NOTES ON THE ISOMORPHIC MODULAR GROUP ALGEBRAS
OF P-SPLITTING AND P-MIXED ABELIAN GROUPS

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Throughout the present paper, let $KG$ be the group algebra of a multiplicative abelian group $G$ with $p$-primary component $G_p$ over a field $K$ of characteristic $p \neq 0$. Denote by $S(KG)$ the $p$-component of torsion of the group $V(KG)$ of all normalized (i.e. of augmentation 1) units. In [3] and, more generally, in [4] we proved the following.

**Theorem A.** If $G$ is $p$-splitting, that is, $G_p$ is a direct factor of $G$, $G_p$ is simply presented and $K$ is perfect, then $S(KG)/G_p$ is simply presented. Moreover, $S(KG)$ is simply presented with a direct factor $G_p$. In particular, if $G$ is in addition $p$-mixed, that is, the only torsion is $p$-torsion, $G$ is a direct factor of $V(KG)$ with simply presented complement.

With the aid of this statement, the following isomorphism assertion was also proved there.

**Theorem B.** If $G$ is $p$-splitting whose $G_p$ is simply presented and $KH \cong KG$ as $K$-algebras for some other group $H$, then $H_p \cong G_p$. In particular, if in addition $G$ is $p$-mixed, $H \cong G$.

Likewise, in [2] we established the following affirmation.

**Theorem C.** Suppose that $G$ is a coproduct of countable abelian groups and $K$ is perfect. Then $S(KG)/G_p$ is a coproduct of countable abelian groups and $G_p$ is a direct factor of $S(KG)$. Thus $S(KG)$ is a coproduct of countable abelian groups. Moreover, if $KH \cong KG$ as $K$-algebras for some group $H$, then $H_p \cong G_p$. In particular, if $G$ is in addition $p$-mixed, $G$ is a direct factor of $V(KG)$ with the same complementary factor. Thus $V(KG)$ is a coproduct of countable abelian groups. Moreover, if $KH \cong KG$ as $K$-algebras for some group $H$, there exists a coproduct of countable abelian $p$-groups $T$ such that $H \times T \cong G \times T$. Thus $H$ is a coproduct of $p$-mixed countable abelian groups.

With this at hand, we obtained in [5] the following isomorphism claim.

**Theorem D.** Let $G$ be a $p$-mixed coproduct with finite torsion-free rank of countable abelian groups and $KH \cong KG$ as $K$-algebras for another group $H$. Then $H \cong G$.

Our aim in this article is to give an independent smooth confirmation of both Theorems B and D by using the following simple but significant technicality of [6].

**Proposition E.** $KG \cong KH$ as $K$-algebras implies that $K(S(KG)/G_p) \cong K(S(KH)/H_p)$ as $K$-algebras.
And so, we are now prepared to give the desired proofs, which are, indeed, more direct than the existing ones and give another strategy in the advantage of greater clarity.

**Proof of Theorem B.** Without loss of generality we may assume that $KG = KH$ because $KG \cong KH$ yields that $KG = KH_j$ for some $H_j \cong H$. According to Theorem A we have that $S(KG)/G_p$ is simply presented, hence Proposition E combined with [9] lead us to $S(KH)/H_p$ is simply presented. Since $H_p$ is balanced in $S(KH)$, it is its direct factor with simply presented complement. But by Theorem A we have $S(KG) = S(KH)$ is simply presented, whence so is $H_p$ as a direct factor. Furthermore, [8] applies to get that the maximal divisible subgroups of $G_p$ and $H_p$ are respectively isomorphic as well as the Ulm-Kaplansky invariants of $G_p$ and $H_p$ are respectively equal. Consequently, $G_p$ and $H_p$ are isomorphic.

Next, if $G$ is $p$-mixed, $KG = KH$ obviously implies that $H$ is $p$-mixed. Besides, $G = G_p \times M$ for some subgroup $M$ secures that $V(FG) = S(FG) \times V(FM)$ and hence $V(FH) = S(FH) \times V(FM)$. Since by what we have shown above $H_p$ is a direct factor of $S(KH)$, it is readily seen that $H_p$ is a direct factor of $V(KH)$, thus of $H$ (cf. [1] and [4] too). Finally, $G \cong G_p \times (G/G_p)$ and $H \cong H_p \times (H/H_p)$. Employing [8], we deduce that $G/G_p \cong H/H_p$. That is why, $G \cong H$ and we are done.

As usual, we denote by $I(KG; G_p)$ the relative augmentation ideal of $KG$ with respect to $G_p$. If the isomorphism implication $G_p \cong H_p \Rightarrow (1+I(KG; G_p))/G_p \cong (1+I(KG; H_p))/H_p$ holds true, another idea for a proof without Proposition E may be like this: Owing to Theorem A, we find that $S(KG) = S(KH)$ is simply presented. So, we apply [7] to infer that $H_p$ is simply presented. Henceforth, we can copy the idea from the preceding proof to get $G_p \cong H_p$. Furthermore, observing that $S(KG) = 1+I(KG; G_p)$ and $S(KH) = 1+I(KH; H_p)$, where $KG = KH$, by what we have assumed above we derive that $S(KG)/G_p = (1+I(KG; G_p))/G_p \cong (1+I(KG; H_p))/H_p = S(KH)/H_p$ and thus, in virtue of Theorem A, $S(KH)/H_p$ is simply presented. Further, the proof goes on in the foregoing proof.

**Proof of Theorem D.** Invoking to Theorem C, $S(KG)/G_p$ is a direct sum of countable groups. But Proposition E along with [9] allows us to conclude that $S(KH)/H_p$ is also a direct sum of countable groups. The balanced property of $H_p$ in $S(KH)$ insures that $H_p$ is a direct factor of $S(KH)$. But, because $H$ is $p$-mixed as is $G$, it is immediate that $V(KH) = HS(KH)$ and thus $H$ is a direct factor of $V(KH)$. Again Theorem C enables us that both $V(KH)$ and $H$ are direct sums of countable groups. Hereafter, the proof goes on in the same way as in [5].

With the aid of Theorem C the same idea for proof of Theorem D, without Proposition E, may also be demonstrated.

**Remark.** Results (a) and (c) from [10] imitated in a weaker form our theorems in [3] and [4] (compare with the statements presented above) as well as results from [7].

**REFERENCES**


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