THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,*
*Volume 7, Number 1/2006, pp. 000-000*

# OPTIMALITY CONDITIONS FOR NONLINEAR PROGRAMMING WITH MIXED CONSTRAINTS AND $\rho$-LOCALLY ARCWISE CONNECTED FUNCTIONS

Ioan M. STANCU-MINASIAN, Andreea Mădălina STANCU

The Romanian Academy, Institute of Mathematical Statistics and Applied Mathematics, Calea 13 Septembrie nr. 13, Ro-050711, Bucharest 5, Romania, E-mail: stancum@csm.ro

A nonlinear programming problem with mixed constraints is considered, where the functions involved are $\rho$-locally arcwise connected, $\rho$-locally *Q*-connected, $\rho$-locally *P*-connected and locally *PQ*-connected (a notion introduced in this paper) and differentiable with respect to an arc. Sufficient optimality conditions are obtained in terms of the right differentials with respect to an arc of the functions. Our results generalize to the case of mixed constraints the results obtained by Stancu-Minasian [10] and Stancu-Minasian and Andreea Mădălina Stancu [11].

## 1. PRELIMINARIES

In this section we introduce the notation and definitions which are used throughout the paper.

Let $\mathbf{R}^n$ be the *n*-dimensional Euclidean space and $\mathbf{R}^n_+$ its nonnegative orthant $\{x \in \mathbf{R}^n, x_j \geqq 0, j = 1,\ldots, n\}$. Throughout the paper the following conventions for vectors in $\mathbf{R}^n$ will be followed:

$x > y$ if and only if $x_i > y_i$, $i = 1, \ldots, n$,

$x \geqq y$ if and only if $x_i \geqq y_i$, $i = 1,\ldots, n$,

$x \geq y$ if and only if $x_i \geqq y_i$, $i = 1,\ldots, n$, but $x \neq y$.

Also, all definitions and theorems are numbered consecutively in a single numeration system in each section.

Let $X^0 \subseteq \mathbf{R}^n$ be a nonempty and compact subset of $\mathbf{R}^n$.

**Definition 1.1.** *Let* $\bar{x}$, $x \in X^0$. *A continuous mapping* $H_{\bar{x},x} : [0,1] \to \mathbf{R}^n$ *with*

$$H_{\bar{x},x}(0) = \bar{x},\ H_{\bar{x},x}(1) = x$$

*is called an arc from* $\bar{x}$ *to* $x$.

**Definition 1.2** ([4]). *We say that the set* $X^0 \subseteq \mathbf{R}^n$ *is a locally arcwise connected set at* $\bar{x}$ ($\bar{x} \in X^0$) ($X^0$ *is* LAC($\bar{x}$), *for short*) *if for any* $x \in X^0$ *there exist a positive number* $a(x,\bar{x})$, *with* $0 < a(x,\bar{x}) \leqq 1$, *and a continuous arc* $H_{\bar{x},x}$ *such that* $H_{\bar{x},x}(\lambda) \in X^0$ *for any* $\lambda \in (0, a(x,\bar{x}))$.

We say that the set $X^0$ is locally arcwise connected if $X^0$ is locally arcwise connected at any $x \in X^0$.

If we choose the function $H_{\bar{x},x}$ of the form $H_{\bar{x},x}(\lambda) = (1-\lambda)\bar{x} + \lambda x$, we retrieve the definition of locally starshaped set as given by Ewing [2].

Recommended by Marius IOSIFESCU, member of the Romanian Academy

**Definition 1.3** ([7]). *Let* $f : X^0 \to \mathbf{R}$ *be a function, where* $X^0 \subseteq \mathbf{R}^n$ *is a locally arcwise connected set at* $\bar{x} \in X^0$ *with the corresponding function* $H_{\bar{x},x}(\lambda)$ *and a maximum positive number* $a(x,\bar{x})$ *satisfying the required conditions* (*from Definition* 1.2). *Also, let* $\rho \in \mathbf{R}$ *and* $\mathrm{d}(\cdot,\cdot) : X^0 \times X^0 \to \mathbf{R}_+$ *such that* $\mathrm{d}(x,\bar{x}) \neq 0$ *for* $x \neq \bar{x}$. *We say that* $f$ *is*:

($\mathrm{i}_1$) $\rho$-locally arcwise connected *at* $\bar{x}$ ($f$ *is* $\rho$-$LCN(\bar{x})$), *for short*) *if for any* $x \in X^0$ *there exist a positive number* $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$ *and an arc* $H_{\bar{x},x}$ *in* $X^0$ *on* $[0, \mathrm{d}(x,\bar{x})]$ *such that*

$$f(H_{\bar{x},x}(\lambda)) \leqq \lambda f(x) + (1-\lambda) f(\bar{x}) - \rho\lambda\, \mathrm{d}(x,\bar{x}),\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}). \tag{1.1}$$

($\mathrm{i}_2$) $\rho$-locally Q-connected *at* $\bar{x}$ ($\rho$-$LQCN(\bar{x})$) *if for any* $x \in X^0$ *there exist a positive number* $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$) *and an arc* $H_{\bar{x},x}$ *in* $X^0$ *on* $[0, \mathrm{d}(x,\bar{x})]$ *such that*

$$\left.\begin{matrix} f(x) \leqq f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{matrix}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) - f(\bar{x}) \leqq -\rho\lambda \mathrm{d}(x,\bar{x}).$$

($\mathrm{i}_3$) $\rho$-locally P-connected *at* $\bar{x}$ ($\rho$-$LPCN(\bar{x})$) *if for any* $x \in X^0$ *there exist a positive number* $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$, *an arc* $H_{\bar{x},x}$ *in* $X^0$ *on* $[0, \mathrm{d}(x,\bar{x})]$, *and a positive number* $\gamma_{\bar{x},x}$ *such that*

$$\left.\begin{matrix} f(x) < f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{matrix}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) \leqq f(\bar{x}) - \lambda\gamma_{\bar{x},x} - \rho\lambda \mathrm{d}(x,\bar{x}).$$

($\mathrm{i}_4$) $\rho$-locally strictly P-connected *at* $\bar{x}$ ($\rho$-$LSTPCN(\bar{x})$) *if for any* $x \in X^0$ *there exist a positive number* $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$, *an arc* $H_{\bar{x},x}$ *in* $X^0$ *on* $[0, \mathrm{d}(x,\bar{x})]$, *and a positive number* $\gamma_{\bar{x},x}$ *such that*

$$\left.\begin{matrix} x \neq \bar{x}, f(x) < f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{matrix}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) < f(\bar{x}) - \lambda\gamma_{\bar{x},x} - \rho\lambda \mathrm{d}(x,\bar{x}).$$

The function $f$ is said to be $\rho$-locally strictly arcwise connected at $\bar{x} \in X^0$($\rho$-$LSCN(\bar{x})$) if for each $x \in X^0$, $x \neq x^0$, inequality (1.1) is strict.

If $f$ is $\rho$-$LCN(\bar{x})$ ($\rho$-$LSCN(\bar{x})$) at each $\bar{x} \in X^0$, then $f$ is said to be $\rho$-$LCN$ ($\rho$-$LSCN$) on $X^0$.

If $f$ is $\rho$-$LQCN$ at each $\bar{x} \in X^0$, then $f$ is said to be $\rho$-$LQCN$ on $X^0$.

If $f$ is $\rho$-$LPCN$ at each $\bar{x} \in X^0$, then $f$ is said to be $\rho$-$LPCN$ on $X^0$.

**Definition 1.4.** *Let* $f : X^0 \to \mathbf{R}$ *be a function, where* $X^0 \subseteq \mathbf{R}^n$ *is a locally arcwise connected set at* $\bar{x} \in X^0$ *with the corresponding function* $H_{\bar{x},x}(\lambda)$ *and a maximum positive number* $a(x,\bar{x})$ *satisfying the required conditions* (*from Definition* 1.2). *We say that* $f$ *is locally* PQ-*connected at* $\bar{x}$ ($LPQCN(\bar{x})$) *if for any* $x \in X^0$ *there exist a positive number* $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$ *and an arc* $H_{\bar{x},x}$ *in* $X^0$ *on* $[0, \mathrm{d}(x,\bar{x})]$ *such that*

$$\left.\begin{matrix} f(x) = f(\bar{x}) \\ 0 < \lambda < \mathrm{d}(x,\bar{x}) \end{matrix}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) - f(\bar{x}) \leqq 0.$$

**Definition 1.5** ([3]). *Let* $f : X^0 \to \mathbf{R}$ *be a function, where* $X^0 \subseteq \mathbf{R}^n$ *is a locally arcwise connected set at* $\bar{x} \in X^0$, *with the corresponding function* $H_{\bar{x},x}(\lambda)$ *and a maximum positive number*

$a(x,\bar{x})$ *satisfying the required conditions. The right differential of* $f$ *at* $\bar{x}$ *with respect to the arc* $H_{\bar{x},x}(\lambda)$ *is defined as*

$$(\mathrm{d}f)^{+}(\bar{x}, H_{\bar{x},x}(0^{+})) = \lim_{\lambda\to 0^{+}} \frac{1}{\lambda}[f(H_{\bar{x},x}(\lambda)) - f(\bar{x})] \qquad (1.1)$$

*provided the limit exists.*

If $f$ is differentiable at any $\bar{x} \in X^{0}$, then $f$ is said to be differentiable on $X^{0}$.

## 2. SUFFICIENT OPTIMALITY CRITERIA

Consider the nonlinear programming problem

$$\text{(P)} \qquad \begin{cases} \text{Minimize } f(x) \\ \text{subject to } g(x) \leqq 0, h(x) = 0, x \in X^{0}, \end{cases}$$

where

i) $X^{0} \subseteq \mathbf{R}^{n}$ is a nonempty open locally arcwise connected set;

ii) $f : X^{0} \to \mathbf{R}$;

iii) $g = (g_i)_{1\le i\le m} : X^{0} \to \mathbf{R}^{m}$;

iv) $h = (h_j)_{1\le j\le k} : X^{0} \to \mathbf{R}^{k}$;

v) the right differentials of $f$, $g_i, i = 1,\ldots,m,$ and $h_j, j = 1,...,k$ at $\bar{x}$ exist with respect to the same arc $H_{\bar{x},x}(\lambda)$.

Let $X = \{x \in X^{0} \mid g(x) \leqq 0,\ h(x) = 0\}$ be the set of all feasible solutions to (P). Let

$$N_{\varepsilon}(\bar{x}) = \{x \in \mathbf{R}^{n} \mid \|x - \bar{x}\| < \varepsilon\}.$$

**Definition 2.1.** a) $\bar{x}$ *is said to be a local minimum solution to problem* (P) *if* $\bar{x} \in X$ *and there exists* $\varepsilon > 0$ *such that* $x \in N_{\varepsilon}(\bar{x}) \cap X \Rightarrow f(\bar{x}) \leqq f(x)$.

b) $\bar{x}$ *is said to be the minimum solution to problem* (P) *if* $\bar{x} \in X$ *and* $f(\bar{x}) = \min_{x\in X} f(x)$.

For $\bar{x} \in X$ we denote by $I = I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$ the set of indices of active constraints at $\bar{x}$, by $J = J(\bar{x}) = \{i \mid g_i(\bar{x}) < 0\}$ the set of indices of nonactive constraints at $\bar{x}$, and set $g_I = (g_i)_{i\in I}$. Obviously, $I \cup J = \{1,2,...,m\}$.

Let $u \in \mathbf{R}^{m}$ be such that $u \geqq 0$ and $u^{T} g(\bar{x}) = 0$. Obviously, $u_I \geqq 0$ and $u_J = 0$, where $u_I$ and $u_J$ denotes the subvectors of $u$ corresponding to the index sets $I$ and $J$, respectively.

Let $K = \{i \in I : u_i > 0\}$ and $L = \{i \in I : u_i = 0\}; K \cup L = I$. Let $g_K$ and $g_L$ be the subvectors of $g_I$ corresponding to the index sets $K$ and $L$, respectively.

In this section we give sufficient optimality theorems for problem (P).

First, we give a sufficient optimality theorem of the Kuhn-Tucker type. The functions $f, g$ and $h$ are not differentiable but are directional differentiable with respect to the same arc $H_{\bar{x},x}(\lambda)$ at $\lambda = 0$.

Let $\{K_1, K_2, K_3\}$ be a partition of the index set $K$; thus $K_i \subset K$ for each $i = 1,2,3, K_r \cap K_s = \emptyset$ for each $r, s \in \{1,2,3\}$ with $r \neq s$, and $\bigcup_{i=1}^{3} K_i = K$.

The next result does not require the function $h$ to be directionally differentiable.

**Theorem 2.2** *Let $\bar{x} \in X^0 \subseteq \boldsymbol{R}^n$, where $X^0$ is a locally arcwise connected set and let $\bar{u} \in \boldsymbol{R}^m$. Assume that there exist the right differentials at $\bar{x}$ with respect to the same arc $H_{\bar{x},x}$ of $f$ and $g$ and $(\bar{x},\bar{u})$ satisfies the conditions below*:

$$(\mathrm{d}f)^{+}(\bar{x},\mathrm{H}_{\bar{x},x}(0^{+}))+\bar{u}^{T}(\mathrm{d}g)^{+}(\bar{x},\mathrm{H}_{\bar{x},x}(0^{+}))\geqq 0,\ \forall\ \ x\in X, \tag{2.1}$$

$$\bar{u}^{T}g(\bar{x})=0, \tag{2.2}$$

$$g(\bar{x})\leqq\ 0\ , h(\bar{x})=0 \tag{2.3}$$

$$\bar{u}\ \geqq\ 0,\ \bar{u}\neq 0 \tag{2.4}$$

*Assume furthermore that*

i$_1$) $$g_i, i\in K_1, \text{ is } \alpha_i\text{-}LQCN(\bar{x}), \tag{2.5}$$

i$_2$) $$\bar{u}_{K_2}^{T}g_{K_2} \text{ is } \beta\text{-}LQCN(\bar{x}) \tag{2.6}$$

i$_3$) $$f+\bar{u}_{K_3}^{T}g_{K_3} \text{ is } \gamma\text{-}LPCN(\bar{x}) \tag{2.7}$$

i$_4$) $$\sum_{i\in K_1}\alpha_i\bar{u}_i+\beta+\gamma\geq 0. \tag{2.8}$$

*Then $\bar{x}$ is a minimum solution to Problem (P).*

The following result is a special case of Theorem 2.2, where the conditions are special cases of (2.5) through (2.8).

**Theorem 2.3.** *Let $\bar{x} \in X^0 \subseteq \boldsymbol{R}^n$, where $X^0$ is a locally arcwise connected set and let $\bar{u} \in \boldsymbol{R}^m$. Assume that there exist the right differentials at $\bar{x}$, with respect to the same arc $H_{\bar{x},x}$ of $f$ and $g$ and $(\bar{x},\bar{u})$ satisfies conditions* (2.1) - (2.4).

*Assume furthermore that any one of the hypotheses below is satisfied.*

i$_1$) $f+\bar{u}_K^T g_K$ *is* $\gamma\text{-}LPCN(\bar{x})$, *where* $\gamma\geqq 0$;

i$_2$) a) $g_i, i\in K$, *is* $\alpha_i\text{-}LQCN(\bar{x})$,

b) $f$ *is* $\gamma\text{-}LPCN(\bar{x})$,

c) $\sum_{i\in K}\alpha_i\bar{u}_i+\gamma\geq\ 0$;

i$_3$) a) $\bar{u}_K^T g_K$ *is* $\beta\text{-}LQCN(\bar{x})$,

b) $f$ *is* $\gamma\text{-}LPCN(\bar{x})$,

c) $\beta+\gamma\geq\ 0$;

i$_4$) a) $\bar{u}_{K_2}^T g_{K_2}$ *is* $\beta\text{-}LQCN(\bar{x})$,

b) $f+\bar{u}_{K_3}^T g_{K_3}$ *is* $\gamma\text{-}LPCN(\bar{x})$, *where* $\{K_2,K_3\}$ *is a partition of* $K$,

c) $\beta+\gamma\geq\ 0$;

i$_5$) a) $g_i$, $i\in K_1$, *is* $\alpha_i\text{-}LQCN(\bar{x})$,

b) $f+\bar{u}_{K_3}^T g_{K_3}$ *is* $\gamma\text{-}LPCN(\bar{x})$, *where* $\{K_1,K_3\}$ *is a partition of* $K$,

c) $\sum_{i\in K_1}\alpha_i\bar{u}_i+\gamma\geq\ 0$;

i$_6$) a) $g_i, i\in K_1$, *is* $\alpha_i\text{-}LQCN(\bar{x})$,

b) $\bar{u}_{K_2}^T g_{K_2}$ *is* $\beta$ - $LQCN(\bar{x})$,

c) $f$ *is* $\gamma$ - $LPCN(\bar{x})$,

d) $\sum_{i\in K_1} \alpha_i \bar{u}_i + \beta + \gamma \geq 0$, *where* $\{K_1, K_2\}$ *is a partition of* $K$.

*Then* $\bar{x}$ *is a minimum solution to problem* (P).

Let $v \in \mathbf{R}^k$ and define $P = \{i \mid v_i > 0\}$ and $Q = \{i \mid v_i < 0\}$. Let $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ be partitions of the sets $P$ and $Q$, respectively.

Let $h_{P_i}$ and $h_{Q_i}$ ($i$=1,2,3) be the subvectors of $h$ corresponding to the index sets $P_i$ and $Q_i$ $(i = 1, 2, 3)$, respectively. Let $v_{P_i}$ and $v_{Q_i}$ $(i = 1,2,3)$ be the subvectors of $v$ corresponding to the index sets $P_i$ and $Q_i (i = 1, 2, 3)$, respectively.

The next result does not require the function $g$ to be directionally differentiable.

**Theorem 2.4.** *Let* $\bar{x} \in X^0 \subseteq \mathbf{R}^n$, *where* $X^0$ *is a locally arcwise connected set and let* $\bar{v} \in \mathbf{R}^k$. *Assume that there exist the right differentials at* $\bar{x}$ *with respect to the same arc* $H_{\bar{x},x}$ *of* $f$ *and* $h$ *and* $(\bar{x}, \bar{v})$ *satisfies the condition*

$$(\mathrm{d}f)^+(\bar{x}, H_{\bar{x},x}(0^+)) + \bar{v}^T (\mathrm{d}h)^+(\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \ \forall \ x \in X.$$

*Assume furthermore that*

i$_1$) $h_i, i \in P_1$, is $LPQCN(\bar{x})$,

i$_2$) $-h_i, i \in Q_1$, is $LPQCN(\bar{x})$,

i$_3$) $\bar{v}_{P_2}^T h_{P_2} + \bar{v}_{Q_2}^T h_{Q_2}$ is $LPQCN(\bar{x})$

i$_4$) $f + \bar{v}_{P_3}^T h_{P_3} + \bar{v}_{Q_3}^T h_{Q_3}$ is $\tau$ - $LPCN(\bar{x}), (\tau \geq 0)$

*Then* $\bar{x}$ *is a minimum solution to Problem* (P)

Let $u \in \mathbf{R}^m$. Let $L = \{i | u_i > 0\}$. Let $v \in \mathbf{R}^k$ and define $P = \{i | v_i > 0\}$ and $Q = \{i | v_i < 0\}$. Let $\{L_1, L_2, L_3, L_4\}, \{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$ be partitions of the sets $L, P$ and $Q$, respectively.

The following result is a combination of Theorems 2.2 and 2.4.

**Thorem 2.5.** *Let* $\bar{x} \in X^0 \subseteq \mathbf{R}^n$, *where* $X^0$ *is a locally arcwise connected set and let* $\bar{u} \in \mathbf{R}^m$ *and* $\bar{v} \in \mathbf{R}^k$. *Assume that there exist the right differentials at* $\bar{x}$, *with respect to the same arc* $H_{\bar{x},x}$ *of* $f, g$ *and* $h$ *and* $(\bar{x}, \bar{u}, \bar{v})$ *satisfies the conditions below:*

$$(\mathrm{d}f)^+(\bar{x}, H_{\bar{x},x}(0^+)) + \bar{u}^T (\mathrm{d}g)^+(\bar{x}, H_{\bar{x},x}(0^+)) + \bar{v}^T (\mathrm{d}h)^+(\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \forall \ x \in X,$$

$$\bar{u}^T g(\bar{x}) = 0,$$

$$g(\bar{x}) \leqq 0, h(\bar{x}) = 0,$$

$$\bar{u} \geqq 0, \ \bar{u} \neq 0.$$

*Assume furthermore that*

i$_1$) $g_i, i \in L_1,$ , is $\alpha_i$ - $LQCN(\bar{x})$,

i$_2$) $h_i, i \in P_1$, is $LPQCN(\bar{x})$,

i$_3$) $-h_i, i \in Q_1$, is $LPQCN(\bar{x})$,

i$_4$) $\bar{u}_{L_2}^T g_{L_2}$ is $\beta$ - $LQCN(\bar{x})$,

i$_5$) $\bar{v}_{P_2}^T h_{P_2} + \bar{v}_{Q_2}^T h_{Q_2}$ is $LPQCN(\bar{x})$

i$_6$) $\bar{u}_{L_3}^T g_{L_3} + \bar{v}_{P_3}^T h_{P_3} + \bar{v}_{Q_3}^T h_{Q_3}$ is $\delta$ - $LQCN(\bar{x})$

i$_7$) $f + \bar{u}_{L_4}^T g_{L_4} + \bar{v}_{P_4}^T h_{P_4} + \bar{v}_{Q_4}^T h_{Q_4}$ is $\tau$ - $LPCN(\bar{x})$

i$_8$) $\sum_{i \in L_1} \alpha_i \bar{u}_i + \beta + \delta + \tau \geq 0$.

*Then* $\bar{x}$ *is a minimum solution to Problem* (P).

In what follows we consider sufficient optimality conditions of the Fritz John type.

Let $(\bar{x}, v_0, v, w)$ be a Fritz John point, where $\bar{x} \in X^0$ (a locally arcwise connected set), $v_0 \in$ R, $v \in \mathbf{R}^m$ and $w \in$ R$^k$. Assume that $(\bar{x}, v_0, v, w)$ satisfies the conditions

$$v_0 (\mathrm{d}f)^+(\bar{x}, H_{\bar{x},x}(0^+)) + v^T (\mathrm{d}g)^+(\bar{x}, H_{\bar{x},x}(0^+)) + w^T (\mathrm{d}h)^+(\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \ \forall \ x \in X, \tag{2.9}$$

$$v^T g(\bar{x}) = 0, \tag{2.10}$$

$$(v_0, v) \geqq 0, \ (v_0, v, w) \neq 0. \tag{2.11}$$

If $v_0 = 0$ then conditions (2.9) –( 2.11) become

$$v^T (\mathrm{d}g)^+(\bar{x}, H_{\bar{x},x}(0^+)) + w^T (\mathrm{d}h)^+(\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \forall \ x \in X, \tag{2.12}$$

$$v^T g(\bar{x}) = 0, \tag{2.13}$$

$$v \geqq 0, \ (v, w) \neq 0. \tag{2.14}$$

Let *I* and *J* be the sets defined at the beginning of this section. By (2.10) we have $v_I \geqq 0$ and $v_J = 0$. Let $L = \{i \in I : v_i > 0\}$. Let $g_L$ be the subvector of $g_I$ corresponding to the index set *L*. Also, let $v_L$ be the subvector of *v* corresponding to the index set *L*.

Let $w \in \mathbf{R}^k$. Define the index sets $U$ and $V$ by $U = \{i \mid w_i > 0\}$ and $V = \{i \mid w_i < 0\}$. Let $h_U$ and $h_V$ be the subvectors of *h* corresponding to the index sets *U* and *V*, respectively. Also, let $w_U$ and $w_V$ be the vector of $w$ corresponding to the index sets *U* and *V*, respectively.

**Theorem 2.6.** *Let* $\bar{x} \in X^0 \subseteq \mathbf{R}^n$, *where* $X^0$ *is a locally arcwise connected set. Assume that there exist the right differentials at* $\bar{x}$ *with respect to the same arc* $H_{\bar{x},x}$ *of* $f$ *,g and* $h$. *Let* $(\bar{x}, v_0, v, w)$ *be a Fritz John point which satisfy conditions* (2.9)-(2.11).

i) *If* $v_0 > 0$, *let the assumptions of Theorem* 2.5 *hold with* $\bar{u} = v_0^{-1} v, \bar{v} = v_0^{-1} w$.

ii) *If* $v_0 = 0$*, let* $(\bar{x}, 0, v, w)$ *satisfy* (2.12)-(2.14) *and assume that the conditions below hold:*

a) $g_i, i \in L_1$, , *is* $\alpha_i$ *- LQCN*($\bar{x}$),

b) $h_i, i \in U_1$, *is* $LPQCN(\bar{x})$,

c) $-h_i, i \in V_1$, *is* $LPQCN(\bar{x})$,

d) $v_{L_2}^T g_{L_2}$ *is* $\beta$ *- LQCN*($\bar{x}$),

e) $v_{U_2}^T h_{U_2} + w_{V_2}^T h_{V_2}$ *is* $LPQCN(\bar{x})$

f) $v_{L_3}^T g_{L_3} + w_{U_3}^T h_{U_3} + w_{V_3}^T h_{V_3}$ *is* $\delta$ *- LQCN* ($\bar{x}$),

g) $\sum_{i \in L_1} \alpha_i v_i + \beta + \delta \geq 0$.

*Then* $\bar{x}$ *is a global minimum solution to Problem* (P).

The proofs will appear in [12].

## REFERENCES


1. AVRIEL, M., ZANG, I., *Generalized arcwise-connected functions and characterizations of local-global minimum properties*. J. Optim. Theory Appl., **32,** 4, pp. 407-425, 1980.
2. EWING, G. M., *Sufficient conditions for global minima of suitably convex functions from variational and control theory*. SIAM Rev., **19 ,** 2, pp, 202-220, 1977.
3. KAUL, R. N., LYALL, V., *Locally connected functions and optimality*. Indian J. Pure Appl. Math., **22** 2, pp. 99-108, 1991.
4. KAUL, R. N., LYALL, VINOD, KAUR, SURJEET, *Locally connected set and functions*. J. Math. Anal. Appl. **134**, pp. 30-45, 1988.
5. LYALL, V., SUNEJA, S. K., AGGARWAL, S., *Fritz John optimality and duality for non-convex programs*. J. Math.Anal.Appl. **212 ,**1, pp. 38-50, 1997.
6. ORTEGA, J. M., RHEINBOLDT, W. C., *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York, NY 1970.
7. PREDA, V., NICULESCU, CRISTIAN, *On duality minmax problems involving* ρ*-locally arcwise connected and related functions*. Analele Universităţii Bucureşti, Matematica, **49**, 2, pp.185-195, 2000.
8. STANCU-MINASIAN, I. M., *Fractional Programming. Theory, Methods and Applications*. Kluwer Academic Publishers, Dordrecht, 1997.
9. STANCU-MINASIAN, I. M., *Duality for nonlinear fractional programming involving generalized locally arcwise connected functions*. Rev. Roumaine Math.Pures Appl.., **49,** 3, pp.287-298, 2004.
10. STANCU-MINASIAN, I. M., *Optimality conditions for nonlinear programming with generalized locally arcwise connected functions*. Proc. Ro. Acad., Series A, **5**, pp.123-127, 2004.
11. STANCU-MINASIAN, I. M., STANCU, Andreea-Mădălina, *Sufficient optimality conditions for nonlinear programming with* ρ*-locally arcwise connected and related functions*. Submitted.
12. STANCU-MINASIAN, I. M., STANCU, Andreea-Mădălina, *Sufficient optimality conditions for nonlinear programming with mixed constraints and generalized* ρ*-locally arcwise connected functions*. Submitted.


*Received January 26, 2006*