \geq 0 \ \forall e_i \in \Delta(1) \}.$$

It contains the origin of M . If $a_i > 0$ for every i , denote by P the convex hull of $\frac{e_i}{a_i}$ ($e_i \in \Delta(1)$). We compute $P^* = \square$. By duality,

$$\inf \{ t > 0; e \in tP \} = -h_{\square}(e) \ \forall e \in N_{\mathbb{R}}.$$

6.2 Toric weak log Fano varieties

Suppose moreover that X is proper and $-K_X - B$ is \mathbb{Q} -semiample. Note that $-K_X - B$ is \mathbb{Q} -semiample if and only if it is nef. And if $a_i > 0$ for every i , the \mathbb{Q} -divisor $-K_X - B$ is necessarily big.

The \mathbb{Q} -semiampleness condition is equivalent to $-\psi_\sigma \in \square$, for every $\sigma \in \Delta(\text{top})$. And $-K_X - B$ is \mathbb{Q} -ample if and only if the vertices of \square are precisely $(-\psi_\sigma)_{\sigma \in \Delta(\text{top})}$. Let $e \in N^{\text{prim}}$, let $\sigma \in \Delta(\text{top})$ contain e . Then $\square + \psi_\sigma \subseteq \sigma^\vee$. Therefore $\langle -\psi_\sigma, e \rangle = h_{\square}(e)$. Therefore

$$a(E_e; X, B) = -h_{\square}(e) = -\inf_{m \in \square} \langle m, e \rangle.$$

We obtain the global formula $\text{mld}(X, B) = -\sup_{0 \neq e \in N} h_{\square}(e)$. By duality,

$$\text{mld}(X, B) = \inf \{ t > 0; \{0\} \subsetneq N \cap tP \}.$$

By Corollary 4.5, the α -invariant of $-K_X - B$ with respect to (X, B) is computed by the formula

$$\gamma(X, B; -K_X - B) = \gamma(0 \in \square).$$

By duality, $\gamma(0 \in \square) = \gamma(0 \in \square^*)$. Denote $d = \dim X$.

Theorem 6.1. Let q be a positive integer. If $\text{mld}(X, B) \geq \frac{1}{q}$, then $\gamma(0 \in \square) \geq \frac{q}{u_{d+1,q}}$. \square

Proof. The assumption gives $a_i \geq \frac{1}{q}$ for every i . Suppose by contradiction that $\gamma < \frac{q}{u_{d+1,q}}$. By the simplex trick, there exists a simplex S , with vertices among those of \square^* , such that $0 \in \text{relint} S$ and $\gamma(0 \in S) \leq \gamma(0 \in \square^*)$. Therefore

$$\gamma(0 \in S) < \frac{q}{u_{d+1,q}}.$$

Let $d' \leq d$ be the dimension of S , and $\frac{e_i}{a_i}$ ($0 \leq i \leq d'$) its vertices. Let $\gamma_0, \dots, \gamma_{d'} > 0$, $\sum_{i=0}^{d'} \gamma_i = 1$ and $0 = \sum_{i=0}^{d'} \gamma_i \frac{e_i}{a_i}$. We have $\gamma(0 \in S) = \min_{i=0}^{d'} \gamma_i$. Say γ_0 is minimal. Then $\gamma_0 < \frac{q}{u_{d+1,q}} \leq \frac{q}{u_{d'+1,q}}$, that is $\sum_{i=1}^{d'} \gamma_i > \sum_{i=1}^{d'} \frac{q}{u_{d'+1,q}}$. Note that $1 \leq qa_i \leq q$. By Theorem 5.6, there exists $z \in \mathbb{N}^{d'} \setminus 0$ such that

$$\frac{qa_i z_i}{1 + \sum_{j=1}^{d'} qa_j z_j} < \gamma_i \quad (1 \leq i \leq d').$$

These inequalities are equivalent to

$$\frac{1}{q} > \max_{i=1}^{d'} \frac{a_i z_i}{\gamma_i} - \sum_{j=1}^{d'} a_j z_j.$$

The right-hand side is the smallest $t > 0$ such that $e = \sum_{i=1}^{d'} -z_i e_i$ belongs to tS . Therefore qe belongs to the relative interior of S , so in the interior of P as well.

We obtain $0 \neq e \in N \cap \text{int}(\frac{1}{q}P)$. Therefore $\text{mld}(X, B) < \frac{1}{q}$, which is a contradiction. \blacksquare

Corollary 6.2. If $\text{mld}(X, B) \geq \frac{1}{q}$, then $\sqrt[q]{(-K_X - B)^d} \leq \frac{d}{q} u_{d+1,q}$. \square

Proof. By Proposition 3.10, $\gamma \cdot \sqrt[q]{(-K_X - B)^d} \leq d$. \blacksquare

Example 6.3. The lower bound in Theorem 6.1 is sharp. In dimension one, $(\mathbb{P}^1, \frac{q-1}{q} \cdot \infty)$ is the only example which attains it. A higher dimensional example is constructed as follows. Let e_1, \dots, e_d be the standard basis of \mathbb{Z}^d , and

$$e_0 = \sum_{i=1}^d -\frac{u_{d+1,q}}{q(1+u_{i,q})} e_i.$$

Then e_0, \dots, e_d are primitive vectors in the lattice $N = \mathbb{Z}^d$, which they generate. Set $a_0 = \frac{1}{q}$ and $a_1 = \dots = a_d = 1$. This defines a toric log Fano variety $(X^d, (1 - \frac{1}{q})E_0)$

with

$$\begin{aligned}\square_{-K_X-B} &= \left\{ m \in \mathbb{R}_{\geq -1}^d; \sum_{i=1}^d \frac{u_{d+1,q}}{1+u_{i,q}} m_i \leq 1 \right\} \\ &= (-1, \dots, -1) + \left\{ m \in \mathbb{R}_{\geq 0}^d; \sum_{i=1}^d \frac{1}{1+u_{i,q}} m_i \leq \frac{1}{q} \right\}.\end{aligned}$$

The simplex $\square_{-K_X-B} = \text{Conv}(v_0, \dots, v_d)$ contains in interior the origin of M , with barycentric coordinates

$$0 = \frac{q}{u_{d+1,q}} v_0 + \sum_{i=1}^d \frac{q}{1+u_{i,q}} v_i.$$

So $\gamma(X, B; -K_X - B) = \frac{q}{u_{d+1,q}}$. One checks that $\text{mld}(X, B) = \frac{1}{q}$, $\text{Bs}| -q(K+B)| = \emptyset$ and $(-q(K_X + B))^d = \frac{u_{d+1,q}}{q}$. In particular, the upper bound in Corollary 6.2 is not sharp. \square

Theorem 6.4 (cf. [8]). Fix $d \geq 1$ and $\epsilon \in (0, 1]$. Consider toric proper varieties X such that $\dim X = d$ and there exists an invariant effective \mathbb{Q} -divisor B such that $-K_X - B$ is \mathbb{Q} -semiample and $\text{mld}(X, B) \geq \epsilon$. Then X belongs to finitely many isomorphism types. \square

Proof. Let $\gamma = \gamma(0 \in \square_{-K_X-B}) = \gamma(0 \in P_{-K_X-B})$. By Theorem 6.1, $\gamma \geq \gamma(d, \epsilon) > 0$. We have $\epsilon \gamma(P - P) \subseteq \epsilon P$. Therefore $\{0\} = N \cap \text{int}(\epsilon \gamma(P - P))$. Theorem 5.1 gives

$$\text{vol}_N(P - P) \leq \frac{1}{\gamma^d \epsilon^d}.$$

Let C be the convex hull of $\Delta(1)$. Since $C \subseteq P$, we deduce that $\text{vol}_N(C - C)$ is bounded above. Since C is a lattice polytope, it follows that the pair $(N, C - C)$ belongs to finitely many isomorphism types. There exist only finitely many fans Δ with given $\Delta(1)$. Therefore (N, Δ) belongs to finitely many isomorphism types. \blacksquare

6.3 Examples

Consider toric log Fano varieties (X, B) , of Picard number one, and such that X has no nontrivial toric finite covers which are étale in codimension one. Let $X = T_N \text{emb}(\Delta)$, $\dim N = d$. Then $\Delta(1) = \{e_0, \dots, e_d\}$, where $e_0, \dots, e_d \in N$ are primitive and generate the lattice N , and no d of them are linearly dependent. The log discrepancies in invariant prime divisors are rational numbers $a_0, \dots, a_d \in [0, 1]$, not all zero. We have

$$0 = \sum_{i=0}^d x_i e_i \left(x_i \in \mathbb{Q}_{>0}, \sum_i x_i = 1 \right).$$

The vector e_i is primitive in the lattice generated by e_0, \dots, e_d if and only if $n_i = 1$, where

$$n_i = \frac{\gcd(qx_0, \dots, \widehat{qx_i}, \dots, qx_d)}{\gcd(qx_0, \dots, qx_d)} \quad (q \geq 1, qx \in \mathbb{Z}^{d+1}).$$

Up to isomorphism, (X, B) is uniquely determined by $(x_i)_i$ and $(a_i)_i$. The dual polytope $\square_{-K_X-B}^*$ is the simplex with vertices e_i/a_i .

The dual description of (X, B) is: $0 \in \square \subset M_{\mathbb{R}}$ is a rational simplex with vertices v_0, \dots, v_d , and 0 lies at distance a_i to the face of \square opposite v_i (distance measured with respect to e_i , the primitive interior direction normal to the face opposite to v_i). Let

$$0 = \sum_{i=0}^d \gamma_i v_i \quad \left(\gamma_i \geq 0, \sum_i \gamma_i = 1 \right).$$

Then $\gamma_i = \frac{a_i}{w_i}$, where $w_i = \text{width}(\square; e_i)$. The above data are related as follows:

- $\langle v_i, e_j \rangle + a_j = 0$ for $i \neq j$, and $\langle v_j, e_j \rangle + a_j = w_j$.
- $(\sum_i a_i x_i)(\sum_i \frac{1}{w_i}) = 1$.
- $x_j = \frac{\frac{1}{w_j}}{\sum_{i=0}^d \frac{1}{w_i}}, \gamma_j = \frac{a_j x_j}{\sum_{i=0}^d a_i x_i}, w_j = \frac{\sum_i a_i x_i}{x_j}$.

We have $K_X + B + \sum_{i=0}^d a_i E_i \sim 0$. For $0 \leq i \leq d$, set $D_i = (\chi^{v_i}) - K_X - B = w_i E_i$. So $K_X + B + D_i \sim_{\mathbb{Q}} 0$ and $\text{lct}(X, B; D_i) = \gamma_i$.

Note that $B = 0$ if and only if $a_i = 1$ for all i , and then $x_i = \gamma_i = \frac{1}{w_i}$ for all i .

Lemma 6.5. $r(K_X + B)$ is Cartier if and only if $|\text{Bs}| - r(K_X + B) = \emptyset$, if and only if $ra_i, rw_i \in \mathbb{Z}$ for every i . □

Lemma 6.6. $\gamma(X, B; -K_X - B) = \min_{i=0}^d \gamma_i$. □

Lemma 6.7. Let $0 < \epsilon \leq \min_{i=0}^d a_i$. Then $\text{mld}(X, B) \geq \epsilon$ if and only if there does not exist an integer $n \geq 0$ such that $\sum_i \{(n+1)x_i\} = 1$, $\{(n+1)x\} \neq x$, and

$$\sum_{i=0}^d a_i \{(n+1)x_i\} - \epsilon < \min_{j=0}^d w_j \{(n+1)x_j\}.$$

The fractional parts of vectors are defined componentwise. □

Proof. We clearly have $\text{mld}(X, B) = a \leq \min_{i=0}^d a_i$. Log discrepancies in toric valuations are computed as follows: if $e = \sum_{i=0}^d t_i e_i$ is a primitive vector in N , the induced toric

valuation E_e has log discrepancy

$$a(E_e; X, B) = \inf \left\{ \epsilon > 0; \sum_{i=0}^d t_i e_i \in \epsilon \cdot \square^* \right\} = \max_{j=0}^d \sum_{i=0}^d a_i t_i - w_j t_j.$$

We have $a \cdot \square^* \subseteq C = \text{Conv}(e_0, \dots, e_d)$. Therefore, the primitive vectors which attain a must be contained in C . But $N \cap C$ is parametrized as follows:

$$N \cap C \setminus \{e_i; i\} = \left\{ \sum_i \{(n+1)x_i\} e_i = \sum_i \lfloor (n+1)x_i \rfloor e_i; n \geq 0, \sum_i \{(n+1)x_i\} = 1 \right\}.$$

Note $\sum_i \{(n+1)x_i\} = 1$ if and only if $\sum_i \lfloor (n+1)x_i \rfloor = n$. And $\sum_i \{(n+1)x_i\} e_i = 0$ if and only if $\{(n+1)x\} = x$, if and only if $nx \in \mathbb{Z}^{d+1}$. We obtain

$$a = \min \left(\min_{i=0}^d a_i, \min_{\sum_i \{(n+1)x_i\}=1, \{(n+1)x\} \neq x} \sum_i a_i \{(n+1)x_i\} - \min_j w_j \{(n+1)x_j\} \right). \quad \blacksquare$$

Lemma 6.7 has a geometric reformulation: let $S = \{x \in \mathbb{R}_{\geq 0}^{d+1}; \sum_{i=0}^d x_i = 1\}$, with vertices P_0, \dots, P_d . Our point $x = \sum_i x_i P_i$ lies in the interior of S . Let $S_\epsilon(x)$ be the simplex with vertices $Q_i = (1 - \frac{\epsilon}{a_i})x + \frac{\epsilon}{a_i} P_i$ ($0 \leq i \leq d$). It is a neighborhood of x in S obtained by sliding each vertex of S toward x , with a certain weight. Then:

- $\text{mld}(X, B) \geq \epsilon$ if and only if $\text{int} S_\epsilon(x)$ contains no $\{(n+1)x\}$ other than x .
- $x = \sum_i \gamma_i Q_i$.

So both $\text{mld}(X, B)$ and $\gamma(X, B; -K_X - B)$ are encoded by the neighborhood $x \in S_\epsilon(x)$. The condition $\text{mld}(X, B) \geq \epsilon$ means that the rational point x is badly approximated by the fractional parts of its multiples, and Theorem 6.4 states that such x belong to a finite set.

Lemma 6.8. Let $\epsilon \leq \min_{i=0}^d a_i$. Then $\text{mld}(X, B) < \epsilon$ if and only if one of the following equivalent conditions hold:

- (a) There exists $n \geq 1$ such that $\sum_{i=0}^d \{(n+1)x_i\} = 1$, $\{(n+1)x\} \neq x$, and

$$\frac{\sum_{i=0}^d a_i \{(n+1)x_i\} - \epsilon}{\sum_{i=0}^d a_i x_i} < \min_{j=0}^d \frac{\{(n+1)x_j\}}{x_j}.$$

- (b) There exists $z \in \mathbb{N}^{d+1}$ such that $\max_j w_j z_j - \sum_i a_i z_i \in (0, \epsilon)$.

Moreover, $z = \lfloor (1 + |z|)x \rfloor$. So if z exists, it is uniquely determined by $|z|$ and x . If $qx \in \mathbb{Z}^{d+1}$, (b) can be decided by considering only the multiples $x, 2x, \dots, (q-1)x$. \square

Example 6.9. Suppose $B = 0$. That is $a_i = 1$, $x_i = \gamma_i = \frac{1}{w_i}$. Then $\text{mld}(X, 0) < \epsilon$ if and only if there exists $n \geq 1$ such that $\{(n+1)x\} \neq x$ and $1 - \epsilon < \min_j \frac{\{(n+1)x_j\}}{x_j}$, if and only if there exists $z \in \mathbb{N}^{d+1} \setminus 0$ such that $\frac{z_j}{\epsilon + \sum_{i=0}^d z_i} < x_j$ for every j . \square

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