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A GENERAL COMPUTER-AIDED METHOD OF OBTAINING THE INFLUENCE COEFFICIENTS MATRIX

Carmen-Ecaterina EISINGER-BORCIA

Institute of Solid Mechanics, Ctin Mille 15, 0101412 Bucharest
E-mail: c_eisinger@yahoo.com

This paper presents a general computer-aided method of obtaining the influence coefficients for any statically determined or undetermined straight beams of a constant cross section, modeled as a p-lumped beam for all the combinations of loading and boundary conditions, unrelated of how big p is. The method can easily be developed into a non-constant cross section. We present the comparisons with known cases in field literature and, for example, we give the influence coefficients matrix for eight grades of freedom. Noting that the direct computation formulas for the influence coefficients are extremely infrequent, the great potential of this Computer-Aided way to acquire the influence coefficients can be understood.

Key words: bending, lumped beam, influence coefficients matrix, computer-aided method.

1. INTRODUCTION

When, in a very large and thorough study about the elasto-dynamic behavior of the real multi-branched mechanisms [1], I wanted to analyze the bending effect of the main camshaft on the entire system, I realized that it is problematic to do this for a large number of branches. Theoretically, a lumped beam model seems to be very suitable, but practically it was limited to a very small number of branches. To study the bending behavior of a lumped beam, the influence coefficients are to be known. So, for a real mechanism with p branches with the main camshaft modeled as a (p+1)-lumped beam under certain loading and boundary conditions, a (p+1) square influence coefficients matrix is to be known. It's well known that the specialty literature [4] offers computing methods for only a very small number, p, of concentrated masses and not for any kind of boundary conditions. Thus, to find a way to compute influence coefficients matrix for a lumped beam with any finite number, p, of concentrated masses and in any boundary conditions, appears to be very challenging.

In this paper, our response to this challenge is presented. We chose to do this by using the method of initial parameters [1, 4].

2. FLEXIBILITY INFLUENCE COEFFICIENTS

In the literature, the concept of "influence coefficients" denotes both the stiffness influence coefficients and the flexibility influence coefficients, which are intimately related - they describe the manner in which the mechanical system deforms under the forces. We will deal only with the flexibility influence coefficients, which will be named, shortly, the influence coefficients. To define the influence coefficients, let us consider a simple discrete system, with no damping, consisting of p masses m_i occupying the position $x_i, i=1,p$ and being in equilibrium. Forces F_j act upon each mass m_i (this can be assumed without loosing generality) so that the masses undergo displacements z_i. Thus, the flexibility influence coefficient e_{ij} is the displacement of the point x_i due to a unit force $F_j=1$ applied at x_j. Note that the flexibility influence coefficients e_{ij} have the appropriate units corresponding to the type of loading: [LF] -for torsion, and [LF-1] -for forces. For a linear system, using the principle of superposition, the flexibility influence coefficients e_{ij} allow to obtain the displacement at the point x_i due to all the forces $F_j, \left(j=1,p \right)$ acting on the system, as:

$$z_\mathrm{i} = F_j e_{ij}, i, j = 1, p. \tag{2.1}$$

In (2.1), Einstein's summation convention is used.

3. EQUATION OF MOTION IN TERMS OF INFLUENCE COEFFICIENTS

To determine the flexibility influence coefficients for a bending beam, based on the method of initial parameters [1], the usual notations from the bending beams theory are used: $T = T(x)$: the shear force; $M = M(x)$: the bending moment; $z = z(x)$: the elastic deflection; $\theta = \theta(x) = \dot{z}(x)$: the slope (of the elastic deflection). Likewise, for a straight beam of constant cross section, the following equations between deflection, shear force, and bending moment hold:

$$f\frac{d^4 z}{dx^4} = \frac{1}{EI} \tag{2.2}$$

$$\frac{d^3 z}{dx^3} = \frac{1}{EI}T \tag{2.3}$$

$$\frac{dM}{dx} = T \tag{2.4}$$

where E is Young's module, I is the moment inertia of the constant cross section, and f is an externally applied load.

The Eq. (2.2) with the appropriate boundary conditions allows to obtain the deflection $z = z(x)$ for any given external loading. The boundary conditions are to be written for each range: between any pair of external concentrated forces/moments and for each portions of the beam on which the distributed forces are applied. The relationships (2.2) and (2.3) serve as continuity conditions.

Looking for the homogenous solution of (2.2) as

$$z(x) = A\frac{x^3}{3!} + B\frac{x^2}{2!} + Cx + D$$

the slope, the bending moment and the shear force are:

$$\theta(x) = A\frac{x^2}{2!} + Bx + C \quad ; \quad M\left(x\right) = -EI\left(Ax + B\right) \quad ; \quad T\left(x\right) = -EI \cdot A$$

Denoting by index 0 the values at the point $x = 0 \Rightarrow z_0 = z\left(0\right)$; $\theta_0 = \theta\left(0\right)$; $M_0 = M\left(0\right)$; $T_0 = T\left(0\right)$ the integration constants become:

$$A = -\frac{1}{EI}T_0;\ B = -\frac{1}{EI}M_0\ ;\ C = \theta_0;\ D = z_0$$

and the homogenous solution of (2.2) is:

$$z(x) = -\frac{1}{EI}T_0\frac{x^3}{3!} - \frac{1}{EI}M_0\frac{x^2}{2!} + \theta_0 x + z_0 \tag{2.5}$$

Since the relationship (2.5) connects the current values of the deflections $z(x)$ to the loading parameters at the origin ($x=0$), the method is known by the name "the initial parameters method".

Now, the particular solutions, typical to each type of external loading, are to be added to the homogenous solution (2.5). Let us consider the beam of long l ($0 \le x \le l$) with typical external loading, as shown in Fig.3.1.

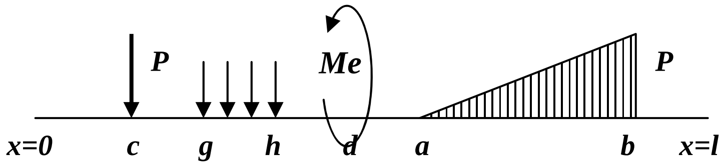

Fig.3.1. Typical external loading

If the concentrated force P is acting in the section $x=c$, the appropriate particular solution is:

$$z_c(x)=0 \qquad \text{if } x<c$$
$$z_c(x)=\frac{1}{EI}P\frac{(x-c)^3}{3!} \qquad \text{if } c \le x \tag{2.6}$$

If the moment Me is acting in the section $x=d$, the appropriate particular solution is:

$$z_c(x)=0 \qquad \text{if } x<d$$
$$z_c(x)=-\frac{1}{EI}Me\frac{(x-d)^2}{2!} \qquad \text{if } d \le x \tag{2.7}$$

(The moments were considered to be positive in a clockwise sense and the external forces in the descendent sense of the vertical axis.)
If the distributed force p is acting on the portion $x \in (g,h)$ of the beam section, the appropriate particular solution is:

$$z_c(x)=0 \qquad \text{if } x<g$$
$$z_c(x)=\frac{1}{EI}p\frac{(x-g)^4}{4!} \qquad \text{if } g \le \mathrm{x} \le \mathrm{h} \tag{2.8}$$
$$z_c(x)=\frac{1}{EI}p\left[\frac{(x-g)^4}{4!}-\frac{(x-h)^4}{4!}\right] \qquad \text{if } h<x$$

If the distributed force with linear variation, *p(x-a)/(x-b),* is acting on the portion $x \in (a,b)$ of the beam section, the appropriate particular solution is:

$$z_c(x)=0 \qquad \text{if } x<a$$
$$z_c(x)=\frac{1}{EI}p\frac{(x-a)^5}{5!} \qquad \text{if } a \le x \le b \tag{2.9}$$
$$z_c(x)=\frac{1}{EI}\frac{p}{(b-a)}\left[\frac{(x-a)^5}{5!}-\frac{(x-b)^5}{5!}-\frac{(x-b)^4}{4!}\right] \qquad \text{if } b<x$$

Let us consider that the beam is loaded by:

(1) p_i distributed forces acting in the section $x \in (a_i, b_i)$ of the beam, (i=1,n_1);

(2) P_j concentrated forces acting in $x=a_j$, (j=1,n_2);

(3) M_k external moments acting in the section $x = a_k$ of the beam (k=1,n$_3$).

Using the superposition principle, the general solution obtained combining the homogenous solution and the three- type of particular solutions (explained before), is:

$$\begin{gathered} z(x) = -\frac{1}{EI}T_0\frac{x^3}{3!} - \frac{1}{EI}M_0\frac{x^2}{2!} + \theta_0 x + z_0 + \frac{1}{EI}\frac{1}{4!}\sum_{i=1}^{n_1} p_i\left(\left\langle x-a_i\right\rangle^4 - \left\langle x-b_i\right\rangle^4\right) - \\ \frac{1}{EI}\frac{1}{3!}\sum_{j=1}^{n_2} P_j\left\langle x-a_j\right\rangle^3 + \frac{1}{EI}\frac{1}{2!}\sum_{k=1}^{n_3} M_k\left\langle x-a_k\right\rangle^2 \end{gathered} \tag{2.10}$$

where $z_0 = z(x_0)$; $\theta_0 = \theta(x_0)$; $M_0 = M(x_0)$; $T_0 = T(x_0)$ are the initial parameters and

$$\left\langle x-\alpha\right\rangle^n = \begin{cases} (x-\alpha)^n & \text{if} \quad x \ge \alpha \\ 0 & \text{if} \quad x < \alpha \end{cases} \tag{2.11}$$

The relationships (2.10) determine the elastic deflection, $z = z(x)$ for a straight beam of constant cross section under the external loading (1), (2) and (3) as function of initial parameters $z_0,\ \theta_0, M_0,\ T_0$.

4. REVERSE-ENGINEERING TASK: COMPUTER-AIDED WAY TO ACQUIRE THE INFLUENCE COEFFICIENTS

Important to note that the formula (2.10) is not dependent on the type of external loading and number and / or kind of boundary conditions or intermediate supports. Also, it's important to note that the method does not require, as a separate step, the static determination of the reactions at the supports. So, it is applicable without any restrictions to the case of a statically undetermined beam.

So, we can conclude that we can find the influence coefficients e_{ij} beside the deflection, the shear force, and the bending moment for a statically determined or undetermined beam, based on the "initial parameters" method.

The routine can be concentrated in the following important procedural steps that are to be followed:

1. Analysis of the loading conditions, in order to establish the boundary conditions at the ends of the beam, as well as at the intermediate supports;
2. Removal of the intermediate supports and their replacement by the intermediate reactions; the intermediate reactions will be regarded as a part of the loading;
3. Computation of the terms appearing necessary in the conditions established for step (1) and writing the conditions. Thus, we obtain a set of algebraic equations in which the unknowns are the values of the initial parameters, $z_0,\ \theta_0, M_0,\ T_0$, and the intermediate reactions;
4. Obtaining the intermediate reactions in terms of initial parameters by solving the equations established in step (3);
5. Determination of the deflection, the shear force, and the bending moment in terms of values of the initial parameters- $z_0,\ \theta_0, M_0,\ T_0$.

Once the elastic deflection ($z = z(x)$) is known, the flexibility influence coefficients e_{ij} will be known, too - by definition, the flexibility influence coefficient e_{ij} is the displacement of the point x_i due to a unit force ($F_j = 1$) applied at point x_j .

It's very important to remark that the method does not require, as a separate phase, the static determination of the reactions at the supports and this is very important in the case of a statically undetermined beam. Thus, the method allows both statically determined and undetermined beams to be treated in the same way.

5. THE ALGORITHM

A general algorithm for the method above was developed and its schematic flow diagram is shown in Fig. 5.1. It could be easily translated into any programming language.

After our analysis, we concluded that all the combinations of boundary conditions can be grouped in five different cases (see, Fig. 5.2) identified by the input data "***tipb***" which must be 1, 2, 3, 4 or 5.

A MATHEMATICA-based application package (see [1], subsection 8.5.2 and Appendix A.ben) to compute the influence coefficients of a p-lumped beam for all the combinations of loading and boundary conditions was written.

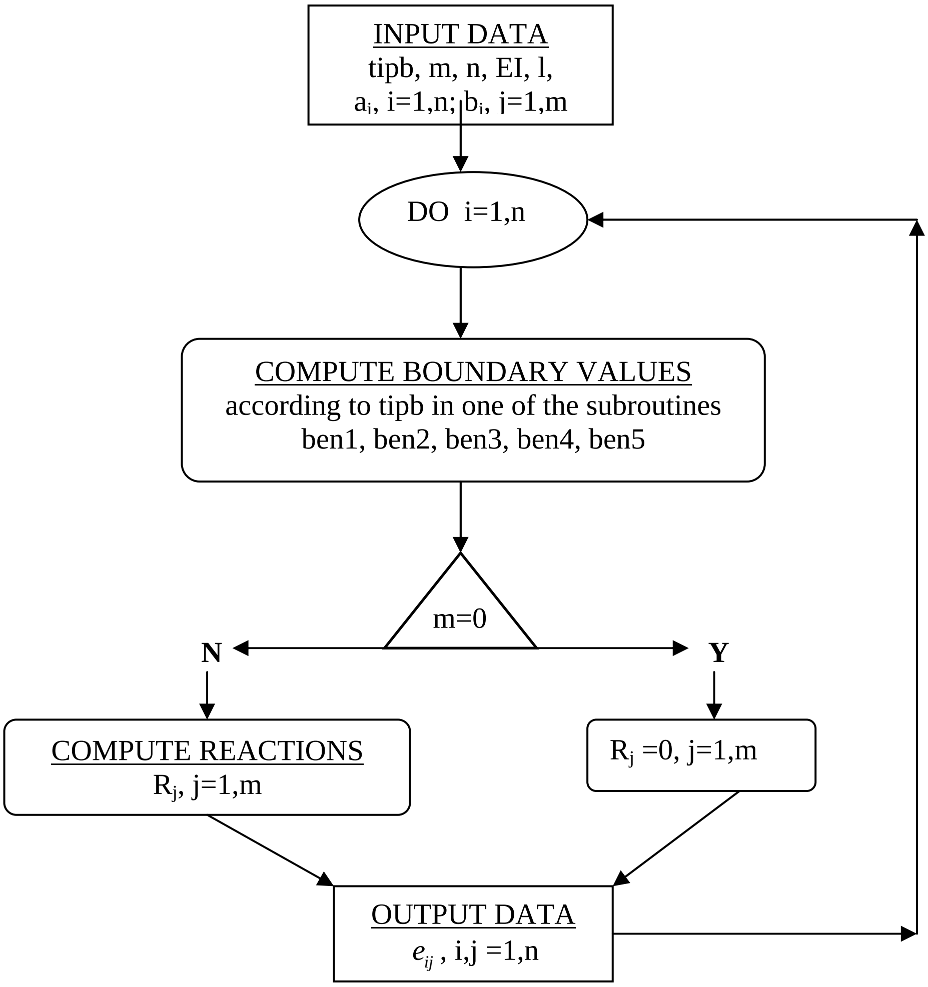

Fig. 5.1 – General flow-diagram for computing the influence coefficients

tipb	***The boundary conditions at point***		
	A	***B***	
1	$x=0, \omega_A=0, \theta_A=0, M_A\neq 0, T_A\neq 0$	$x=l, \omega_B\neq 0, \theta_B\neq 0, M_B=0, T_B=0$	Fig. 5.2a
1	$x=0, \omega_A=0, \theta_A=0, M_A\neq 0, T_A\neq 0$	$x=l, \omega_B=0, \theta_B\neq 0, M_B=0, T_B\neq 0$	Fig. 5.2b
2	$x=0, \omega_A=0, \theta_A=0, M_A\neq 0, T_A\neq 0$	$x=l, \omega_B=0, \theta_B=0, M_B\neq 0, T_B\neq 0$	Fig. 5.2c
3	$x=0, \omega_A=0, \theta_A\neq 0, M_A=0, T_A\neq 0$	$x=l, \omega_B=0, \theta_B\neq 0, M_B=0, T_B\neq 0$	Fig. 5.2d
4	$x=0, \omega_A=0, \theta_A\neq 0, M_A=0, T_A\neq 0$	$x=l, \omega_B\neq 0, \theta_B\neq 0, M_B=0, T_B=0$	Fig. 5.2e
5	$x=0, \omega_A\neq 0, \theta_A\neq 0, M_A=0, T_A=0$	$x=l, \omega_B\neq 0, \theta_B\neq 0, M_B=0, T_B=0$	Fig. 5.2f

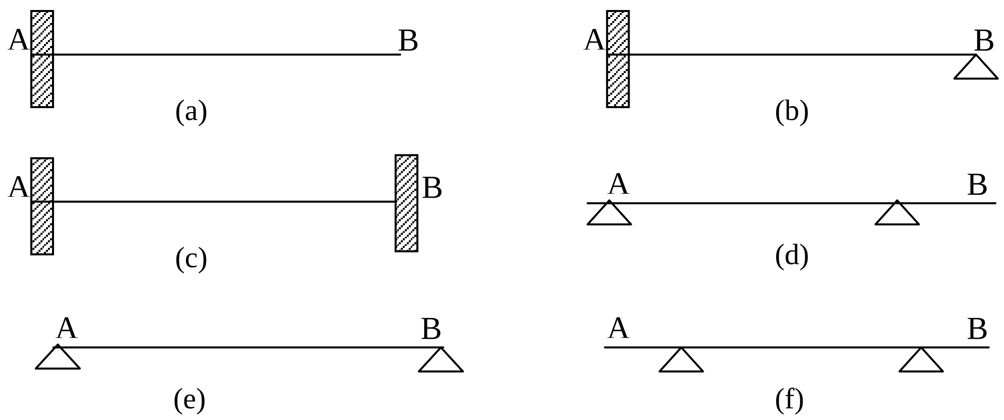

Fig. 5.2 – The boundary conditions

Being very flexible, the package can be run independently or can request other package. For an independent running the Input Data requested are the following:

- *kodben=1* : code for independent running;
- *tipb* = 1;2;3;4 or 5 ;
- *ei* = E I (E-Young's modulus; I- the moment inertia of the constant cross section);
- *l* : length of beam AB;
- *n* : number of masses {M_1,M_2,...,M_n} between A and B;
- *m* : number of reactions {R_1,R_2,...,R_m} between A and B;
- *a* : vector -distances between A and position of masses {M_1,M_2,...,M_n} ({n,1}-Array);
- *b* : vector -distances between A and position of reactions {R_1,R_2,...,R_m} ({m,1}-Array);

Warning: If a reaction exists on B, then b[m]=l (l - length of the beam AB). Therefore:

- m>0 for tipb = 2,3 or 4;
- m>1 and a[[1]]=0 for tipb = 5.

The Output Data is ***mee*** – an symmetric {n,n} matrix that contains the computed influence coefficients, e_{ij} .

The program can be run for all the combinations of boundary conditions and different numbers of masses.

6. EXAMPLE

To check our method, algorithm and program we run the program for all the cases for which we find the direct computation formula for the influence coefficients in strength materials textbooks/papers-frequently found only for the 2-lumped beams (for example, see [4]). Our results were identical with those obtained with the direct computation formula. For the 4-lumped beam shown in Fig. 6.1, the calculated influence coefficients are presented in the 4x4 matrix in Fig. 6.2. The concentrated masses are positioned at the distances given in the vector **a** in Fig. 6.2 and the reactions are placed at the distances given in the vector **b** in Fig. 6.2.

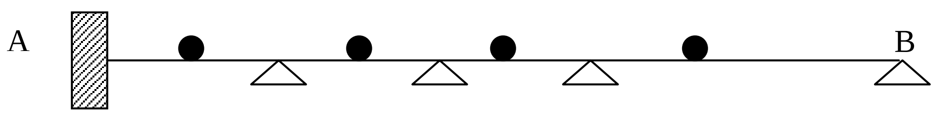

Fig. 6.1 A 4-lumped beam.

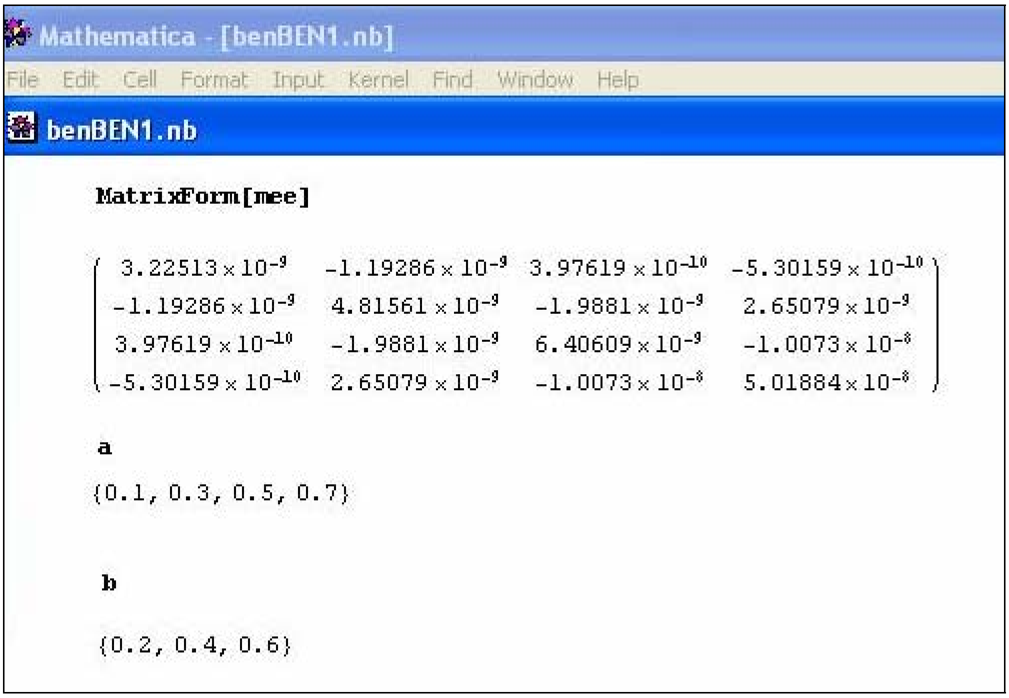

Fig. 6.2 The calculated influence coefficients

CONCLUDING REMARKS

Noting that, the direct computation formulas for the influence coefficients are extremely infrequent, the great potential of this Computer-Aided way to acquire the influence coefficients can be understood. Thus, the method is a very powerful tool, especially in the case of a large number of concentrated masses and/or intermediate supports.

The method can easily be generalized to a non-uniform cross-section.

Implemented in a Computer-Aided tool, the method can lead to a better understanding, learning and teaching.

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