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BRANCHING PROCESSES AND INSURANCE

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Somebody wants to ensure **all** its descendants as follows: at the end of the year of birth of a descendant, this receives one monetary unit (1 MU). Suppose that the insurer knows the probability distribution of ξ_x – the number of descendants of x and of T_x – the age of the parent of x when x is born. What premium insurance should be paid in order that the probability that the insurer will lose money is a given ε? We give some estimation for that and prove that if T_x is bounded below by some $t_0 > 0$ then the sum paid by the insurer is a very short tailed random variable.

1. NOTATION AND STATEMENT OF THE PROBLEM

Let $\mathbf{N}$ be the set of all positive integers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Let $\Gamma = \bigcup_{k=0}^{\infty} \mathbf{N}^k$ be the set of all finite words with letters from $\mathbf{N}$. The void word corresponding to $k = 0$ will be denoted by o. "o" is the common ancestor of all the words. For $k \geq 1$, an element x from $\mathbf{N}^k$ is a word of length k which will be denoted componentwise by $(x_1, x_2, \ldots, x_k)$. The operators naturally involved with Γ which will be used are

– the "delete first letter" operator θ_L: $\Gamma \setminus \{o\} \to \Gamma$ defined by $\theta_L(x_1, x_2, \ldots x_k) = (x_2, \ldots x_k)$

– the "delete last letter" operator θ_R: $\Gamma \setminus \{o\} \to \Gamma$ defined by $\theta_R(x_1, x_2, \ldots x_k) = (x_1, \ldots x_{k-1})$.

Clearly, "L" from θ_L comes from "Left" and "R" from θ_R comes from "Right".

The mth iterate of θ_L and θ_R will be denoted by θ_L^m and θ_R^m. Let us agree that $\theta_L^0(x) = \theta_R^0(x) = x$. Of course, $x \in \mathbf{N}^k \Rightarrow \theta_L^k(x) = \theta_R^k(x) =$ o. The meaning of $x = (x_1, x_2, \ldots x_k)$ is as follows : x is the descendant of $\theta_R(x)$ which is the descendant of $\theta_R^2(x)$ which is the descendant of $\theta_R^3(x)$,…., which is the descendant of $\theta_R^{k-1}(x) = \underline{x_1}$ which, finally, is one of the descendants of o. We can think of x as being names which, as in the old times, described the genealogy of x.

Now, suppose that with every $x \in \Gamma \setminus \{o\}$ we associate two random variables, on some probability space (Ω, K, P), namely, $\xi_x : \Omega \to \mathbf{N}_0$ and $T_x : \Omega \to (0,\infty)$. Their meaning is that ξ_x is the number of descendants of x and T_x is the age of the parent of x – thus the age of $\theta_R(x)$ – when x was born. Then we can express the time τ_x when $x \in \mathbf{N}^k$ was born as

$$\tau_x = T_{\underline{x_1}} + T_{\underline{x_1 x_2}} + .. + T_{\underline{x_1 x_2 \ldots x_k}} = \sum_{m=0}^{k-1} T_{\theta_R^m(x)} . \tag{1.1}$$

Suppose now that the insurance premium is put in a bank with a (possible variable) instantaneous interest rate δ. This means that 1 MU at time 0 values $\exp\left(\int_0^t \delta(u)\mathrm{d}u\right)$ MU at time t. Let $\alpha(t) =$

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$\exp\left(-\int_0^t \delta(u)\mathrm{d}u\right)$ be the value of 1 MU at time t actualized at time 0. Then, according to the deal, the cost of x for the insurer is $\alpha(\tau_x)$

Thus, the total cost of the business for the insurer is $X = \sum_{k=1}^{\infty} X_k$, where X_k is the cost of the kth generation, $X_k = \sum_{x \in G_k} \alpha(\tau_x)$. The kth generation of ensured people can be defined as

$$G_k = \{x \in \mathbf{N}^k \mid x_1 \le \xi_o, x_2 \le \xi_{\underline{x_1}}, x_3 \le \xi_{\underline{x_1 x_2}}, \ldots, x_k \le \xi_{\underline{x_1 x_2 \ldots x_{k-1}}}\}. \tag{1.2}$$

Remark that G_k is a *random* set – possible void if $\xi_o = 0$. Its section at $\omega \in \Omega$ will be denoted by $G_k(\omega)$. Precisely, according to our conventions,

$$G_k(\omega) = \{x \in \mathbf{N}^k \mid x_1 \le \xi_o(\omega), x_2 \le \xi_{\underline{x_1}}(\omega), x_3 \le \xi_{\underline{x_1 x_2}}(\omega), \ldots, x_k \le \xi_{\underline{x_1 x_2 \ldots x_{k-1}}}(\omega)\}. \tag{1.3}$$

Notice that, according to our assumptions about the random variables ξ_x the possibility that $\xi_x = \infty$ is excluded, hence $G_k(\omega)$ is a ***finite*** set. As the family of all finite sub-sets of $\mathbf{N}_0$ is countable, we can speak about the distribution of G_k.

Definition. Let $G \subset \Omega \times \Gamma$ be a set such that $G(\omega) := \{x \in \Gamma \mid (\omega, x) \in G\}$ is finite for any ω. Suppose that for every finite $J \subset \Gamma$ the sets $A_J(G) := \{\omega \in \Omega \mid G(\omega) = J\}$ are in K. Then the distribution of G is the system of numbers $(\pi_J)_{J \subset \Gamma,\ J \text{ finite}}$, where $\pi_J = P(G(\omega) = J)$; G will be called a ***random set with finite sections.***

If G_1 and G_2 are two random sets with finite sections, we say that G_1 and G_2 are ***identically distributed,*** and write $G_1 \sim G_2$, if $P(G_1(\omega) = J) = P(G_2(\omega) = J)\ \forall J \subset \Gamma$, J finite.

If $(G_n)_n$ is a sequence of random sets with finite sections we say that they are ***independent*** if the sets $(A_{J_n}(G_n))_n$ are independent for any collection $(J_n)_n$ of finite subsets of Γ.

The (possibly not realistic) maximal goal is to find the distribution of X and compute the number Π (the premium) such that $P(X > \Pi) = \varepsilon$. This is ***the real problem***. A minimal goal is to compute EX and Var(X). An intermediary goal is to say something about $\boldsymbol{m_X}$ **–** the moment generating function of X.

We shall denote by E_k the expectation of X_k and by V_k its variance.

2. ADDITIONAL HYPOTHESES AND STRAIGHTFORWARD CONSEQUENCES

We shall suppose in the sequel that

H1. All the random variables ξ_x are i.i.d. and $\xi_x \sim \xi$, where ξ is one of them. We shall denote $\mathrm{E}\xi := \mu$, Var $\xi = \sigma^2$.

H2. All the random variables T_x are i.i.d. and $T_x \sim T$, where T is one of them. Moreover, T has positive integer values.

H3. δ is constant. Then $\alpha(t) = v^t$ with $v = e^{-\delta}$. Denote by L_n the quantities $\mathrm{E}(v^T)^n = \mathrm{E}v^{nT}$. We shall often write L instead of L_1 and $s^2 = \mathrm{Var}(v^T) = L_2 - L^2$. If $\delta > 0$ it is obvious that $1 > L > L_2 > L_3 > \ldots$

H4. The random variables $(\xi_x)_{x \in \Gamma}$ and $(T_x)_{x \in \Gamma}$ are independent.

Assumptions H1 and H4 are natural. About assumption H2: it is easy to accept that T_x are independent and that T assumes positive integer values, since we agreed that the insurance is paid at the end of the year. It is more complicated to accept that T_x are identically distributed. However, if x has m descendants, born at ages $T_1 \le T_2 \le .. \le T_m$, we can assume that these random variables arise from the same distribution T by means of order statistics. Precisely, we can say that there exist some i.i.d. random variables $T_1', \ldots, T_m'$ and that $T_i = T'_{(i)}$ is the i th order statistics attached to them. In this way we solved the problem of twins, too: if the random variables are discrete, it is possible that some of the order statistics coincide. Finally, H3 is hardly acceptable, but we do not know what can one say in its absence.

Under these assumptions the cost of the kth generation is

$$X_k = \sum_{x\in G_k} v^{\tau_x} . \tag{2.1}$$

Proposition 2.1. *Assume only* H1,H2 *and* H4 . *Let* $x \in \mathbf{N}^k$, $k \geq 2$.

(i) *We have* $\tau_x \sim T + \tau_y$, *where* y *is some element from* $\mathbf{N}^{k-1}$ *and* τ_y *is independent on* T. *Moreover, all the random variables* $(\tau_x)_{x\in\mathbf{N}^k}$ *are identically distributed. They are not independent, but if* $x(n) =(x_1(n),x_2(n),\ldots,x_k(n))$ *is a sequence of elements of* $\mathbf{N}^k$ *such that* $m \neq n \Rightarrow x_1(m) \neq x_1(n)$, *then* $(\tau_{x(n)})_n$ *are indeed independent.*

(ii) *For two arbitrary elements* $x,y \in \mathbf{N}^k$, *the correlation coefficient between* τ_x *and* τ_y *is* $r(\tau_x, \tau_y) = \frac{l(x,y)}{k}$, *where* $l(x,y) = \max\{j \leq k \mid x_i = y_i\ \forall\ i \leq j\}$.

Proof. According to (1.1), $\tau_x = T_{x_1} + T_{(x_1,x_2)} + .. + T_{(x_1,x_2,\ldots,x_k)}$, where all k summands are i.i.d. and distributed as T. We can take $y = \theta_L(x)$.

Proposition 2.2. *Assume all hypotheses* H1 – H4. *Then*

$$G_k = \bigcup_{n\leq\xi_o} G_{k-1,n}, \tag{2.2}$$

where $G_{k-1,n} = \{x \in \mathbf{N}^k \mid x_1 = n,\ x_2 \leq \xi_{x_1},\ x_3 \leq \xi_{(x_1x_2)},\ldots,\ x_k \leq \xi_{(x_1x_2\ldots x_{k-1})}\}$ *are disjoint and* $\theta_L(G_{k,n})$ *are independent random sets distributed as* G_{k-1}. *Hence, the vector* $Z_k := (X_1,\ldots,X_k)$ *can be written as*

$$Z_k = \sum_{n\leq\xi_o} v^{T_n}(1, Z^*_{k-1,n}) , \tag{2.3}$$

where $Z^*_{k-1,n}$ *are independent copies of* Z_{k-1} *and* $T_n \sim T$ *are i.i.d.*

Remark. After all, the meaning of (2.2) and (2.3) is that the set of the descendants of generation k of "o" is the union of the descendants of generation k-1 of its children and that the cost for the insurance of its descendants is the cost of the insurance for its children and the descendants of the children.

Proof. Relation (2.2) is obvious by (1.2). Now, by the definition of θ_L, $\theta_L(G_{k,n}) =\{\theta_L(x) \mid x \in G_{k,n}\} = \{(x_2, x_3,\ldots,x_k) \in \mathbf{N}^{k-1} \mid x_2 \leq \xi_n,\ x_3 \leq \xi_{(n,x_2)},\ldots,\ x_k \leq \xi_{(n,x_2,\ldots,x_{k-1})}\}$ are independent (since all the random variables $(\xi_{(n,y)})_n$ are i.i.d.) and have the same distribution as G_{k-1}.

As to (2.3), write $X_k = \sum_{x\in G_k} v^{\tau_x} = \sum_{x\in \bigcup_{n\leq\xi_o} G_{k,n}} v^{\tau_x} = \sum_{n\leq\xi_o}\sum_{x\in G_{k,n}} v^{\tau_x} = \sum_{n\leq\xi_o} v^{T_n} \sum_{y\in\theta_L G_{k,n}} v^{\tau_y} = \sum_{n\leq\xi_o} v^{T_n} X^*_{k-1,n}$

where $X^*_{k-1,n} = \sum_{y\in\theta_L G_{k,n}} v^{\tau_y}$ are distributed as X_{k-1} and are independent.

Now, we compute the expectation and the variance of X_k. Recall the notation: $E_k = \mathrm{E}X_k$, $V_k = \mathrm{Var}(X_k)$, $\mu = \mathrm{E}\xi$, $\sigma^2 = Var(\xi)$, $L_n = \mathrm{E}v^{nT}$, $L = L_1$, $s^2 = L_2 - L^2$.

Corollary 2.3. *Under the assumptions* H1-H4 *we have*

$$E_k = (\mu L)^k, \tag{2.4}$$

$$V_1 = \mu s^2 + L^2\sigma^2. \tag{2.5}$$

For $k \geq 2$, V_k satisfies the recurrence

$$V_k = \mu L_2 V_{k-1} + (\mu L)^{2k-2} V_1. \tag{2.6}$$

Hence, if we put

$$\rho = \frac{L_2}{\mu L^2}, \tag{2.7}$$

then

$$\rho = 1 \Rightarrow V_k = V_1 k(\mu L)^{2k-2} \text{ and } \rho \neq 1 \Rightarrow V_k = V_1(\mu L)^{2k-2}\frac{\rho^k - 1}{\rho - 1}. \tag{2.8}$$

Moreover,

$$\mu L > 1 \Leftrightarrow E_k \to \infty,\ V_k \to \infty \text{ as } k \to \infty. \tag{2.9}$$

Finally,

$$\mathrm{E}X < \infty \Leftrightarrow \mu L < 1 \text{ and in this case } \mathrm{E}X = \frac{\mu L}{1 - \mu L}. \tag{2.10}$$

Proof. Apply (2.3). As $X_k = \sum_{n \leq \xi_o} v^{T_n} Y_n = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} v^{T_i} Y_i\right) 1_{\{\xi_0 = n\}}$, T_i and Y_i are independent, it follows that

$\mathrm{E}X_k = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \mathrm{E}v^{T_i}\,\mathrm{E}Y_i\right) P(\xi_o = n) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} L\mathrm{E}X_{k-1}\right) P(\xi_o = n)$ (since $T_i \sim T$ and $Y_i \sim X_{k-1}$) $=$

$\sum_{n=1}^{\infty} nLE_{k-1}\ P(\xi_o = n) = LE_{k-1}\mathrm{E}\xi_o = (\mu L)E_{k-1}$.

As $E_0 = 1$, (2.4) follows at once.

As to (2.5), we have

$$V_1 = \mathrm{E}X_1^2 - (\mathrm{E}X_1)^2 = \mathrm{E}\Big(\sum_{n \leq \xi_o} v^{T_n}\Big)^2 - \mu^2 L^2 = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \mathrm{E}v^{T_i}\right)^2 P(\xi_o = n) - \mu^2 L^2 =$$

$$\sum_{n=1}^{\infty}\left(nL_2 + n(n-1)L^2\right) P(\xi_o = n) - \mu^2 L^2 = \mu L_2 + L^2(\mathrm{E}\xi^2 - \mathrm{E}\xi) = \mu(s^2 + L^2) + L^2(\mu^2 + \sigma^2 - \mu) - \mu^2 L^2$$

$$= \mu s^2 + L^2\sigma^2.$$

As to (2.7), we have $\mathrm{E}X_k^2 = \sum_{n=1}^{\infty}\mathrm{E}\left(\sum_{i=1}^{n} v^{T_i} Y_i\right)^2 P(\xi_o = n) = \sum_{n=1}^{\infty}\mathrm{E}\left(\sum_{i,i'=1}^{n} v^{T_i + T_{i'}} Y_i Y_{i'}\right) P(\xi_o = n) =$

$\sum_{n=1}^{\infty}\mathrm{E}\left(\sum_{i=1}^{n} v^{2T_i} Y_i^2 + \sum_{1 \leq i \neq i' \leq n} v^{T_i} v^{T_{i'}} Y_i Y_{i'}\right) P(\xi_o = n) = \sum_{n=1}^{\infty}\left(nL_2\mathrm{E}X_{k-1}^2 + n(n-1)L^2\mathrm{E}^2 X_{k-1}\right) P(\xi_o = n) = \mu L_2 \mathrm{E}(X_{k-1})^2 +$
$L^2\mathrm{E}^2 X_{k-1}\,(\mathrm{E}\xi^2 - \mathrm{E}\xi) = \mu L_2(V_{k-1} + \mathrm{E}^2 X_{k-1}) + L^{2k}\mu^{2k-2}(\mu^2 + \sigma^2 - \mu)$, hence

$$V_k = \mathrm{E}X_k^2 - \mathrm{E}^2 X_k = \mu L_2(V_{k-1} + \mu^{2k-2}L^{2k-2}) + L^{2k}\mu^{2k-2}(\sigma^2 - \mu) = \mu L_2 V_{k-1} + \mu^{2k-2}L^{2k-2}(\mu L_2 + L^2\sigma^2 - \mu L^2)$$

and (2.6) follows. Next, (2.8) can be proved by induction and (2.9) is obvious.

3. **CORRELATION COEFFICIENTS BETWEEN X_M AND X_N THE VARIANCE OF X**

If we want to find a more precise estimation of Var(X), we have to compute the covariances $c_{m,n}$ = EX_mX_n – EX_mEX_n . Let us put by convention $X_0 = 1$. Then $c_{m,0} = c_{0,n} = 0$.

Proposition 3.1. *The covariances $c_{m,n}$ satisfy the recurrence relation*

$$c_{m,n} = \mu L_2 c_{m-1,n-1} + V_1(\mu L)^{m+n-2}. \tag{3.1}$$

Therefore,

$$\rho = 1 \Rightarrow c_{m,n} = V_1 \min(m,n)(\mu L)^{m+n-2}\ ,\ \rho \neq 1 \Rightarrow c_{m,n} = V_1 \frac{\rho^{\min(m,n)} - 1}{\rho - 1}(\mu L)^{m+n-2}. \tag{3.2}$$

Proof. Suppose $m \geq n \geq 1$.According to (2.3), write

$$X_m = \sum_{i \leq \xi} v^{T_i} Y_i\ ,\ X_n = \sum_{i' \leq \xi} v^{T_{i'}} Z_{i'}\ , \tag{3.3}$$

where Y_i are i.i.d., distributed as X_{m-1} and $Z_{i'}$ also are i.i.d. and distributed as X_{n-1}. If $i \neq i'$ then Y_i are independent on $Z_{i'}$ – they represent the descendants of "i " in generation m-1 and the descendants of "i' " in generation n -1. If $i = i'$ they are not independent – they represent the descendants of "i" in generations m and n, but the pair (Y_i, Z_i) is distributed as (X_{m-1}, X_{n-1}). That's why we can write

$$\begin{aligned}
\mathrm{E}X_mX_n &= \mathrm{E}[(\sum_{i\leq\xi} v^{T_i}Y_i)(\sum_{i'\leq\xi} v^{T_{i'}}Z_{i'})] = \sum_{k=1}^{\infty} \mathrm{E}[(\sum_{i\leq k} v^{T_i}Y_i)(\sum_{i'\leq k} v^{T_{i'}}Z_{i'})]P(\xi_o = k)\\
&= \sum_{k=1}^{\infty} \mathrm{E}[\sum_{i\leq k} v^{2T_i}Y_iZ_i + \sum_{1\leq i\neq i'\leq k} v^{T_{i'}}v^{T_{i'}}Y_iZ_{i'}]P(\xi_o = k)\\
&= \sum_{k=1}^{\infty} [(\sum_{i\leq k} \mathrm{E}v^{2T_i}\mathrm{E}Y_iZ_i) + \sum_{1\leq i\neq i'\leq k} \mathrm{E}v^{T_{i'}}\mathrm{E}v^{T_{i'}}\mathrm{E}Y_i\mathrm{E}Z_{i'}]P(\xi_o = k)\\
&= \sum_{k=1}^{\infty} [(\sum_{i\leq k} L_2\mathrm{E}(X_{m-1}X_{n-1})) + \sum_{1\leq i\neq i'\leq k} L^2\mathrm{E}X_{m-1}\mathrm{E}X_{n-1}]P(\xi_o = k)\\
&= \sum_{k=1}^{\infty} [(kL_2\mathrm{E}(X_{m-1}X_{n-1})) + k(k-1)L^2\mathrm{E}_{m-1}\mathrm{E}_{n-1}]P(\xi_o = k)\\
&= L_2\mathrm{E}(X_{m-1}X_{n-1})\mathrm{E}\xi + L^2(\mu L)^{m+n-2}(\mathrm{E}\xi^2 - \mathrm{E}\xi)\ .
\end{aligned}$$

If we write E$X_mX_n = c_{m,n} + E_mE_n = c_{m,n} + (\mu L)^{m+n}$, then it follows that $c_{m,n} - (\mu L)^{m+n} = \mu L_2[c_{m-1,n-1} + (\mu L)^{m+n-2}] + (\mu^2 + \sigma^2 - \mu) L^2(\mu L)^{m+n-2}$ or

$$c_{m,n} = \mu L_2 c_{m-1,n-1} + (\mu L)^{m+n-2}(\mu L_2 + \sigma^2 L^2 - \mu L^2) \tag{3.4}$$

and this is precisely (3.1) since $\mu(L_2 - L^2) + \sigma^2 L^2 = V_1$. To check (3.2), let us write (3.1) as

$$c_{m+1,n+1} = ac_{m,n} + V_1 q^{m+n} \tag{3.5}$$

with $a = \mu L_2$ and $q = \mu L$. Notice that $q < 1$, if we want EX to be finite. Let also $Q = q^2$. Then, for $n = 0$ we get $c_{m+1,1} = V_1 q^m \Leftrightarrow c_{m.1} = V_1 q^{m-1}$. For $n = 1$ it follows that $c_{m+1,2} = aV_1 q^{m-1} + V_1 q^m$, hence $c_{m,2} = V_1 q^{m-2}(a + Q)$. By iteration, we find that

$$m \geq n \Rightarrow c_{m,n} = V_1 q^{m-n}(a^{n-1} + a^{n-2}Q + \ldots + aQ^{n-2} + Q^{n-1}) \tag{1.1}$$

and that is precisely (3.2), since $\rho = 1$ is the same as $a = Q \Leftrightarrow L_2 = \mu L^2$.

Proposition 3.2. *The correlation coefficients $r_{m,n} = r(X_m, X_n)$ are given by*

$$\rho = 1 \Rightarrow r_{m,n} = \sqrt{\frac{\min(m,n)}{\max(m,n)}}\ ,\ \rho \neq 1 \Rightarrow r_{m,n} = \sqrt{\left|\frac{\rho^{\min(m,n)}-1}{\rho^{\max(m,n)}-1}\right|}\ . \tag{3.7}$$

Proof. We have $r_{m,n} = \dfrac{c_{m,n}}{\sqrt{c_{m,m}c_{n,n}}}$. Suppose that $m \geq n$ and $\rho \neq 1$. Then , by (3.6) we have

$c_{m,n} = V_1 q^{m-n}(a^{n-1} + a^{n-2}Q + \ldots + aQ^{n-2} + Q^{n-1})$, $c_{m,m} = V_1(a^{m-1} + a^{m-2}Q + \ldots + aQ^{m-2} + Q^{m-1})$,

$c_{m,m} = V_1(a^{n-1} + a^{n-2}Q + \ldots + aQ^{n-2} + Q^{n-1})$, thus $r^2_{m,n} = \dfrac{Q^{m-n}(a^{n-1}+a^{n-2}Q+\ldots+Q^{n-1})^2}{(a^{m-1}+a^{m-2}Q+\ldots+Q^{m-1})(a^{n-1}+a^{n-2}Q+\ldots+Q^{n-1})}$

$$= \frac{Q^{(m-1)-(n-1)}(a^{n-1}+a^{n-2}Q+\ldots+Q^{n-1})}{(a^{m-1}+a^{m-2}Q+\ldots+Q^{m-1})} = \frac{Q^{-(n-1)}(a^{n-1}+a^{n-2}Q+\ldots+Q^{n-1})}{Q^{-(m-1)}(a^{m-1}+a^{m-2}Q+\ldots+Q^{m-1})}.$$

As $a/Q = \rho$, we can further write $r^2_{m,n} = \dfrac{1+\rho+\ldots+\rho^{n-1}}{1+\rho+\ldots+\rho^{m-1}}$ and the proof is complete.

Corollary 3.3.. $\mathrm{Var}(X) < \infty \Leftrightarrow \mu L < 1$ *and in this case*

$$\mathrm{Var}(X) = \frac{V_1}{(1-\mu L)^2(1-\rho(\mu L)^2)} = \frac{V_1}{(1-\mu L)^2(1-\mu L_2)}\ . \tag{3.8}$$

Proof. We have $\mathrm{Var}(X) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{cov}\,(X_m, X_n)$. Let $x = \mu L$. If $\rho \neq 1$, then according to (3.2) we have

$$\mathrm{Var}(X) = \frac{V_1}{\rho-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^{m+n-2}(\rho^{m\wedge n}-1) = \frac{V_1}{\rho-1}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^{m+n-2}\rho^{m\wedge n} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^{m+n-2}\right) =$$

$$\frac{V_1}{\rho-1}[(1+2x+2x^2+\ldots)(\rho+\rho^2x^2+\rho^3x^4+\rho^4x^6+\ldots) - (1+x+\ldots)^2) = \frac{V_1}{\rho-1}[(\frac{2}{1-x}-1)\frac{\rho}{1-\rho x^2} - \frac{1}{(1-x)^2}]$$

$$= \frac{V_1}{\rho-1}[\frac{1+x}{1-x}\cdot\frac{\rho}{1-\rho x^2} - \frac{1}{(1-x)^2}] = \frac{V_1}{\rho-1}[\frac{\rho(1-x^2)-(1-\rho x^2)}{(1-x)^2(1-\rho x^2)}] = \frac{V_1}{(1-x)^2(1-\rho x^2)}.$$

The second equality follows from the definition of ρ: $\rho x^2 = \dfrac{L_2}{\mu L^2}\mu^2 L^2$. If $\rho = 1$ the result is the same:

$$\mathrm{Var}(X) = V_1\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m\wedge n)x^{m+n-2} = V_1(1+2x+2x^2+2x^3+\ldots)(1+2x^2+3x^4+4x^6+..) =$$

$$V_1\frac{1+x}{1-x}\cdot\frac{1}{(1-x^2)^2} = \frac{V_1}{(1-x)^2(1-x^2)} = \frac{V_1}{(1-x)^2(1-\rho x^2)}\ .$$

Remark. Let $N_k = |G_k|$ be the number of descendants of "o" in generation k. It is well known that $\mathrm{E}N_k = \mu^k$, thus $\mu > 1 \Leftrightarrow \mathrm{E}N_k \to \infty$. Relation (2.9) points out that it is possible that the total cost of the insurer have a finite expectation, if the interest rate δ is great enough to ensure the inequality $\mu L < 1$.

Moreover, if we let $\delta \to 0$, we find the classical results of branching process theory (see [1], [3]) as particular cases. Thus, $\delta = 0 \Rightarrow v = 1 \Rightarrow L = L_2 = 1$, $\rho = \dfrac{1}{\mu}$, $V_1 = \sigma^2$.Then $X_k = N_k$ = the number of members of G_k and X is the total number of descendents of "o". We have E$X = \dfrac{\mu}{1-\mu}$, Var $(X) = \dfrac{\sigma^2}{(1-\mu)^3}$ and the correlation coefficient of N_m and N_n is $r_{m,n} = \sqrt{\mu^{|m-n|}\dfrac{1-\mu^{m\wedge n}}{1-\mu^{m\vee n}}}$.

Example. For instance, if $\delta = \ln 1.05$, thus $v = 0.953$, and $T = 14 + Z$, $Z \sim$ Binomial $(114,p)$, $p = .1$ (just for fun, x may procreate between age 14 and 114 years with an expected age of procreation at 24 years !), then $L = \mathrm{E}v^T = v^{14}(q+pv)^{100} \approx 0.3133$ thus E$X < \infty$ as long as $\mu < 1/L \approx 3.19$. Further on, $L_2 = \mathrm{E}v^{2T} = v^{28}(q+pv^2)^{100} \approx 0.1002$. Now, if, for instance, $\xi \sim$ Bin(6,1/3), then $\mu = 2$, $\sigma^2 = 4/3$, $\mu L \approx 0.6267$, hence E$X \approx 1.679$. As $V_1 = \mu(L_2 - L^2) + L^2\sigma^2 \approx 2s^2 + 0.1745 \approx 0.1787$ and $\mu L \approx .6266$, $\mu L_2 \approx .2004$, by (4.7) we get

$$\mathrm{Var}(X) = \frac{.1787}{(1-.6266)^2(1-.2004)} \approx 1{,}6029 \Rightarrow \sigma(X) \approx 1.266.$$

From now on we can apply Tchebyshev's inequality to estimate $P(X > C)$: $P(X > \mathrm{E}X + k\sigma(X)) < k^{-2}$. The problem is that such as estimation is bad. For $k = 5$ we get $P(X > 1.679 + 5\cdot 1.266) = P(X > 8.0092) < 1/25 = 4\%$, but surely this probability is much less than that. We think that the premium $\Pi = 8$ MU is very safe for the insurer under the conditions assumed. However, more sophisticated techniques are necessary to find better bounds.

What can one say about the exponential premium principle? In order to be able to make estimations one has to study some moment generating functions.

4. MOMENT GENERATING FUNCTION OF $(X_1,\ldots, X_K)$ AND EXPONENTIAL PREMIUM

In the sequel we assume H1 – H4.

Let $\boldsymbol{m_k}(t_1,\ldots,t_k) = \mathrm{E}\exp(t_1X_1 + t_2X_2 + \ldots + t_kX_k)$ or, for short, $\boldsymbol{m_k}(\boldsymbol{t}) = \mathrm{E}\, e^{<\boldsymbol{t},Z_k>}$, be the moment generating function of the vector $(X_1,\ldots,X_k)$ of costs for the first k generations of descendants, actualized at moment $t = 0$. Let also $\varphi_k(t) = \boldsymbol{m_k}(t,\ldots,t) = \mathrm{E}\exp[t(X_1 + X_2 + \ldots + X_k)]$ and $\Pi_k(t) = \log[\varphi_k(t)]\,/\,t$. Then, for $t > 0$, $\Pi_k(t)$ is the very definition of the *exponential premium demanded by an insurer with constant risk-aversion t in order to ensure the first k generations.* (see [2], [4], [6])

We are interested in the function $\Pi(t) = \log(\mathrm{E}e^{tX})\,/\,t$. Since we only deal with positive random variables, it is clear that $\Pi = \lim_{k\to\infty}\Pi_k$.

Proposition 4.1. *Let* $g(x) = \mathrm{E}x^{\xi}$ *be the generating function of* ξ_o *and* $\boldsymbol{m}(t) = \mathrm{E}e^{tv^T}$, *where* $T = T_o$. *Then*

(i) $\boldsymbol{m}_1(t) = g(\boldsymbol{m}(t)) = g[\mathrm{E}e^{tv^T}]$;

(ii) $k \geq 2 \Rightarrow \boldsymbol{m_k}(t_1,\ldots,t_k) = g[\mathrm{E}(e^{t_1v^T} m_{k-1}(t_2v^T, t_3v^T, \ldots, t_kv^T))]$;

(iii) $\varphi_k(t) = g[\mathrm{E}(e^{tv^T}\varphi_{k-1}(tv^T))]$;

(iv) *the* m.g.f. $\varphi(t) = \mathrm{E}e^{tX}$ *of X, satisfies the relation* $\varphi(t) = g[\mathrm{E}(e^{tY}\varphi(tY))]$ with $Y = v^T$;

(v) *the exponential premium* $\Pi(t)$ *satisfies the relation* $t\Pi(t) = \log[g(\mathrm{E}e^{t(Y+\Pi(tY))})]$, $Y = v^T$.

Proof. (i) $\boldsymbol{m_1}(t) = \mathrm{E}\exp(tX_1) = \mathrm{E}\exp(t\sum_{i\leq\xi_o} v^{T_j}) = \sum_{n=0}^{\infty}\mathrm{E}e^{t\sum_{i\leq n}v^{T_i}} P(\xi_o = n) = \sum_{n=0}^{\infty}(\mathrm{E}e^{tv^T})^n\ P(\xi_o = n) =$

$$g(\mathrm{E}e^{tv^T}) = g(\boldsymbol{m}(t)).$$

(ii) Apply (2.3). For $k \geq 2$, write $\boldsymbol{t} = (t_1,\ldots,t_k)$ as $\boldsymbol{t} = (t_1,\boldsymbol{u})$ with $\boldsymbol{u} = (t_2,\ldots,t_k)$. Then $t_1X_1 + t_2X_2 + \ldots + t_kX_k = <\boldsymbol{t}, Z_k> = <\boldsymbol{t}, \sum_{n\leq\xi_o} v^{T_n}(1, Z^*_{k-1,n})> = \sum_{n\leq\xi_o} v^{T_n}(t_1 + <\boldsymbol{u}, Z^*_{k-1,n}>)$, where $Z^*_{k-1,n}$ are independent copies of Z_{k-1}.
Write this relation as

$$t_1X_1 + t_2X_2 + \ldots + t_kX_k = \sum_{N\geq 0}(\sum_{n\leq N} v^{T_n}(t_1 + <\boldsymbol{u}, Z^*_{k-1,n}>))1_{\{\xi_o = N\}}. \tag{4.1}$$

Then $\exp(t_1X_1 + t_2X_2 + \ldots + t_kX_k) = \sum_{N\geq 0}\exp(\sum_{n\leq N} v^{T_n}(t_1 + <\boldsymbol{u}, Z^*_{k-1,n}>))1_{\{\xi_o = N\}}$, hence

$\boldsymbol{m_k}(\boldsymbol{t}) = \sum_{N\geq 0} \mathrm{E}\exp(\sum_{n\leq N} v^{T_n}(t_1 + <\boldsymbol{u}, Z^*_{k-1,n}>))P(\xi_o = N)$.

As $(T_n, Z^*_{k-1,n})_n$ are i.i.d., we can write

$$\boldsymbol{m_k}(\boldsymbol{t}) = \sum_{N\geq 0}(\mathrm{E}\exp(v^{T_1}(t_1 + <\boldsymbol{u}, Z^*_{k-1,1}>))^n P(\xi_o = N) = g[\mathrm{E}\exp(v^{T_1}(t_1 + <\boldsymbol{u}, Z^*_{k-1,1}>))]. \tag{4.2}$$

As T_1 is independent on $Z^*_{k\text{-}1,1}$ we have

$$\mathrm{E}(\exp(v^{T_1}(t_1 + <\boldsymbol{u}, Z^*_{k-1,1}>)) = \mathrm{E}(\exp(t_1 v^{T_1})\exp(<\boldsymbol{u}, Z^*_{k-1,1}>)) =$$
$$\mathrm{E}[\,\mathrm{E}(\exp(t_1 v^{T_1})\exp(v^{T_1} <\boldsymbol{u}, Z^*_{k-1,1}>)|T_1)\,] = \mathrm{E}[\,\exp(t_1 v^{T_1})\mathrm{E}(\exp(<v^{T_1}\boldsymbol{u}, Z^*_{k-1,1}>)|T_1)\,] =$$
$$\mathrm{E}[\,\exp(t_1 v^{T_1})\boldsymbol{m}_{k-1}(v^{T_1}\boldsymbol{u})\,].$$

(iii) is a special case of (ii) for $t_1 = t_2 = \ldots = t_k = t$. while, for (iv) let $k \to \infty$ in (iii). Finally, (v) is obvious.

In the special case $\delta = 0 \Leftrightarrow v = 1$ we obtain classical results (see [1],[3]). In this case, $X_k = N_k$ and $X = N$ is the total number of descendants of "o". We have

Corollary 4.2. *Let $g(x) = \mathrm{E}x^{\xi}$ be the generating function of ξ_o and $\boldsymbol{g}_k$ the generating function of the vector $(N_1,\ldots,N_k)$ defined by $\boldsymbol{g}_k(x_1,x_2,\ldots,x_k) = \mathrm{E}[x_1^{N_1}x_2^{N_2}\ldots x_k^{N_k}]$. Let also $\mathrm{c}_k(x) = \boldsymbol{g_k}(x,x,\ldots,x)$ be the generating function of $S_k := N_1 + N_2 + \ldots + N_k$ and $c = \lim_{k\to\infty}c_k$ the generating function of the total number of N descendants of "o". Then*

(i) $\boldsymbol{g}_1 = g$;
(ii) $k \geq 2 \Rightarrow \boldsymbol{g_k}(x_1,\ldots,x_k) = g[x_1\boldsymbol{g}_{k-1}(x_2,\ldots,x_k)]$, $x_i > 0\ \forall\ 1 \leq i \leq k$;
(iii) $c_k(x) = g[xc_{k-1}(x)]\ \forall\ x > 0$;
(iv) $c(x) = g[xc(x)]\ \ \forall\ x > 0$;
(v) *the exponential premium of N*, $\Pi(t) = [\log c(e^t)]\,/\,t$ *satisfies the equation* $t\Pi(t) = \log[g(e^{t(1+\Pi(t))})]$.

Proof. In Proposition 4.1. take $v^T = 1$ and replace $\exp(t_j)$ by x_j.

Remark. It is difficult to find natural cases when the functional equation (iv) from Pproposition 4.1 has computable solutions. However, it provides another way to prove (2.10) and (3.8) concerning EX and Var(X), if we can prove somehow that $\varphi(t) < \infty$ in a neighborhood of 0. If we denote by Y the random variable v^T, $0 < Y < 1$, then equation (iv) from Proposition 4.1 becomes

$$\varphi(t) = g(\psi(t)) \text{ with } \psi(t) = \mathrm{E}(e^{tY}\varphi(tY)). \tag{4.3}$$

Then $\varphi'(0) = g'(\psi(0)\cdot\psi'(0) = g'(1)(1 + \varphi'(0))\mathrm{E}Y$. As $g'(1) = \mathrm{E}\xi_o = \mu$ and $\mathrm{E}Y = L$, we get the equation $\varphi'(0) = \mu L(1 + \varphi'(0))$; as $\varphi'(0) = \mathrm{E}X$, we rediscover (2.10). If we differentiate (4.3) twice, after some (tedious) computations we rediscover (3.8), since $\varphi''(0) = \mathrm{E}X^2$.

Definiton. Let F be a probability distribution on $[0,\infty)$. Consider its moment generating function $\boldsymbol{m_F}(t) = \int e^{tx}\,\mathrm{d}F(x)$. Let $\mathrm{Dom}(F) := \{t \mid \boldsymbol{m_F}(t) < \infty\}$. Then F is called ***short tailed*** if $(-\infty,0] \in \mathrm{Int}(\mathrm{Dom}(F))$; or, in other words, if $\boldsymbol{m_F}(t) < \infty\ \forall\ t < t_0$ for some positive t_0. For instance, any F with bounded support is short tailed; Poisson(λ), Negbin(ν,λ) and Gamma(ν,λ) all are short tailed. Call F to be ***very short tailed*** (and write $F \in$ VST) if $\mathrm{Dom}(F) = \Re$. For instance, Binomial(n,p) and Poisson(λ) **are** very short tailed but Negbin(ν,λ) and Gamma(ν,λ) are not.

From a practical point of view, a distribution F is VST if its exponential premium $\Pi_F(t)$ defined as $[\log \boldsymbol{m_F}(t)] / t$ is finite at any risk-aversion coefficient $t > 0$ (or, to use the slang, if "F can be insured").

If $F \notin$ VST then there exists $t_0 > 0$ such that $\Pi_F(t) = \infty$ if $t > t_0$. (F "cannot be ensured" if the risk-aversion coefficient of the ensurer is too big).

Here is a main difference between N and X.

Proposition 4.3. *N is **never** very short tailed (if we let aside the trivial case $\xi = 0$ (a.s.), but **is** short tailed if ξ is short tailed and we are in the subcritical case , i.e.,* $\mu < 1$ (see [1]).

Proof. If $\mu = \mathrm{E}\xi \geq 1$, then $\mathrm{E}N = \infty$ hence N cannot be short tailed. Suppose that $\mu < 1$ (the subcritical case). Let $\xi \sim \begin{pmatrix} 0 & 1 & \cdots & n & \cdots \\ p_0 & p_1 & \cdots & p_n & \cdots \end{pmatrix}$. Then $p_0 > 0$ and $p_n > 0$ for some $n \geq 1$. Let $\varphi(t) = c(e^t)$ be the m.g.f. of N. By Corollary 5.2 (iv) φ satisfies the equation

$$\varphi(t) = g(e^t\varphi(t)). \tag{4.4}$$

Hence $\varphi(t) > p_0 + p_n e^{nt}\varphi^n(t)$. Let $t > 0$ be such that $\alpha := p_n e^{nt} > 1$. Then $\varphi(t) > \alpha\varphi^n(t)$. As $\varphi(t) > 1$, we have $\varphi(t) < \varphi^n(t)$, hence $\varphi^n(t) > \varphi(t) > \alpha\varphi^n(t)$, which can only hold if $\varphi(t) = \infty$. The second assertion is a special case (α=1) of the next result.

Proposition 4.4. *Let $Y = e^T$, $L = \mathrm{E}Y$ and $\mu = \mathrm{E}\xi_0$. Suppose that*

$$\mu L < 1. \tag{4.5}$$

(i) *If ξ is short tailed, then X is short tailed, too*;

(ii) *If ξ is* VST ***and*** ess sup $Y < 1$ *then S is* VST, *too*;

(iii) N is short tailed $\Leftrightarrow$ ξ *is short tailed and* $\mu < 1$ (*the **subcritical** case*).

Proof. (i) We have to prove that there exists $t_* > 0$ such that $t < t_* \Rightarrow \mathrm{E}e^{tX} < \infty$. Keep the same notation as in Proposition 4.1.Thus $\varphi_k(t) = \mathrm{E}\exp[t(X_1 + X_2 + \ldots + X_k)]$ satisfies the recurrence relations

$$\varphi_1(t) = g(\mathrm{E}e^{tY}) \text{ and } k \geq 2 \Rightarrow \varphi_k(t) = g(\mathrm{E}(e^{tY}\varphi_{k-1}(tY)). \tag{4.6}$$

From the very definition of φ_k the sequence $(\varphi_k(t))_k$ is increasing. Hence it has a limit, $\varphi(t) = \mathrm{E}e^{tX}$, such that $\varphi(t) = g(\mathrm{E}(e^{tY}\varphi(tY))$.

As we agreed that ξ is short tailed, there exists $x_0 > 1$ such that $x < x_0 \Rightarrow g(x) < \infty$. Let then $t_0 = \dfrac{\ln x_0}{\alpha}$. Thus $t_0 > 0$ and $t < t_0 \Rightarrow \psi(t) < \infty$. Remark that $\varphi'_1(0) = 1$ and $\varphi_1'(0) = \mu L < 1$. As φ_1 is increasing and convex, the equation $\varphi_1(t) = 1 + \beta t$ has exactly one positive solution $t(\beta)$ for any $\beta > \varphi_1'(0) = \mu L$. .

Let $\beta \in (\mu L, 1)$ be fixed and let t_1 be the unique positive solution of $\varphi_1(t) = 1 + \beta t$. As the line $t \mapsto 1 + \beta t$ is a chord and ψ is convex, it follows that

$$\varphi_1(t) < 1 + \beta t \;\forall\; t \in (0, t_1). \tag{4.7}$$

Let $t_* = (1 - \beta)t_1$. Then $t_* > 0$ and we claim that

$$t < t_* \Rightarrow \varphi(t) \leq 1 + \frac{\beta t}{1-\beta} < \infty. \tag{4.8}$$

In order to prove that, we shall check by induction that

$$t < t_* \Rightarrow \varphi_k(t) < 1 + t(\beta + \beta^2 + \ldots + \beta^k). \tag{4.9}$$

For $k = 1$ the assertion is true. Let $k \geq 2$ and $t < t_*$. Then

$$\varphi_k(t) = g(\mathrm{E}(e^{tY}\varphi_{k-1}(tY)) \leq g(\mathrm{E}(e^{tY}(1 + tY(\beta + \beta^2 + \ldots + \beta^{k-1})))) \tag{4.10}$$

(since $tY < t < t_*$) . But $1 + tY(\beta + \beta^2 + \ldots + \beta^{k-1}) < \exp(tY(\beta + \beta^2 + \ldots + \beta^{k-1}))$, hence (4.10) implies the inequality $\varphi_k(t) \leq g(\mathrm{E}\exp(tY + tY(\beta + \beta^2 + \ldots + \beta^{k-1}))) = \varphi_1(t(1 + \beta + \ldots + \beta^k))$. Next, $t(1 + \beta + \ldots + \beta^{k-1}) <$

$t_*(1 + \beta + \ldots + \beta^{k-1}) < t_* / (1 - \beta) = t_1$. Then (4.2) holds. Hence $\varphi_k(t) < 1 + \beta t(1 + \beta + \ldots + \beta^{k-1})$. Therefore (4.9) holds thus (4.8) holds, too.

(ii) Suppose that ξ is VST. Let α = ess sup $Y < 1$. The news is that now $\varphi_1(t) < \infty \ \forall\ t > 0$. We want to prove that $\varphi(t) < \infty \ \forall\ t > 0$. Suppose for a contradiction that $\varphi(t) = \infty$. As $\varphi(t) = g(\mathrm{E}(e^{tY}\varphi(tY))$ and $1 \le e^{tY} \le e^t$ that would imply the fact that $\mathrm{E}\varphi(tY) = \infty$. But $Y \le \alpha \Rightarrow \varphi(tY) \le \varphi(\alpha t) \Rightarrow \mathrm{E}\varphi(tY) \le \varphi(\alpha t) \Rightarrow \varphi(\alpha t) = \infty$. Repeating the arguments $\varphi(\alpha t) = \infty \Rightarrow \varphi(\alpha^2 t) < \infty \Rightarrow \ldots \Rightarrow \varphi(\alpha^k t) < \infty \ \forall\ k \Rightarrow \varphi(t) = \infty \ \forall\ t > 0$ and that contradicts the existence of $t_* > 0$ such that $\varphi(t) < \infty$ for $t < t_*$.

(iii) "$\Rightarrow$". Of course $\xi_o \le N$, hence if N is short tailed, ξ_o is short tailed, too. If $\mu_1 \ge 1$, then $\mathrm{E}N = \infty$, thus N cannot be short tailed. The converse implication, "$\Leftarrow$" is a particular case of (i)., for α=1.

Corollary 4.5. *Let $Y = v^T$. Suppose that there exists $a > 0$ such that $T \ge a$ a.s. Let $\alpha = v^a < 1$. Suppose that $\alpha\mu < 1$ and ξ is* VST. *Then X is* VST, *too.*

We shall give now a lower bound for $\varphi(t)$. We need the following

Definition (See [5], [6]). Let Y and Y' be two non-negative random variables. Then Y ***is dominated by*** Y' in the ***increasing convex order*** (denoted by $Y \prec_{\mathbf{icx}} Y'$) if $\mathrm{E}u(Y) \le \mathrm{E}u(Y')$ for any $u : [0,\infty) \to [0,\infty)$ non-decreasing and convex. We shall use the following properties of this stochastic dominance: (some of them are obvious).

(i) $\mathrm{E}Y \prec_{\mathbf{icx}} Y$.

(ii) $Y \prec_{\mathbf{icx}} Y'$, ψ is non-decreasing and convex $\Rightarrow \psi(Y) \prec_{\mathbf{icx}} \psi(Y')$.

(iii) Invariance w.r. to compounding: if $(Y_n)_n$ and $(Y'_n)_n$ are i.i.d. and N, N' are two counters independent on both, then $Y_1 + \ldots + Y_N \prec_{\mathbf{icx}} Y'_1 + \ldots + Y'_{N'}$.

(iv) If $Y \prec_{\mathbf{icx}} Y'$ then $\boldsymbol{m_Y(t)} \le \boldsymbol{m_{Y'}(t)} \ \forall\ t \ge 0$, thus $\Pi_Y \le \Pi_{Y'}$.

We shall need another property, as well, for which we do not know references.

Lemma 4.6.. *Let Y_i, Z_i (i = 1,2) be positive and independent. Suppose that $Y_1 \prec_{\mathbf{icx}} Y_2$ and $Z_1 \prec_{\mathbf{icx}} Z_2$. Then $Y_1Z_1 \prec_{\mathbf{icx}} Y_2Z_2$.*

Proof. Let F_i, G_i be the distributions of Y_i, Z_i. Let also u be non- decreasing, convex and positive. Then $\mathrm{E}u(Y_1Z_1) = \iint u(yz)\mathrm{d}F_1(y)\mathrm{d}G_1(z)$. As the mapping $y \mapsto u(yz)$ is non-decreasing and convex and $Y_1 \prec_{\mathbf{icx}} Y_2$, it follows that $\int u(yz)\mathrm{d}F_1(y) \le \int u(yz)\mathrm{d}F_2(y)$. Let $w(z) = \int u(yz)\mathrm{d}F_2(y)$. Then w also is non-decreasing and convex (obvious), hence $\mathrm{E}u(Y_1Z_1) \le \int w(z)\mathrm{d}G_1(z) \le \int w(z)\mathrm{d}G_2(z) = \mathrm{E}u(Y_2Z_2)$.

We now prove a result concerning comparisons of two different progeny scenarios of progeny. Recall that $\Gamma = \bigcup_{k=0}^{\infty} \mathbf{N}^k$.

Proposition 4.7. *Let* $\boldsymbol{\xi} = (\xi_x)_{x\in\Gamma}$ *and* $\boldsymbol{\xi}' = (\xi'_x)_{x\in\Gamma}$ *be two families of i.i.d. random variables associated with the progeny of* "o" *under two scenarios. Let also* $\boldsymbol{T}' = (T_x)_{x\in\Gamma}$ *and* $\boldsymbol{T}' = (T'_x)_x$ *be i.i.d., denoting the independent birth times of the same progeny . Let X_k , X'_k be the costs of the kth generation according to both scenarios and X, X' be the total costs of the insurance. Suppose that $v^{T_x} \prec_{\mathbf{icx}} v^{T'_x}$ and $\xi_x \prec_{\mathbf{icx}} \xi'_x \ \forall\ x \in \Gamma$. Then $X_k \prec_{\mathbf{icx}} X'_k \ \forall\ k$* and *$X \prec_{\mathbf{icx}} X'$.*

Proof. Induction on k. Let $Y_x = v^{T_x}$ and $Y'_x = v^{T'_x}$. For $k = 1$, $X_1 = \sum_{i\le\xi_o} Y_i$ is increasing, convex and dominated by $X'_1 = \sum_{i\le\xi'_o} Y'_i$ by invariance to compounding (property (iii)). We shall now use (3.4): $X_k =$

$\sum_{n\le\xi_0} Y_n X^*_{k-1,n}$ with $X^*_{k-1,n}$ i.i.d. distributed as X_{k-1} and $X'_k = \sum_{n\le\xi_0} Y_n X'^*_{k-1,n}$ with $X'^*_{k-1,n}$ i.i.d. distributed as X'_{k-1}. According to Lemma 5.5. and to invariance to compounding, we have $X_k \prec_{\mathbf{icx}} X'_k$ by the induction hypothesis. In the same way one can prove that $X \prec_{\mathbf{icx}} X'$.

Now, here is a lower bound for the exponential premium Π.

Corollary 4.8.. *Let* $L = \mathrm{E}v^T$. *Then* $LN_1 + L^2N_2 + \ldots + L^kN_k \prec_{\mathbf{icx}} X_1 + X_n + \ldots + X_k$ *for any* k . *It follows that* $\varphi_k \ge \psi_k$ *and* $\varphi \ge \psi$, *where* $\psi_k(t) = \mathrm{E}\exp(t(LN_1 + L^2N_2 + \ldots + L^kN_k))$, $\psi = \lim_{k\to\infty}\psi_k$.

Hence $\Pi(t) \ge (\ln\psi(t)) / t$. *Moreover*, ψ *is the unique solution of the equation* $\psi(t) = g(e^{tL}\psi(tL))$.

Proof. Use the fact that $L = \mathrm{E}v^T \prec_{\mathbf{icx}} v^T$ and Proposition 4.7 with $T = T'$, $Y = L$ and $Y' = v^T$.

5. OPEN PROBLEMS

1. Estimate $r = r(\xi,T) := \sup\{t \mid \varphi(t) < \infty\}$. This constant would mean the maximum risk-aversion coefficient that an insurer can allow.
2. Find a better estimation of $P(X > C)$ than that given by Tchebycheff's inequality.

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