THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 8 , Number 1/2007, pp.000-000

SOME EXISTENCE RESULTS FOR A CLASS OF RELATIVELY B-PSEUDOMONOTONE VARIATIONAL INEQUALITIES OVER PRODUCT SETS

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We consider a general class of relatively B-pseudomonotone variational inequalities over product sets. To prove the existence of a solution of problems, the concept of pseudomonotonicity is extended. Some recent results in this field are thus generalized.

Keywords: Variational inequalities, System of variational inequalities, Relatively B-pseudomonotone maps.

1. INTRODUCTION

The problems which imply variational inequality over product sets have many applications. Thus, the Nash equilibrium problem for differentiable functions, various equilibrium-type problems, as traffic equilibrium, spatial equilibrium and general equilibrium programming problems from economics, game theory, operations research, mathematical physics, for example, can be modelled as variational inequality problems [4, 10, 13, 14, 8, 12, 15]. For new results relative to this field see, for example, Pang [15], Konnov [9], Ansari and Yao [1,2], Yang and Yao [18], Ansari and Khan [3]. Thus in [3], without using arguments from generalized monotonicity, a concept of relatively B-pseudomonotonicity in the sense of Brezis [5,6] is introduced. In this paper, we consider a general class of variational inequalities over product sets, which extend many known results in this field. Then, using a fixed point theorem of Chowdhury and Tan [7], some existence results on the solution of problems in our class are derived.

2. PRELIMINARIES

Let $I=\{1,2,...,n\}$ be a finite index set and, for each $i\in I$, X_i a topological vector space and Y_i an arbitrary set. For $i\in I$, let K_i a nonempty convex subset of X_i, $K=\prod_{i\in I} K_i$, $X=\prod_{i\in I} X_i$. For $x\in X$ we write $x=(x_i)_{i\in I}$ where $x_i\in X_i, i\in I$. For each $i\in I$, let $f_i : K\to Y_i$ and $\Psi_i : Y_i\times K_i\times K_i\to R$ be nonlinear maps and $\Psi=(\Psi_i, i\in I)$. Consider the Ψ-variational inequality problem over product sets

(Ψ-VIPPS) Find $\bar{x}\in K$ such that $\sum_{i\in I}\Psi_i\left(f_i(\bar{x}),\bar{x}_i;y_i\right)\geq 0,$ for all $y_i\in K_i, i\in I$.

Also, consider the Ψ-system of variational inequalities

(Ψ - SVIP) Find $\bar{x}\in K$ such that $\Psi_i\left(f_i(\bar{x}),\bar{x}_i;y_i\right)\geq 0,$ for all $y_i\in K_i, i\in I$.

We see that (Ψ-SVIP) implies (Ψ-VIPPS). If for each $i\in I$ we have $\Psi_i\left(f_i(\bar{x}),\bar{x}_i;\bar{x}_i\right)=0,$ then (Ψ-VIPPS) implies (Ψ-SVIP). Hence, in this case, problems (Ψ-VIPPS) and (Ψ-SVIP) are equivalent.

If by $\langle\cdot,\cdot\rangle$, we denote the pairing between X_i^* and X_i and $f_i : K \to X_i^*$ is a nonlinear map and $\Psi_i(f_i(\bar{x}), \bar{x}_i; y_i) = \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle$ for each $i \in I$, then (Ψ-VIPPS) reduces to the (VIPPS) problem considered by, for example, Konnov [10] and Ansari, Khan [3]. In this case, Konnov proved some existence results for a solution of (VIPPS) under relatively pseudomonotonicity or strongly relative pseudomonotonicity assumptions in the setting of Banach space. Also in this case, problem (Ψ-SVIP) reduces to the (SVIP) problem.

Now, for each $i \in I$ let $F_i : K \to 2^{X_i^*}$ be a multivalued map with nonempty values and $F = (F_i; i \in I)$, $F : K \to 2^{X^*}$ a multivalued map with nonempty values and $\Psi = (\Psi_i; i \in I)$, where $\Psi_i : X_i^* \times K_i \times K_i \to R$. Then, define the multivalued Ψ-variational inequality problem (Ψ-MVIPPS) over product sets and system of multivalued Ψ-variational inequalities (Ψ-SMVIP), respectively, by

(Ψ-MVIPPS) Find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that $\sum_{i \in I} \Psi_i(\bar{u}_i, \bar{x}_i; y_i) \geq 0$, for all $y_i \in K_i$, $i \in I$,

where u_i is the i component of u and

(Ψ-MSMVIP) Find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that $\Psi_i(\bar{u}_i, \bar{x}_i; y_i) \geq 0$, for all $y_i \in K_i, i \in I$.

In order to prove the equivalence of the above problems we can proceed similarly to the case of the equivalence of problems (Ψ-VIPPS) and (Ψ-SVIP). For establishing the main result on existence of solutions of problems (Ψ-VIPPS) and (Ψ-SVIP), we use a fixed point theorem of Chowdhury and Tan [7]. For a set A we denote by 2^A the family of all subsets of A and by $\Im(A)$ the family of all finite subsets of A. Next, co A denotes the convex hull of A.

Theorem 2.1 ([7]). *Let K be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) X and $T : K \to 2^K$ a multivalued map. Assume that the following conditions hold*:

i1) *for all $x \in K$, $T(x)$ is convex*;

i2) *for each $A \in \Im(K)$ and for all $y \in \operatorname{co} A$, $T^{-1}(y) \cap \operatorname{co} A$ is open in $\operatorname{co} A$*;

i3) *for each $A \in \Im(K)$ and for all $x, y \in \operatorname{co} A$ and every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in K converging to x such that $ty + (1-t)x \notin T(x_\alpha), \forall \alpha \in \Gamma, \forall t \in [0,1]$ we have $y \notin T(x)$*;

i4) *there exist a nonempty closed compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$, $\forall x \in K \setminus D$*;

i5) *for all $x \in D$, $T(x)$ is nonempty.*

Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$..

3. EXISTENCE RESULTS FOR (Ψ-VIPPS) AND (SΨ-VIP)

In the first part of this section we define the concept of relatively B-pseudomonotonicity for mapping Ψ, which extends the concept of pseudomonotonicity in the sense of Brezis [5,6] and the concept of relatively B-pseudomonotonicity introduced by Ansari and Khan [3].

Definition 3.1. *We say that f is relatively* B-*pseudomonotone* (*respectively, relatively demimonotone*) *with respect to Ψ if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha \in \Gamma}$ in K converging to x* (*respectively, converging weakly to x*) *with* $\liminf_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i(f_i(x^\alpha), x_i^\alpha; x_i) \geq 0$, *we have*

$$\sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) \geq \limsup_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i(f_i(x^\alpha), x_i^\alpha; y_i) \text{ for all } y \in K.$$

We note that for $\Psi_i(f_i(x), x_i, y_i) = \langle f_i(x), y_i - x_i \rangle, i \in I$, this definition reduces to the definition of relatively B-pseudomonotonicity (respectively, relatively demimonotonicity) of f. Also, if the set I is a singleton set, we get the concept of pseudomonotonicity defined by Brezis [5]. The next theorem gives us an existence result for (Ψ-VIPPS).

Theorem 3.1. *Assume that*
i1) f *is relatively* B-*pseudomonotone with respect to* Ψ;
i2) *for any* $A \in \Im(K)$, *the mapping* $x \to \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i)$ *is upper semicontinuous on* co A;
i3) *there exists a nonempty closed compact subset* D *of* K *and* $\widetilde{y} \in$ D *such that* $\sum_{i \in I} \Psi_i(f_i(x), x_i; \widetilde{y}_i) < 0$ *for all* $x \in K \setminus D$;
i4) *the mapping* $y \mapsto \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i)$ *is quasiconvex for any* $x \in K$;
i5) $\sum_{i \in I} \Psi_i(f_i(x), x_i; x_i) = 0$, $\forall x \in K$;
Then (Ψ-VIPPS) *has a solution.*

Proof: Let $T: K \to 2^K$ be the multivalued map given by $T(x) = \{y \in K : \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) < 0\}$, $x \in K$. We prove that T verifies the conditions of the part (i) of our Theorem 2.1.

By (i4) we get that $T(x)$ is a convex set for any $x \in K$. Using (i2) we see that for $A \in \Im(K)$ and $y \in \text{co}\, A$, the set $[T^{-1}(y)]^c \cap \text{co}\, A = \{x \in \text{co}\, A : \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) \geq 0\}$ is closed in co A, thus the set $T^{-1}(y) \cap$ co A is an open set in co A i.e. (i2) from Theorem 2.1.

Now let us establish that (i3) of Theorem 2.1 is satisfied. Let $x, y \in$ co A and a net $\{x^\alpha\}_{\alpha \in \Gamma}$ in K converging to x such that

$$\sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; t y_i + (1-t) x_i\right) \geq 0 \text{ for all } \alpha \in \Gamma,\ t \in [0,1]. \tag{3.1}$$

By (3.1), for $t = 0$ we get $\sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; x_i\right) \geq 0$ for all $\alpha \in \Gamma$. Hence

$$\liminf_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; x_i\right) \geq 0$$

and then by (i1) we obtain

$$\sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) \geq \limsup_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; y_i\right) \tag{3.2}$$

Also, by (3.1) for $t = 1$, we have

$$\sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; y_i\right) \geq 0 \text{ for all } \alpha \in \Gamma.$$

This inequality yields

$$\liminf_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; y_i\right) \geq 0 \tag{3.3}$$

Combining (3.2) and (3.3), we obtain $\sum_{i \in I} \Psi_i\left(f_i\left(x^\alpha\right), x_i^\alpha; y_i\right) \geq 0$, i.e. $y \notin T(x)$.

We see that (i3) from Theorem 3.1 corresponds to (i4) from Theorem 2.1.

Also, according to (i3) and definition of T, the set $T(x)$ is nonempty for all $x \in \mathrm{D}$. Thus, (v) of Theorem 2.1 is satisfied.

Hence all the conditions of Theorem 2.1 are satisfied. It follows that there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, i.e. $\sum_{i \in I} \Psi_i(f_i(\hat{x}), \hat{x}_i; \hat{x}_i) < 0$, inequality which contradicts (i5). Thus, there exists $\bar{x} \in K$ such that $T(\bar{x}) = \Phi$, i.e. $\sum_{i \in I} \Psi_i(f_i(\bar{x}), \bar{x}_i; y_i) \geq 0$, for all $y_i \in K_i$, $i \in I$, i.e. $\bar{x}$ is a solution of (Ψ-VIPPS), and the theorem is proved.

The case of real reflexive Banach spaces is considered below.

Theorem 3.2. *For* $i \in I$, *let* X_i *be a real reflexive Banach space and* K_i *be a nonempty closed convex subset of* X_i. *Assume* (i2), (i4) *and* (i5) *of Theorem* 3.1 *and*

i6) f *is relatively demimonotone wrt* Ψ;

i7) *there exists* $\tilde{y} \in K$ *such that*

$$\lim_{\substack{\|x\| \to \infty \\ x \in K}} \sum_{i \in I} \Psi_i(f_i(x), x_i; \tilde{y}_i) < 0 \tag{3.4}$$

where $\|\cdot\|$ *is the product norm on* X. *Then* (Ψ-VIPPS) *has a solution*.

Proof: We should verify condition (i3) of Theorem 3.1.

Let γ be the left-hand-side expression of (3.4). According to (i7) we have $\gamma < 0$.

Let $\varepsilon > 0$ be such that $\|\tilde{y}\| \leq \varepsilon$ and $\sum_{i \in I} \Psi_i(f_i(x), x_i; \tilde{y}_i) < \frac{\gamma}{2}$ for any $x \in K, \|x\| > \varepsilon$. Now, for each $i \in I$, let $K_i^\varepsilon = \{x_i \in K_i : \|x_i\|_i \leq \varepsilon\}$ where $\|\cdot\|_i$ is the norm on X_i. If $K^\varepsilon = \prod_{i \in I} K_i^\varepsilon$ then K^ε is a nonempty weakly compact subset of K. Moreover, we note that, for any $x \in K \setminus K^\varepsilon$, we have $\sum_{i \in I} \Psi_i(f_i(x), x_i; \tilde{y}_i) < \frac{\gamma}{2} < 0$. Thus (i3) of Theorem 3.1 is satisfied with $D = K^\varepsilon$, and the conclusion follows immediately.

Remark 3.1. According to Section 2, if for each $i \in I$, $\Psi_i(f_i(x), x_i; x_i) = 0$, then problems ($\Psi$-VIPPS) and ($\Psi$-SVIP) are equivalent. Hence the existence of solutions for the (Ψ-SVIP) problem is proved under the assumptions of Theorems 3.1 or 3.2.

4. EXISTENCE RESULTS FOR (Ψ-MVIPPS) AND (Ψ-SMVIP)

In this section we consider the case of multivalued maps, i. e., the case of problems (Ψ-MVIPPS) and (Ψ-SMVIP) defined in Section 2. We use the technique of Yang and Yao [18] to derive the existence results.

Definition 4.1. *We say that* F *is relatively pseudomonotone, respectively, relatively demimonotone with respect to* Ψ *if for each* $x \in K$ *and every net* $\{x^\alpha\}_{\alpha \in \Gamma}$ *in* K *converging to* x (*respectively, converging weakly to* x) *such that*

$$\liminf_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i(u_i^\alpha, x_i^\alpha; x_i) \geq 0, \text{ for all } u_i^\alpha \in F_i(x^\alpha),$$

for all $u_i \in F_i(x)$ *we then have*

$$\sum_{i\in I}\Psi_i(u_i,x_i;y_i)\geq \limsup_{\alpha\in\Gamma}\sum_{i\in I}\Psi_i(u_i^\alpha,x_i^\alpha;y_i) \text{ for all } u_i^\alpha\in F_i(x^\alpha) \text{ and } y\in K.$$

We derive important results related to the above problems using continuous selections of multivalued maps [17]. Thus, let Z be a topological vector space, U a subset of Z, $G:U\to 2^{Z^*}$ a multivalued map and $g:U\to Z^*$ a single valued map. Then g is said to be a selection of G on U if $g(x)\in G(x)$ for all $x\in U$. The function g is said to be a continuous selection of G on U if it is continuous on U and a selection of G on U. In our case, if $Y=\prod_{i\in I}Y_i$, $Y_i=X_i^*$ for any $i\in I$, then $f:K\to X^*=\prod_{i\in I}X_i^*$ is a selection of $F:K\to 2^{X^*}$ if and only if, for each $i\in I$, $f_i:K\to X_i^*$ is a selection of $F_i:K\to 2^{X_i^*}$ on K.

Lemma 4.1. *If f is a selection of F, then every solution of* (Ψ-VIPPS) *is a solution of* (Ψ-MVIPPS).

Proof:

Let $\bar{x}\in K$ be a solution of (Ψ-VIPPS). Then $\sum_{i\in I}\Psi_i(f_i(\bar{x}),\bar{x}_i;y_i)\geq 0$ for all $y_i\in K_i$, $i\in I$. We take $\bar{u}_i=f_i(\bar{x}), i\in I$. Thus $\bar{u}=f(\bar{x})$. Now, since f is a selection of F, we have $\bar{u}\in F(\bar{x})$. Hence the last inequality is equivalent to $\sum_{i\in I}\Psi_i(\bar{u}_i,\bar{x}_i;y_i)\geq 0$ for all $y_i\in K_i$, $i\in I$, i.e., $(\bar{x},\bar{u})$ is a solution of (Ψ-MVIPPS).

By Definitions 3.1 and 4.1 and the definition of selection for a multivalued map, we have the following result.

Lemma 4.2. *Suppose that f is a selection of F on K and F is relatively* B-*pseudomonotone* (*respectively*, *relatively demimonotone*) *with respect to Ψ. Then f is also relatively* B-*pseudomonotone* (*respectively*, *relatively demimonotone*) *with respect to Ψ.*

Theorem 4.1. *Assume that*

i1) *for each $i\in I$, K_i is a nonempty convex subset of X_i;*

i2) *F is relatively* B-p*seudomonotone with respect to Ψ;*

i3) *f is a continuous selection of F on K;*

i4) *there exist a nonempty closed compact subset D of K and an element $\tilde{y}\in D$ such that $\sum_{i\in I}\Psi_i(f_i(x),x_i;\tilde{y}_i)<0$ for all $x\in K\setminus D$;*

i5) *the map $x\to\sum_{i\in I}\Psi_i(f_i(x),x_i;y_i)$ is continuous on K for any $y\in K$. Then* (Ψ-MVIPPS) *has a solution.*

Proof: Using Lemma 4.2 we see that under our assumptions we can apply Theorem 3.1 for the continuous selection f of F on K. Hence there exists $\bar{x}\in K$ a solution of (Ψ-VIPPS).

Now, for each $i\in I$ let $\bar{u}_i=f_i(\bar{x})\in F_i(\bar{x})$. Then, by Lemma 4.1, $(\bar{x},\bar{u})$ is a solution of (Ψ-MVIPPS) and the theorem is proved.

Now, from Theorems 3.2 and 4.1, we obtain the next result.

Corollary 4.1. *Assume that*

j1) *for each $i\in I$, K_i is a nonempty convex subset of a real reflexive Banach space X_i;*

j2) *F is a relatively demimonotone multivalued map with respect to Ψ;*

j3) *there exists $\tilde{y}\in K$ such that $\lim_{\substack{\|x\|\to\infty\\ x\in K}}\sum_{i\in I}\Psi_i(f_i(x),x_i;\tilde{y}_i)<0$. Then* ($\Psi$-MVIPPS) *has a solution.*

5. SOME SPECIAL CASES

In this section we consider a special form for $\Psi_i, i \in I$. Let $Y_i = X_i^*$ be the dual of $X_i, i \in I$ and $\langle \cdot,\cdot \rangle$ the pairing between X_i^* and X_i. For each $i \in I$, $f_i : K \to X_i^*$ is a nonlinear map. If $\Psi_i(f_i(x), x_i; y_i) = \langle f_i(x), y_i - x_i \rangle$ for $i \in I$, then (Ψ-VIPPS) reduces to the (VIPPS) problem, a variational inequality problem over product sets, considered, for example, by Konnov [9] and Ansari and Khan [3]: find $\bar{x} \in X$ such that $\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0$ for all $y_i \in K_i$, $i \in I$. In this case, Konnov [9] proved some existence results for a solution of (VIPPS) under relatively pseudomonotonicity or strongly relative pseudomonotonicity assumptions in the setting of Banach space.

Also, we note that problem (Ψ-SVIP) reduces to the (SVIP) problem, a system of variational inequalities: find $\bar{x} \in X$ such that $\langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0$ for all $y_i \in K_i$ and $i \in I$.

In this case, for some applications and further results see for example [1,3,8,11,12,15].

By using Theorems 3.1 and 3.2 we obtain

Corollary 5.1 ([3]). *For each $i \in I$, let K_i be a nonempty convex subset of a real topological vector space (not necessarily Hausdorff) X_i. Let $f = (f_i)_{i \in I}$ be relatively* B-*pseudomonotone* [3, *Definition* 3.1] *such that for each $A \in \Im(K)$, $x \to \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on* co *A. Assume that there exist a nonempty closed compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0.$ Then* (VIPPS) *has a solution.*

Corollary 5.2 ([3]). *For each $i \in I$, let K_i a nonempty closed convex subset of a real reflexive Banach space X_i. Let $f = (f_i)_{i \in I}$ be relatively demimonotone* [3] *such that, for each $A \in \Im(K)$, $x \to \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on* co *A. Assume that there exists $\tilde{y} \in K$ such that $\lim_{\substack{\|x\| \to \infty \\ x \in K}} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$. Then* (VIPPS) *has a solution.*

We note that in [3] as an application of Corollary 5.2 the existence of a coincidence point for two families of nonlinear operators is established. In what follows we will present these results.

Corollary 5.3 ([3]). *For each $i \in I$, let X_i be a real reflexive Banach space and $f_i, g_i : K \to X_i^*$ where $g_i : K \to X_i^*$ is a nonlinear operator. Assume that $f - g$, where $f = (f_i)_{i \in I}$ and $g = (g_i)_{i \in I}$, is relatively demimonotone and, for each $A \in \Im(X)$, $x \to \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on* co *A. Further, assume that there exists $\tilde{y} \in X$ such that $\lim_{\substack{\|x\| \to \infty \\ x \in K}} \sum_{i \in I} \langle (f_i - g_i)(x), \tilde{y}_i - x_i \rangle < 0$. Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.*

Corollary 5.4 ([3]). *For $i \in I$, let $(X_i, \langle \cdot,\cdot \rangle)$ be a real Hilbert space and K_i a nonempty closed convex subset of X_i. Assume that f is relatively demimonotone with respect to Ψ there exists $\tilde{y} \in K$ with*

$\lim\limits_{\substack{\|x\|\to\infty \\ x\in K}} \sum_{i\in I}\langle x_i - f_i(x), \tilde{y}_i - x_i\rangle < 0$, *and that for each* $A\in \Im(K)$ *the mapping* $x \to \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ *is lower semicontinuous on* co *A. Then there exists* $\bar{x}\in K$ *such that* $f_i(\bar{x}) = \bar{x}_i$ *for any* $i\in I$.

Concerning the multivalued case, we see that if for each $i\in I$ we take $\Psi_i(u, x_i, y_i) = \langle u_i, y_i - x_i\rangle$, where $u\in F(x)$, $y_i\in K_i$, $x\in K$, problems (Ψ-MVIPPS) and (Ψ-SMVIP) reduce to problems (GVIPPS) and (SGVIP), respectively, considered by Ansari and Khan [3]. Thus, in this case, the results stated in [3] are consequences of our results.

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Received: January 16, 2007