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ESTIMATING PROB(Y<X) FOR THE LAPLACE ASYMMETRIC DISTRIBUTION

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We consider the problem of evaluating the probability $\text{Prob}\{Y < X\}$, where X and Y are two independent random variables having Laplace asymmetric distributions. We obtain a maximum likelihood estimator $\hat{R}_n$, an estimator with the method of moments $\tilde{R}_n$, and a non-parametric estimator $\overline{R}_n$ for the quantity $R = \text{Prob}\{Y < X\}$. We compare the performances of these three estimators using Monte Carlo techniques.

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1. INTRODUCTION

The problem of estimating $R = \text{Prob}(Y < X)$ where X and Y are independent random variables has been studied for the exponential distribution [3], for the double exponential distribution [2], for the Luceño distribution [5], for the power distribution [8](see also [10] for other distributions).

We shall consider the Laplace asymmetric distribution $LA(\theta;k;\sigma)$ with parameters $k,\sigma > 0$, $\theta \in \text{IR}$, that is the distribution on IR with the probability density function:

$$\rho(x;\theta;k,\sigma) = \frac{\sqrt{2}}{\sigma}\frac{k}{1+k^2}\begin{cases} e^{\frac{\sqrt{2}}{k\sigma}(x-\theta)}, & x < \theta \\ e^{-\frac{\sqrt{2}}{\sigma}k(x-\theta)}, & x \geq \theta \end{cases} \tag{1}$$

The purpose of this paper is to estimate $Prob\{Y<X\}$ where X and Y are real valued independent random variables, $X \sim LA(\theta_1;1;\sigma_1)$ and $Y \sim LA(\theta_2;1;\sigma_2)$. In real life both variables are nonnegative, but for theoretical reasons we will consider them over the entire IR. We shall think of X as the strength of a mechanical system being subjected to a stress Y, which leads us to a stress-strength model. The quantity we are estimating is the static reliability of a mechanical system. Also, for the numerical simulation we shall consider the parameters σ_1,σ_2 known, and θ_1,θ_2 unknown.

2. EVALUATION OF THE RELIABILITY MEASURE *R*

Since we have considered $k = 1$, from (1) we have that the density function for $LA(\theta;1;\sigma)$ is

$$\rho(x;\theta,\sigma)=\frac{1}{\sigma\sqrt{2}}\begin{cases} e^{\frac{\sqrt{2}}{\sigma}(x-\theta)}, & x<\theta \\ e^{-\frac{\sqrt{2}}{\sigma}(x-\theta)}, & x\geq\theta \end{cases} \tag{2}$$

Now, from (2) we deduce that the cummulative distribution function for $LA(\theta;1;\sigma)$ is:

$$F_{\theta,\sigma}(x)=\begin{cases} \frac{1}{2}e^{\frac{\sqrt{2}}{\sigma}(x-\theta)}, \ x\leq\theta \\ 1-\frac{1}{2}e^{-\frac{\sqrt{2}}{\sigma}(x-\theta)}, \ x>\theta \end{cases} \tag{3}$$

By (2) and (3) and the two equations

$$R=P(Y<X)=E(P(Y<X\mid X))=\int_{-\infty}^{\infty}P(Y<X\mid X=x)\rho(x;\theta_1;1;\sigma_1)dx \tag{4}$$

$$P(Y<X\mid X)=F_{\theta_2,\sigma_2}(X) \tag{5}$$

we have the following cases:

1. $\theta_1<\theta_2$. In this case,

$$R=\frac{1}{2\sigma_1\sqrt{2}}\int_{-\infty}^{\theta_1}e^{\left(\frac{\sqrt{2}}{\sigma_2}+\frac{\sqrt{2}}{\sigma_1}\right)x-\frac{\sqrt{2}}{\sigma_2}\theta_2-\frac{\sqrt{2}}{\sigma_1}\theta_1}dx+\frac{1}{2\sigma_1\sqrt{2}}\int_{\theta_1}^{\theta_2}e^{\left(\frac{\sqrt{2}}{\sigma_2}-\frac{\sqrt{2}}{\sigma_1}\right)x-\frac{\sqrt{2}}{\sigma_2}\theta_2+\frac{\sqrt{2}}{\sigma_1}\theta_1}dx+\frac{1}{2\sigma_1}\int_{\theta_2}^{\infty}e^{-\frac{\sqrt{2}}{\sigma_1}x+\frac{\sqrt{2}}{\sigma_1}\theta_1}dx-$$
$$-\frac{1}{2\sigma_1\sqrt{2}}\int_{\theta_2}^{\infty}e^{-\left(\frac{\sqrt{2}}{\sigma_2}+\frac{\sqrt{2}}{\sigma_1}\right)x+\frac{\sqrt{2}}{\sigma_2}\theta_2+\frac{\sqrt{2}}{\sigma_1}\theta_1}dx=\frac{1}{4}\frac{\sigma_2}{\sigma_1+\sigma_2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_1-\theta_2)}+\frac{1}{4}\frac{\sigma_2}{\sigma_1-\sigma_2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_1-\theta_2)}-\frac{1}{4}\frac{\sigma_2}{\sigma_1-\sigma_2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_1-\theta_2)}-$$
$$-\frac{1}{4}\frac{\sigma_2}{\sigma_1+\sigma_2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_1-\theta_2)}+\frac{1}{2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_1-\theta_2)},$$

or, finally,

$$R=\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_1-\theta_2)}-\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma 2}(\theta_1-\theta_2)}. \tag{6}$$

2. $\theta_1\geq\theta_2$. In this case:

$$R=\frac{1}{2\sigma_1\sqrt{2}}\int_{-\infty}^{\theta_2}e^{\left(\frac{\sqrt{2}}{\sigma_2}+\frac{\sqrt{2}}{\sigma_1}\right)x-\frac{\sqrt{2}}{\sigma_2}\theta_2-\frac{\sqrt{2}}{\sigma_1}\theta_1}dx+\frac{1}{2\sigma_1}\int_{\theta_2}^{\theta_1}e^{\frac{\sqrt{2}}{\sigma_1}x-\frac{\sqrt{2}}{\sigma_1}\theta_1}dx-\frac{1}{2\sigma_1\sqrt{2}}\int_{\theta_2}^{\theta_1}e^{\left(\frac{\sqrt{2}}{\sigma_1}-\frac{\sqrt{2}}{\sigma_2}\right)x-\frac{\sqrt{2}}{\sigma_1}\theta_1+\frac{\sqrt{2}}{\sigma_2}\theta_2}dx+$$
$$+\frac{1}{2\sigma_1}\int_{\theta_1}^{\infty}e^{-\frac{\sqrt{2}}{\sigma_1}x+\frac{\sqrt{2}}{\sigma_1}\theta_1}dx-\int_{\theta_1}^{\infty}e^{-\left(\frac{\sqrt{2}}{\sigma_1}+\frac{\sqrt{2}}{\sigma_2}\right)x+\frac{\sqrt{2}}{\sigma_1}\theta_1+\frac{\sqrt{2}}{\sigma_2}\theta_2}dx=\frac{1}{4}\frac{\sigma_2}{\sigma_1+\sigma_2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_2-\theta_1)}+\frac{1}{2}-\frac{1}{2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_2-\theta_1)}- \tag{6}$$
$$-\frac{1}{4}\frac{\sigma_2}{\sigma_2-\sigma_1}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_2-\theta_1)}+\frac{1}{4}\frac{\sigma_2}{\sigma_2-\sigma_1}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_2-\theta_1)}+\frac{1}{2}-\frac{1}{4}\frac{\sigma_2}{\sigma_1+\sigma_2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_2-\theta_1)},$$

or, finally,

$$R=1-\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_2-\theta_1)}+\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_2-\theta_1)}. \tag{7}$$

Using (6) and (7), we can write the general expression for R :

$$R(\theta_1,\theta_2,\sigma_1,\sigma_2)=P(Y<X)=\begin{cases}\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_1-\theta_2)}-\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_1-\theta_2)}, & \theta_1<\theta_2\\ 1-\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_1}(\theta_2-\theta_1)}+\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_2}(\theta_2-\theta_1)}, & \theta_1\geq\theta_2\end{cases} \tag{8}$$

Denoting $\theta_1-\theta_2\overset{not}{=}\delta$, we obtain:

$$R(\delta,\sigma_1,\sigma_2)=\begin{cases}\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_1}\delta}-\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{\frac{\sqrt{2}}{\sigma_2}\delta}, & \delta<0\\ 1-\frac{1}{2}\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}e^{-\frac{\sqrt{2}}{\sigma_1}\delta}+\frac{1}{2}\frac{\sigma_2^2}{\sigma_1^2-\sigma_2^2}e^{-\frac{\sqrt{2}}{\sigma_2}\delta}, & \delta\geq 0\end{cases} \tag{9}$$

Let us observe that we have earlier assumed that $\sigma_1\neq\sigma_2$, therefore the (9) only holds for this case. For the case where $\sigma_1=\sigma_2=\sigma$, repeating steps 1. and 2. before, we get:

$$R(\delta,\sigma,\sigma)=\begin{cases}\frac{1}{2}e^{\frac{\sqrt{2}}{\sigma}\delta}\left(1-\frac{\delta}{\sigma\sqrt{2}}\right), & \delta<0\\ 1-\frac{1}{2}e^{-\frac{\sqrt{2}}{\sigma}\delta}\left(1+\frac{\delta}{\sigma\sqrt{2}}\right), & \delta\geq 0\end{cases} \tag{10}$$

In both situations ($\sigma_1=\sigma_2$ or $\sigma_1\neq\sigma_2$) the graph of R has the following appearance:

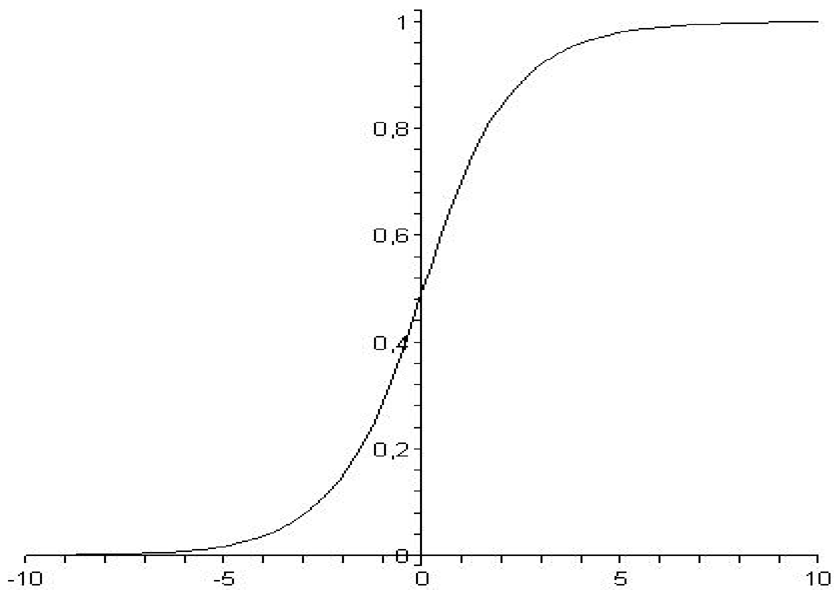

Figure 1: The graph of the reliability R

The function R is continuous and differentiable on IR, indefinitely differentiable on IR\{0}, and:

$$R(\delta,\sigma_1,\sigma_2)=1-R(-\delta,\sigma_1,\sigma_2),\ \forall\delta\in IR,\sigma_1,\sigma_2>0$$

Let us notice that, as for the normal distribution, when σ decreases, the reliability increases, therefore the resistence to a certain stress is larger if the variance of the variable *X* is smaller.

In what follows we will consider that σ_1 and σ_2 are known.

3. ESTIMATION OF *R*=Prob{*Y*<*X*}

Let $Z\sim LA(\theta;1;\sigma)$ be a random variable and $(z_k)_{1\le k\le n}$ a random sample from Z. We then have

$$E(Z)=\frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\theta}x\cdot e^{\frac{\sqrt{2}}{\sigma}(x-\theta)}dx+\frac{1}{\sigma\sqrt{2}}\int_{\theta}^{\infty}x\cdot e^{-\frac{\sqrt{2}}{\sigma}(x-\theta)}dx=I_1+I_2 \tag{11}$$

With the change of variable

$$\frac{\sqrt{2}}{\sigma}(x-\theta)=y,$$

we have

$$I_1=\frac{1}{2}\int_{-\infty}^{0}\left(\frac{\sigma}{\sqrt{2}}y+\theta\right)e^{y}dy=\frac{\theta}{2}-\frac{\sigma}{2\sqrt{2}},I_2=\frac{1}{2}\int_{0}^{\infty}\left(\frac{\sigma}{\sqrt{2}}y+\theta\right)e^{-y}dy=\frac{\sigma}{2\sqrt{2}}+\frac{\theta}{2}$$

and therefore, by (11) we have $E(Z)=\theta$. So, if we know the value of σ, we can apply the method of moments in order to find an estimator for θ, namely

$$\overline{\theta}_n=\frac{1}{n}\sum_{k=1}^{n}z_k=\overline{z}_n \tag{12}$$

Another estimator for θ, when σ is known, is the maximum likelihood estimator (see [9] for details and estimation in more general cases) given by

$$\hat{\theta}_n=z_{j(n):n} \tag{13}$$

where $\mathrm{j(n)}=\left[\frac{\mathrm{n}}{2}\right]+1$, $(x_{j:n})_{1\le j\le n}$ stands for the ordered sample corresponding to the initial sample, and $[x]$ is the integer part of *x*.

Based on the estimators (12) and (13), in the case where we have two samples from the random variables $X\sim LA(\theta_1;1;\sigma_1)$ and $Y\sim LA(\theta_2;1;\sigma_2)$, namely, $(x_k)_{1\le k\le n}$ and $(y_l)_{1\le l\le n}$, and we know the values of σ_1 and σ_2, we find the following parametric estimators of *R*:

$$\overline{R}_n=R(\overline{\theta}_{1,n}-\overline{\theta}_{2,n},\sigma_1,\sigma_2) \tag{14}$$

$$\hat{R}_n=R(\hat{\theta}_{1,n}-\hat{\theta}_{2,n},\sigma_1,\sigma_2) \tag{15}$$

(we note that we have considered the same sample size for both samples). In (14) and (15) $\overline{\theta}_{i,n}$ and $\hat{\theta}_{i,n}$ are the estimators for θ_i by the method of moments and by the method of maximum likelihood, while *R* is obtained from (9) or (10), depending on the relation between σ_1 and σ_2.

4. SIMULATION STUDY

In this section, besides the parametric estimators $\overline{R}_n$ and $\hat{R}_n$, defined in the preceding section, we will consider a non-parametric estimator (see [5],[8]), defined as:

$$\widetilde{R}_n = \frac{card\{(X_i, Y_j) \mid Y_j < X_i,\ 1 \le i, j \le n\}}{n^2} \tag{16}$$

We will study the behaviour of the three estimators previously defined from the perspective of the quantities:

$$MB(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}\left(R(\delta;\sigma_1,\sigma_2) - \mathbf{R}_i\right)$$

$$MSE(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}\left(R(\delta;\sigma_1,\sigma_2) - \mathbf{R}_i\right)^2$$

where $\mathbf{R}_i$ is an estimator for the reliability *R*, depending on the sample (experiment) *i* and N is the number of experiments (samples of same size *n*), in our case estimating R. The first quantity is the *mean bias* and the second is the *mean square error*.

For the numerical simulation we have considered for the parameters involved the values: $n = 100,\ N = 1000, (\theta_1, \theta_2) \in \{(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)\},\ (\sigma_1, \sigma_2) \in \{(1,2),(2,1),(2,2)\}$.

Remarks concerning the mean bias (**MB**):

- For the estimator $\overline{R}_n$, generally, in each of the cases $\sigma_1 > \sigma_2$, $\sigma_1 < \sigma_2$ or $\sigma_1 = \sigma_2$, we noticed that $MB(\overline{R}_n) > 0$ if $\delta < 0$ and $MB(\overline{R}_n) < 0$ if $\delta > 0$. If $\delta = 0$ there is no clear tendency (see Figure 2).
- For the non-parametric estimator $\widetilde{R}_n$, there is no apparent tendency for the MB, regardless of the relationship between σ_1 and σ_2.
- For the maximum likelihood estimator $\hat{R}_n$, if $\sigma_1 > \sigma_2$ then almost always $MB(\hat{R}_n) < 0$ and if $\sigma_1 < \sigma_2$ then $MB(\hat{R}_n) > 0$. Otherwise, if $\sigma_1 = \sigma_2$ the behaviour is the same as for the method of moments estimator (see Figure 3).

Remarks concerning the mean square error (**MSE**).

- For the estimator $\overline{R}_n$, there are no significant differences between the cases $\sigma_1 > \sigma_2$ and $\sigma_1 < \sigma_2$. If $\sigma_1 = \sigma_2$, MSE increases as θ_1 moves away from θ_2 (see Figure 4).
- There are no major differences between the three estimators if we only consider the variation of MSE when the relation between σ_1 and σ_2 changes.

Comparing the three estimators.

- In the three significant cases for the parameters σ_1 and σ_2, all three estimators appear to behave similarly.
- Regardless of the difference $\delta = \theta_1 - \theta_2$, we have noticed that the maximum likelihood estimator $\hat{R}_n$ has the smallest values of the MSE (see Figure 5).
- The estimator for which we have obtained the biggest MSE values is the estimator $\overline{R}_n$ if $\delta = 0$, respectively $\widetilde{R}_n$ if $\delta \ne 0$.

Final remarks.

1. The best estimator from the approximation point of view is $\hat{R}_n$, regardless of the choices made for the parameters.

2. From the computational point of view, the most effective is $\overline{R}_n$, since it requires only two means to be evaluated, as opposed to the maximum likelihood estimator, where we need to partially order the sample for every estimation of the parameters.

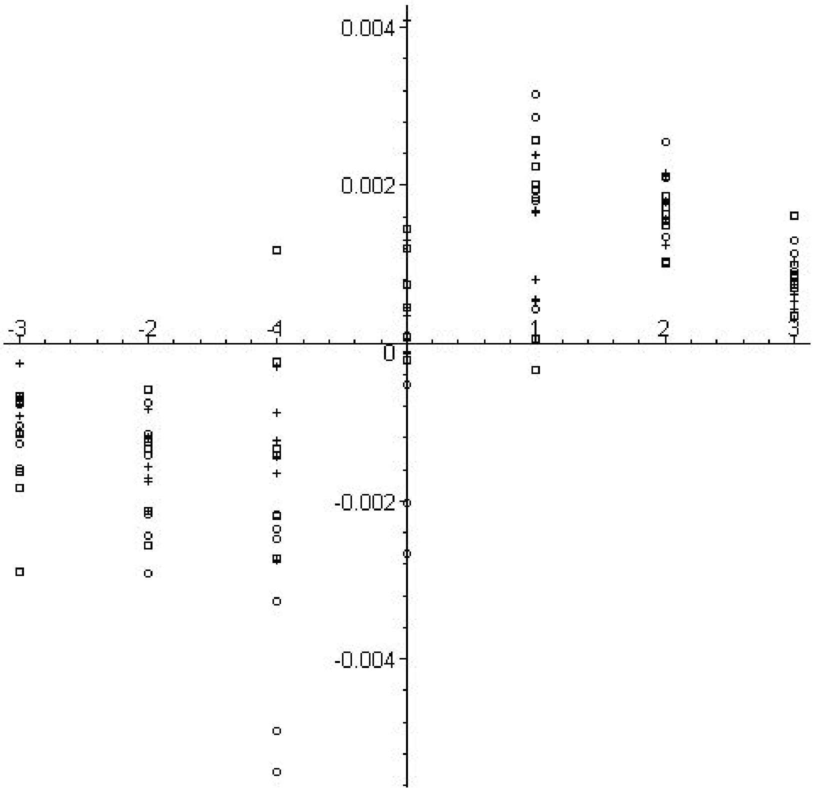

Figure 2: MB for the estimator $\overline{R}_n$:
$+$ - $\sigma_1 > \sigma_2$, ○ - $\sigma_1 < \sigma_2$, □ - $\sigma_1 = \sigma_2$.

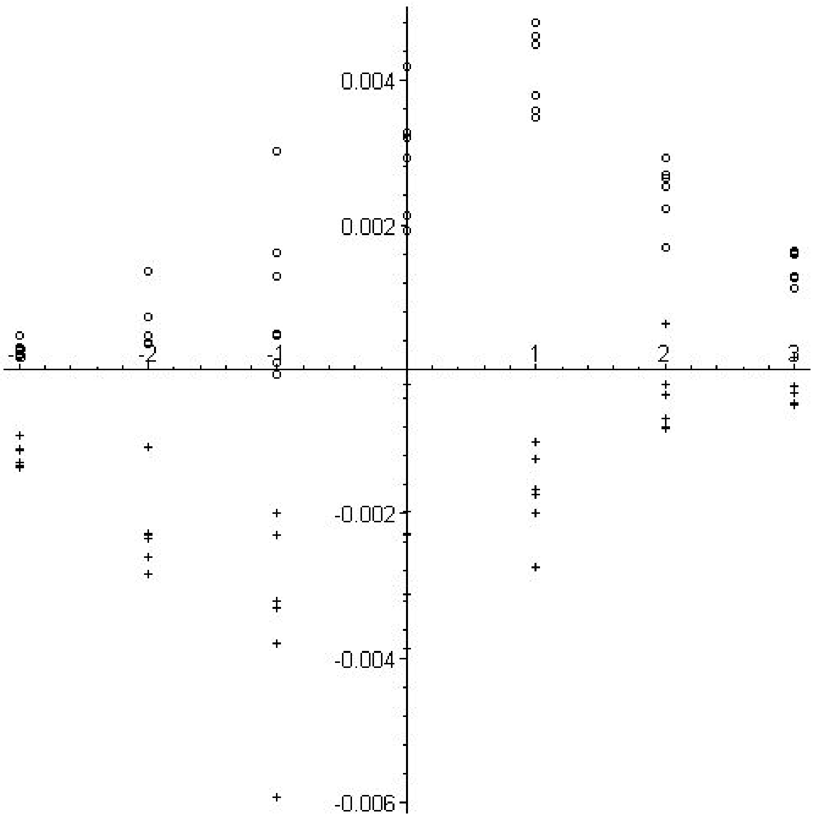

Figure 3: MB for the estimator $\hat{R}_n$:
$+$ - $\sigma_1 > \sigma_2$, ○ - $\sigma_1 < \sigma_2$.

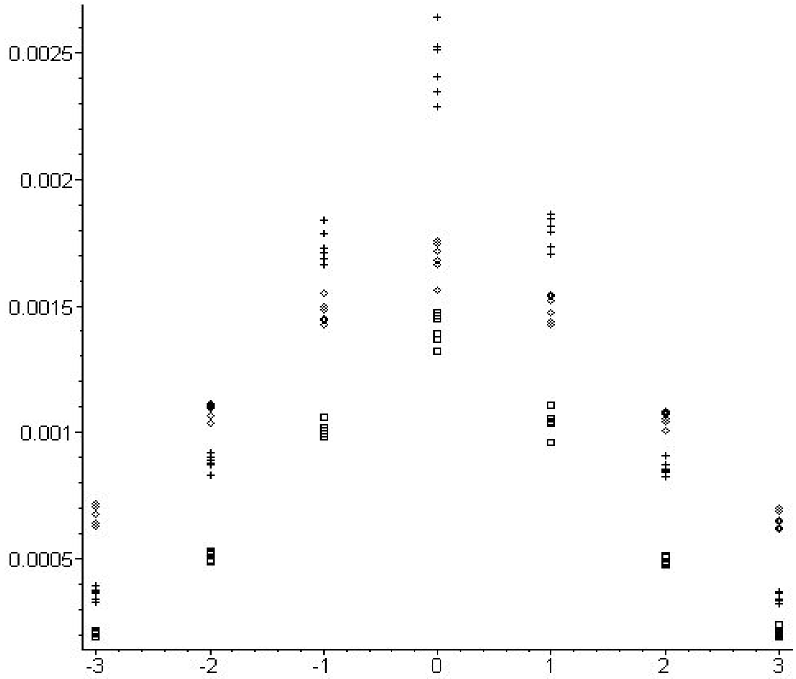

Figure 4: MSE for the estimator $\overline{R}_n$: + - $\sigma_1 > \sigma_2$, ○ - $\sigma_1 < \sigma_2$.

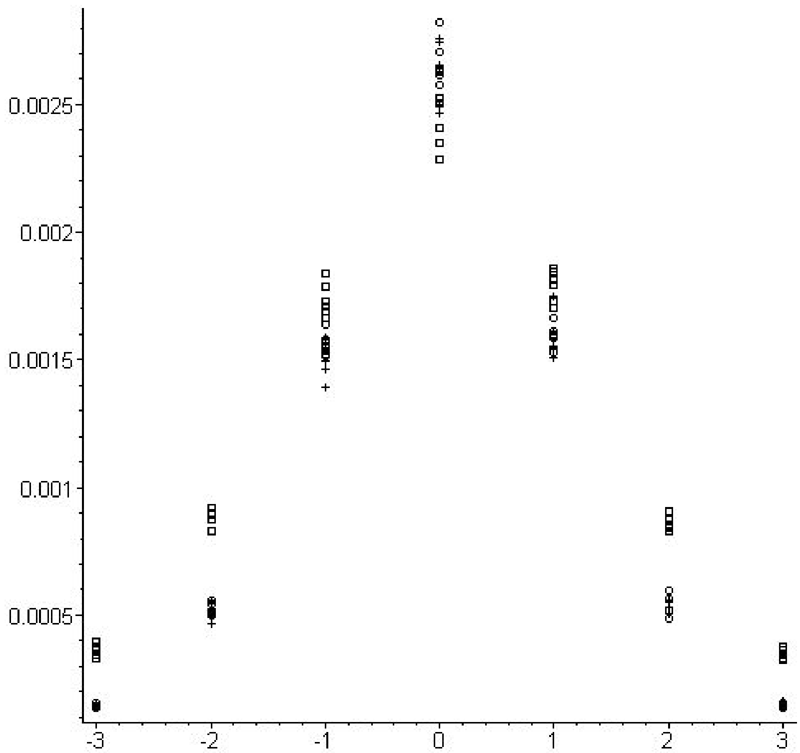

Figure 5: MSE for the three estimators, $\sigma_1 = \sigma_2$: + - $\overline{R}_n$, ◊ - $\widetilde{R}_n$, □ - $\hat{R}_n$.

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