NONLINEAR PROGRAMMING WITH SEMILOCALLY B-PREINVEX AND RELATED FUNCTIONS

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A nonlinear programming problem is considered where the functions involved are \( \eta \)-semidifferentiable. Fritz John type and Karush-Kuhn-Tucker type necessary optimality conditions are obtained. Moreover, a result relative to sufficiency of optimality conditions is given. Wolfe type and Mond-Weir type duality results are formulated in terms of \( \eta \)-semidifferentials. The duality results are given using the concepts of generalized semilocally b-preinvex functions. Our results generalize the results obtained by Preda, Stancu-Minasian and Batatorescu [2], Suneja and Gupta [5], Suneja et al. [6].

1. PRELIMINARIES

In this section, we introduce the notation and definitions which are used throughout the paper.

Let \( \mathbb{R}^n \) be the n-dimensional Euclidean space and \( \mathbb{R}_+^n \) its positive orthant, i.e. \( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n, x = (x_j), x_j \geq 0, j=1,\ldots,n \} \).

For \( x,y \in \mathbb{R}^n \), by \( x \leq y \) we mean \( x_i \leq y_i \) for all \( i \), \( x \leq y \) means \( x_i \leq y_i \) for all \( i \) and \( x_j < y_j \) for at least one \( j, 1 \leq j \leq n \). By \( x < y \) we mean \( x_i < y_i \) for all \( i \), and by \( x \not\leq y \) we mean the negation of \( x \leq y \).

Throughout the paper all definitions, theorems, lemmas, corollaries, remarks are numbered consecutively in a single numeration system in each section.

Let \( 0 \subseteq X^0 \subseteq \mathbb{R}^n \) be a set and \( \eta: 0 \times X^0 \rightarrow \mathbb{R}^n \) a vector function.

**Definition 1.1.** We say that \( 0 \) is \( \eta \)-vex at \( x \in 0 \) if \( x + \lambda \eta(x, x) \in 0 \) for all \( x \in 0 \) and \( \lambda \in [0, 1] \).

We say that \( 0 \) is \( \eta \)-vex if \( 0 \) is \( \eta \)-vex at any \( x \in 0 \).

We remark that if \( \eta(x, \bar{x}) = x - \bar{x} \) for any \( x \in 0 \), then \( 0 \) is \( \eta \)-vex at \( \bar{x} \in X^0 \) iff \( X^0 \) is a convex set at \( \bar{x} \).

**Definition 1.2.** [7] Let \( X^0 \subseteq \mathbb{R}^n \) be a nonempty set. A function \( f: X^0 \rightarrow \mathbb{R} \) is said to be preinvex on \( X^0 \) (with respect to \( \eta \)) if \( f \) is \( \eta \)-vex, for short) if there exists an \( n \)-dimensional vector function \( \eta: X^0 \times X^0 \rightarrow \mathbb{R}^n \) such that for all \( x, u \in X^0 \) and \( \lambda \in [0, 1] \) we have

\[
  f(u + \lambda \eta(x, u)) \leq \lambda f(x) + (1 - \lambda) f(u).
\]

**Definition 1.3.** We say that \( X^0 \subseteq \mathbb{R}^n \) is an \( \eta \)-locally starshaped set at \( \bar{x} \) (\( \bar{x} \in X^0 \)) if for any \( x \in X^0 \) there exists \( 0 < a_\eta(x, \bar{x}) \leq 1 \) such that \( \bar{x} + \lambda \eta(x, \bar{x}) \in X^0 \) for any \( \lambda \in [0, a_\eta(x, \bar{x})] \).

We say that \( X^0 \) is \( \eta \)-locally starshaped if \( X^0 \) is \( \eta \)-locally starshaped at any \( \bar{x} \in X^0 \).

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Definition 1.4. Let $f: X^0 \to \mathbb{R}$ be a function, where $X^0 \subseteq \mathbb{R}^n$ is an $\eta$-locally starshaped set at $\bar{x} \in X^0$, with the corresponding maximum positive number $a_\eta(x, \bar{x})$ satisfying the required conditions. We say that $f$ is:

(i) semilocally $b$-preinvex (slb-preinvex) at $\bar{x}$ if for any $x \in X^0$, there exist a positive number $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$ and a function $b: X^0 \times X^0 \times [0,1] \to \mathbb{R}_+$ such that $f(\bar{x} + \lambda \eta(x, \bar{x})) \geq \lambda b(x, \bar{x}, \lambda)f(x) + (1 - \lambda b(x, \bar{x}, \lambda))f(\bar{x})$ for $0 < \lambda < d_\eta(x, \bar{x})$, $\lambda b(x, \bar{x}, \lambda) \leq 1$.

If $f$ is semilocally $b$-preinvex at each $\bar{x} \in X^0$ for the same $b$, then $f$ is said to be semilocally $b$-preinvex on $X^0$.

(ii) semilocally quasi $b$-preinvex (slbq-preinvex) at $\bar{x}$ if for any $x \in X^0$, there exist a positive number $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$ and a function $b: X^0 \times X^0 \times [0,1] \to \mathbb{R}_+$ such that

\[
\begin{align*}
0 < \lambda < d_\eta(x, \bar{x}) & \implies b(x, \bar{x}, \lambda)f(\bar{x} + \lambda \eta(x, \bar{x})) \leq b(x, \bar{x}, \lambda)f(\bar{x}) \\
\lambda b(x, \bar{x}, \lambda) & \leq 1
\end{align*}
\]

If $f$ is semilocally quasi $b$-preinvex at each $\bar{x} \in X^0$ for the same $b$, then $f$ is said to be semilocally quasi $b$-preinvex on $X^0$.

Definition 1.5. [2],[3] Let $f: X^0 \to \mathbb{R}$ be a function, where $X^0 \subseteq \mathbb{R}^n$ is an $\eta$-locally starshaped set at $\bar{x} \in X^0$. We say that $f$ is $\eta$-semidifferentiable at $\bar{x}$ if $(df)^+ (\bar{x}, \eta(x, \bar{x}))$ exists for each $x \in X^0$, where

\[
(df)^+ (\bar{x}, \eta(x, \bar{x})) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} [f(\bar{x} + \lambda \eta(x, \bar{x})) - f(\bar{x})]
\]

(the right derivative at $\bar{x}$ along the direction $\eta(x, \bar{x})$).

If $f$ is $\eta$-semidifferentiable at any $\bar{x} \in X^0$, then $f$ is said to be $\eta$-semidifferentiable on $X^0$.

Note that semidifferentiable functions correspond to $\eta(x, \bar{x}) = x - \bar{x}$.

Some properties possessed by the semidifferentiable functions are given by Kaul and Lyall [1].

Definition 1.6. Let $f: X^0 \to \mathbb{R}$ be an $\eta$-semidifferentiable function on $X^0 \subseteq \mathbb{R}^n$. We say that $f$ is semilocally pseudo $b$-preinvex (slp-preinvex) at $\bar{x} \in X^0$ if

\[
(df)^+ (\bar{x}, \eta(x, \bar{x})) \geq 0 \implies b(x, \bar{x}, \lambda)f(x) \geq b(x, \bar{x}, \lambda)f(\bar{x}).
\]

If $f$ is semilocally pseudo $b$-preinvex at each $\bar{x} \in X^0$ for the same $b$, then $f$ is said to be semilocally pseudo $b$-preinvex on $X^0$.

Definition 1.7. Let $f: X^0 \to \mathbb{R}$ be an $\eta$-semidifferentiable function on $X^0 \subseteq \mathbb{R}^n$. We say that $f$ is semilocally explicitly $b$-preinvex (sleb-preinvex) at $\bar{x} \in X^0$ if for each $x \in X^0$, $x \neq \bar{x}$, we have

\[
\overline{b} (x, \bar{x}) [f(x) - f(\bar{x})] > (df)^+ (\bar{x}, \eta(x, \bar{x}))
\]

where

\[
\overline{b} (x, \bar{x}) = \lim_{\lambda \to 0^+} b(x, \bar{x}, \lambda).
\]

Definition 1.8. Let $f: X^0 \to \mathbb{R}$ be an $\eta$-semidifferentiable function on $X^0 \subseteq \mathbb{R}^n$. We say that $f$ is semilocally strongly pseudo $b$-preinvex (slspb-preinvex) at $\bar{x} \in X^0$ if
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\[ b(x, \bar{x})(df)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 \Rightarrow f(x) \geq f(\bar{x}) \]

where \( b(x, \bar{x}) \) is defined by (1.1).

If \( f \) is slsb-preinvex at each \( \bar{x} \in X^0 \) for the same \( b \), then \( f \) is said to be slsb-preinvex on \( X^0 \).

For \( b(x, \bar{x}, \lambda) = 1 \) these definitions reduce to those of semilocally preinvex, semilocally quasi-preinvex, semilocally pseudo-preinvex considered by Preda, Stancu-Minasian and Batatorescu [2].

**Theorem 1.9.** Let \( f : X^0 \to \mathbb{R} \) be an \( \eta \)-semidifferentiable function on an \( \eta \)-locally starshaped set \( X^0 \).

\( b(x, \bar{x}) \)

1. **a)** The function \( f \) is slb-preinvex at \( x \in X^0 \) if and only if \( (df)^+(\bar{x}, \eta(x, \bar{x})) \) exists and \( b(x, \bar{x})(f(x) - f(\bar{x})) \geq (df)^+(\bar{x}, \eta(x, \bar{x})) \).

2. **b)** If \( f \) is slqb-preinvex, then \( f(x) \leq f(\bar{x}) \Rightarrow b(x, \bar{x})(df)^+(\bar{x}, \eta(x, \bar{x})) \leq 0 \), where \( b(x, \bar{x}) = \lim_{\lambda \to 0^+} b(x, \bar{x}, \lambda) \) and \( \lambda \) \( b(x, \bar{x}, \lambda) \leq 1 \).

### 2. SUFFICIENT OPTIMALITY CRITERIA

Consider the nonlinear programming problem

\[ \begin{align*}
\text{(NP)} & \quad \text{Minimize } f(x) \\
& \quad \text{subject to: } g(x) \leq 0, \ x \in X^0
\end{align*} \]

where \( X^0 \subseteq \mathbb{R}^n \) is a nonempty \( \eta \)-locally starshaped set and \( f : X^0 \to \mathbb{R}, \ g : X^0 \to \mathbb{R}^m \) are \( \eta \)-semidifferentiable functions.

Let \( X = \{ x \in X^0 : g(x) \leq 0 \} \) be the set of all feasible solutions to (NP).

Let \( N_\varepsilon(\bar{x}) = \{ x \in \mathbb{R}^n : \| x - \bar{x} \| < \varepsilon \} \)

**Definition 2.1.** (a) \( \bar{x} \) is said to be a local minimum solution to problem (NP) if \( \bar{x} \in X \) and there exists \( \varepsilon > 0 \) such that \( x \in N_\varepsilon(\bar{x}) \cap X \Rightarrow f(x) \geq f(\bar{x}) \).

(b) \( \bar{x} \) is said to be the minimum solution to problem (NP) if \( \bar{x} \in X \) and \( f(\bar{x}) = \min_{x \in X} f(x) \).

The next theorem gives a sufficient optimality criterion.

**Theorem 2.2.** Let \( \bar{x} \in X^0 \) and let \( f \) be slb\(_1\)-preinvex at \( \bar{x} \) and \( g \) be slb\(_2\)-preinvex at \( \bar{x} \). If there exists \( \bar{u} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{u}) \) satisfies the conditions

\[ (df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T(g(\bar{x}))^+(\bar{x}, \eta(x, \bar{x})) \geq 0, \forall x \in X, \]

\[ \bar{u}^T g(\bar{x}) = 0, \]

\[ g(\bar{x}) \leq 0, \]

\[ \bar{u} \geq 0, \]

with \( b_1(x, \bar{x}) = \lim_{\lambda \to 0^+} b_1(x, \bar{x}, \lambda) > 0 \), then \( \bar{x} \) is an optimal solution to problem (NP).

**Corollary 2.3.** Let \( \bar{x} \in X^0 \) and let \( f \) be slb\(_1\)-preinvex at \( \bar{x} \) and \( g \) be slb\(_2\)-preinvex at \( \bar{x} \). If there exists \( \bar{u}_0 \in \mathbb{R} \) and \( \bar{u} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{u}_0, \bar{u}) \) satisfy (2.2) and (2.3) of Theorem 2.2., and the conditions
\[
\overline{u}_0 (df)^+ (\overline{x}, \eta(x, \overline{x})) + \overline{u}_0^T (dg)^+ (\overline{x}, \eta(x, \overline{x})) \geq 0, \quad \forall x \in X
\]

\[
(\overline{u}_0, \overline{u}_0^T) \geq 0, \quad (\overline{u}_0, \overline{u}) \neq 0
\]

with \( \overline{b}_i (x, \overline{x}) = \lim_{\lambda \to 0^+} b_i (x, \overline{x}, \lambda) \), then \( \overline{x} \) is an optimal solution to problem (NP).

**Remark 2.4.** In the statement of Corollary 2.3, it suffices to assume only the \( \text{slb}_2 \)-preinvexity of \( g_i (I = \{i \mid g_i (\overline{x}) = 0\}) \), instead of \( g_i (i = 1, \ldots, m) \) at \( \overline{x} \).

**Theorem 2.5.** Let \( \overline{x} \in X^0 \), \( f \) be \( \text{slspb}-\)preinvex and \( g_i \) be \( \eta \)-semidifferentiable and \( \text{slqb}-\)preinvex at \( \overline{x} \). If there exists \( \overline{u} \in \mathbb{R}^m \) such that \( (\overline{x}, \overline{u}) \) satisfy conditions (2.1) - (2.4) of Theorem 2.2, then \( \overline{x} \) is an optimal solution to Problem (NP).

**Theorem 2.6.** Let \( \overline{x} \in X^0 \). We assume that there exists \( \overline{u} \in \mathbb{R}^m \) such that at \( \overline{x} \), \( f \) is \( \text{slspb}-\)preinvex, the numerical function \( \overline{u}_i \), \( g_i \) is \( \eta \)-semidifferentiable and \( \text{slqb}-\)preinvex and such that \( (\overline{x}, \overline{u}) \) satisfies conditions (2.1) - (2.4) of Theorem 2.2. Then \( \overline{x} \) is an optimal solution to Problem (NP).

**Theorem 2.7.** Let \( \overline{x} \in X^0 \). We assume that there exists \( \overline{u} \in \mathbb{R}^m \) such that \( (\overline{x}, \overline{u}) \) satisfies conditions (2.1) - (2.4) of Theorem 2.2 and the numerical function \( f + \overline{u}_i g_i \) is \( \text{slspb}-\)preinvex at \( \overline{x} \). Then \( \overline{x} \) is an optimal solution to Problem (NP).

### 3. NECESSARY OPTIMALITY CRITERIA

**Definition 3.1.** We say that \( g \) satisfies the generalized Slater's constraint qualification (GSQ) at \( \overline{x} \in X \), if \( g_i \) is \( \text{slp}-\)preinvex at \( \overline{x} \) and there exists \( \hat{x} \in X \) such that \( g_i (\hat{x}) < 0 \).

**Lemma 3.2.** Let \( \overline{x} \in X \) be a local minimum solution to (NP). We assume that \( g_i \) is continuous at \( \overline{x} \) for any \( i \in J \), and that \( f_i \), \( g_i \) are \( \eta \)-semidifferentiable at \( \overline{x} \). Then the system

\[
\begin{align*}
(df)^+ (\overline{x}, \eta(x, \overline{x})) &< 0 \\
(dg_i)^+ (\overline{x}, \eta(x, \overline{x})) &< 0
\end{align*}
\]

has no solution \( x \in X^0 \).

**Theorem 3.3.** (Fritz John type necessary optimality criteria) Let us suppose that \( g_i \) is continuous at \( \overline{x} \) for \( i \in J \). Assume also that \( (df)^+ (\overline{x}, \eta(x, \overline{x})) \) and \( (dg_i)^+ (\overline{x}, \eta(x, \overline{x})) \) are preinvex functions of \( x \) on \( X^0 \), which is an \( \eta \)-locally starshaped set at \( \overline{x} \). If \( \overline{x} \) is a local minimum solution to Problem (NP), then there exist \( \overline{u}_0 \in \mathbb{R} \), \( \overline{u} \in \mathbb{R}^m \) such that

\[
\overline{u}_0 (df)^+ (\overline{x}, \eta(x, \overline{x})) + \overline{u}_0^T (dg)^+ (\overline{x}, \eta(x, \overline{x})) \geq 0 \text{ for all } x \in X^0,
\]

\[
(\overline{u}_0, \overline{u}) \neq 0, \quad (\overline{u}_0, \overline{u}) \geq 0.
\]

**Theorem 3.4.** (Kuhn-Tucker type necessary optimality criteria) Let \( \overline{x} \in X \) be a local minimum solution to Problem (NP) and let \( g_i \) be continuous at \( \overline{x} \) for \( i \in J \). Assume also that \( (df)^+ (\overline{x}, \eta(x, \overline{x})) \) and \( (dg_i)^+ (\overline{x}, \eta(x, \overline{x})) \) be preinvex functions of \( x \) on \( X^0 \) - an \( \eta \)-locally starshaped set at \( \overline{x} \). If \( g \) satisfies GSQ at \( \overline{x} \), then there exists \( \overline{u} \in \mathbb{R}^m \) such that

\[
(df)^+ (\overline{x}, \eta(x, \overline{x})) + \overline{u}_0^T (dg)^+ (\overline{x}, \eta(x, \overline{x})) \geq 0 \text{ for all } x \in X^0,
\]

\[
\overline{u}_0^T g(\overline{x}) = 0, \quad g(\overline{x}) \leq 0, \quad \overline{u} \geq 0.
\]
4. WOLFE DUALITY

Relative to the Problem (NP) we consider the Wolfe dual

\[
\begin{align*}
\text{(WD)} & \quad \text{Maximize} & & \Psi(u,y) = f(u) + y^T g(u) \\
& & \text{subject to} & (df)^+(u,\eta(x,u)) + y^T (dg)^+(u,\eta(x,u)) \geq 0, \text{ for all } x \in X, \\
& & & y \geq 0, u \in X^0, y \in \mathbb{R}^m,
\end{align*}
\]

where \( X^0 \) is a nonempty \( \eta \)-locally starshaped set at any \( x \in X^0 \).

Let \( W \) denote the set of all feasible solutions to Problem (WD).

**Theorem 4.1.** (Weak Duality) Let \( \bar{x} \in X \) and \( (\bar{u}, \bar{y}) \in W \). If \( f \) and \( g \) are \( \eta \)-preinvex on \( X^0 \), with \( \bar{b}(\bar{x}, \bar{u}) = \lim_{\lambda \to 0^+} b(\bar{x}, \bar{u}, \lambda) > 0 \), then \( f(\bar{x}) \geq \Psi(\bar{u}, \bar{y}) \).

**Corollary 4.2.** Let \( \bar{x} \in X \) and \( (\bar{u}, \bar{y}) \in W \) such that \( f(\bar{x}) = \Psi(\bar{u}, \bar{y}) \). If the hypotheses of Theorem 4.1 are satisfied, then \( \bar{x} \) and \( (\bar{u}, \bar{y}) \) are the optimal solutions to (NP) and (WD) respectively.

**Theorem 4.3.** (Direct Duality) Let \( \bar{x} \in X \) be an optimal solution to (NP), \( f \) and \( g \) be \( \eta \)-semidifferentiable at \( \bar{x} \) and

1. \( (df)^+(\bar{x},\eta(x,\bar{x})) \) and \( y^T (dg)^+(\bar{x},\eta(x,\bar{x})) \) are preinvex functions of \( x \) on \( X^0 \), an \( \eta \)-locally starshaped set at \( \bar{x} \);
2. \( g_i \) (\( i \in J \)) are continuous at \( \bar{x} \);
3. \( g \) satisfies the generalized Slater’s constraint qualification at \( \bar{x} \).

Then there exists \( \bar{y} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{y}) \in W \) and \( f(\bar{x}) = \Psi(\bar{u}, \bar{y}) \).

Moreover, if the functions \( f \) and \( g \) are \( \eta \)-preinvex on \( X^0 \) and \( \bar{b}(x,u) > 0 \) for all \( (u, y) \in W \), then \( \bar{x} \) is an optimal solution to (NP) and \((\bar{x}, \bar{y})\) is an optimal solution to (WD).

**Theorem 4.4.** (Strict Converse Duality) Let \( \bar{x} \in X \) be an optimal solution to (NP), \( f \) and \( g \) be \( \eta \)-semidifferentiable at \( \bar{x} \) and:

1. \( (df)^+(\bar{x},\eta(x,\bar{x})) \) and \( y^T (dg)^+(\bar{x},\eta(x,\bar{x})) \) are preinvex functions of \( x \) on \( X^0 \), an \( \eta \)-locally starshaped set at \( \bar{x} \);
2. \( g_i \) (\( i \in J \)) are continuous at \( \bar{x} \);
3. \( g \) satisfies the generalized Slater’s constraint qualification at \( \bar{x} \);
4. \( g \) is \( \mathrm{slb} \)-preinvex on \( X^0 \).

If \( (x^*, y^*) \) is an optimal solution of (WD), \( f \) is \( \mathrm{slb} \)-preinvex on \( X^0 \) and \( \bar{b}(\bar{x}, \bar{x}^*) > 0 \), then \( x^* = \bar{x} \), i.e. \( x^* \) is an optimal solution to (NP) and \( f(\bar{x}) = \Psi(x^*, y^*) \).

**Remark 4.5.** If \( \eta(x, u) = x - u \) we obtain the Wolfe dual considered by Suneja and Gupta [5].

5. MOND-WEIR DUALITY

For problem (NP) we consider a general Mond-Weir dual problem

\[
\begin{align*}
\text{(MWD)} & \quad \text{Maximize} & & f(u) \\
& & \text{subject to} & (df)^+(u,\eta(x,u)) + y^T (dg)^+(u,\eta(x,u)) \geq 0, \quad \forall x \in X,
\end{align*}
\]
\[ y^T g(u) \geq 0, \]
\[ y \geq 0, u \in X^0, y \in \mathbb{R}^m. \]

Let \( W_1 \) denote the set of all feasible solutions to Problem (MWD). We assume that \( X^0 \) is a nonempty \( \eta \)-locally starshaped set.

**Theorem 5.1.** (Weak Duality) If \( x \in X, (u, y) \in W_1, f \) is slspb-preinvex and \( y^T g \) is slqb-preinvex on \( X^0 \), then \( f(x) \geq f(u) \).

**Corollary 5.2.** Let \( \bar{x} \in X \) and \( (\bar{u}, \bar{y}) \in W_1 \) such that \( f(\bar{x}) = f(\bar{u}) \). If the hypotheses of Theorem 5.1 hold, then \( \bar{x} \) and \( (\bar{u}, \bar{y}) \) are the optimal solutions to (NP) and (MWD) respectively.

**Theorem 5.3.** (Direct Duality) Let \( \bar{x} \in X \) be an optimal solution to (NP), let \( f \) and \( g \) be \( \eta \)-semidifferentiable at \( \bar{x} \), and assume that

- \( i_1 \) \( (df)^+(\bar{x}, \eta(x, \bar{x})) \) and \( y^T (dg)^+(\bar{x}, \eta(x, \bar{x})) \) are preinvex functions of \( x \) on \( X^0 \), an \( \eta \)-locally starshaped set at \( \bar{x} \);
- \( i_2 \) \( g_i (i \in J) \) are continuous at \( \bar{x} \);
- \( i_3 \) \( g \) satisfies the generalized Slater’s constraint qualification at \( \bar{x} \).

Then there exists \( \bar{y} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{y}) \in W_1 \) and \( f(\bar{x}) = \Psi(\bar{x}, \bar{y}) \).

Moreover, if the hypotheses of Theorem 5.1 are satisfied, then \( \bar{x} \) and \( (\bar{x}, \bar{y}) \) are respectively optimal solutions to (NP) and (MWD).

**Theorem 5.4.** (Converse Duality) Let \( (\bar{u}, \bar{y}) \in W_1 \). If \( f \) is slspb-preinvex and \( \bar{y}^T g \) is slqb-preinvex and there exists \( \bar{x} \in X \) such that \( f(\bar{x}) = f(\bar{u}) \), then \( \bar{x} \) solves the primal problem.

**Remark 5.5.** If \( \eta(x, u) = x - u \) we obtain the Mond-Weir dual considered by Suneja and Gupta [5].

**Remark 5.6.** Since the class of semilocally b-preinvex functions includes the class of semilocally b-convex, the class of semilocally preinvex functions and the class of b-preinvex functions, our results generalize those of Preda, Stancu-Minasian and Batatorescu [2], Suneja and Gupta [5] and Suneja et al. [6]. The proofs of all theorems will appear in [4].

**REFERENCES**


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