We consider a class of nonlinear continuous-time programming problems and a general Mond-Weir dual model for this class. We get that WD-invexity property is a necessary and sufficient condition for weak duality.

Key words: WD-invexity, continuous-time nonlinear programming, Mond-Weir duality, weak duality.

1. INTRODUCTION

In 1953, Bellman [1] introduced a certain class of continuous-time optimization problems. Since then, many new classes of continuous-time nonlinear problems were considered. See De Oliveira and Rojas-Medar [4], Rojas-Medar et al. [5], Zalmai [6] and the references therein.

De Oliveira and Rojas-Medar [4] gave a generalization of the notions of KKT-invexity and WD-invexity for the continuous-time nonlinear programming problem, introduced by Martin [2] for the mathematical programming case. Also, they proved two very interesting results: KKT-invexity is a necessary and sufficient condition for global optimality of a Karush-Kuhn-Tucker point and WD-invexity is a necessary and sufficient condition for weak duality, where a Lagrangian dual (a continuous-time analogue to Wolfe's duality) is considered.

We prove that WD-invexity property of a general Mond-Weir dual for the continuous-time nonlinear programming problem also is a necessary and sufficient condition for weak duality. In this respect, we consider a general WD-invexity concept and a general qualification constraint, generalizations of the notions introduced by Oliveira and Rojas-Medar [4].

2. PRELIMINARIES

We consider the following continuous-time nonlinear programming problem

\[
\begin{align*}
\text{(CNP)} & \quad \text{minimize } \phi(x) = \int_0^T f(x(t),t)dt \\
& \text{subject to } g(x(t),t) \leq 0 \text{ a.e. in } [0,T], x \in X,
\end{align*}
\]

where \( X \) is a nonempty open convex subset of the Banach space \( L^\infty_n[0,T] \), \( \phi : X \to \mathbb{R} \), \( f(x(t),t) = \xi(x)(t) \) and \( g(x(t),t) = \gamma(x)(t) \), with the mappings \( \xi \) and \( \gamma \) from \( X \) into \( \Lambda^n_0[0,T] \) and \( \Lambda^n_0[0,T] \), respectively. Here, \( L^\infty_n[0,T] \) is the space of all \( n \)-dimensional vector-valued Lebesgue

\[
\text{DUALITY FOR A CLASS OF CONTINUOUS-TIME PROGRAMMING PROBLEMS}
\]

Vasile PREDA

University of Bucharest, Faculty of Mathematics and Computer Science, Str. Academiei 14, 010014 Bucharest, Romania
E-mail: vasilepreda0@gmail.com
measurable functions defined on the compact interval \([0, T] \subset \mathbb{R}\) that are essentially bounded, with the norm \(\| \cdot \|_\infty\) defined by

\[
\| x \|_\infty = \max \text{esssup} \left\{|x_j(t)|, 0 \leq t \leq T\right\},
\]

where \(\{x_j(t)\}_{1 \leq j \leq n} = x(t) \in \mathbb{R}^n\); the space \(\Lambda^n_\infty[0, T]\) is the space of all \(m\)-dimensional vector-valued functions defined on \([0, T]\) that are essentially bounded and Lebesgue measurable, with the norm \(\| \cdot \|_1\) defined by

\[
\| y \|_1 = \max_{1 \leq j \leq n} \int_0^T |y_j(t)| \, dt
\]

for \(y(t) = (y_j(t))_{1 \leq j \leq n} \in \mathbb{R}^n\).

Let \(F = \{x \in X : g(x(t), t) \leq 0 \text{ a.e. in } [0, T]\}\) be the set of all feasible solutions of (CNP). We suppose that \(F\) is a nonempty set and all vectors are column vectors. For \(w \in \mathbb{R}^p, w \leq 0\) means that \(w_i \leq 0\) for all \(i = 1, 2, \ldots, p\); \(w < 0\) means that \(w_i < 0\) for \(i = 1, 2, \ldots, p\) and \(w^\prime\) stands for the transposed of \(w\).

Now, for (CNP) problem, we consider a general Mond-Weir dual. We suppose that the functions \(t \mapsto \nabla f(x(t), t)\) and \(t \mapsto \nabla g_i(x(t), t)z(t), i \in I = \{1, 2, \ldots, p\}\), are Lebesgue integrable in \([0, T]\) for all \(x \in X\) and for all \(z \in L^n_\infty[0, T]\). The general Mond-Weir type dual is

\[
\text{(MWDP) } \quad \max \psi(x, \lambda) = \int_0^T \left[ f(x(t), t) + \lambda_{i_0}(t)g_{i_0}(x(t), t) \right] dt
\]

subject to

\[
\int_0^T \left[ \nabla f'(x(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i(x(t), t) \right] z(t) dt = 0,
\]

\[
\lambda_{i_k}'(t)g_{i_k}(x(t), t) \geq 0 \text{ a.e. in } [0, T], \quad k = 1, \ldots, \nu,
\]

\[
\lambda_i(t) \geq 0 \text{ a.e. in } [0, T], \quad i \in I,
\]

\[
z \in L^n_\infty[0, T], \quad x \in X, \quad \lambda \in L^n_\infty[0, T],
\]

where \(\nu \geq 0, I_\alpha \cap I_\beta = \Phi\) for \(\alpha \neq \beta\) and \(\bigcup_{\alpha=0}^\nu I_\alpha = \{1, \ldots, m\}\), \(\lambda_{i_k} = (\lambda_i)_{i \in I_{i_k}}\) and \(g_{i_k} = (g_i)_{i \in I_{i_k}}\) with

\[
\lambda_{i_k}'(t)g_{i_k}(x(t), t) = \sum_{i \in I_{i_k}} \lambda_i'(t)g_i(x(t), t).
\]

This dual problem (MWDP) may be considered as the continuous-time analogue of a general Mond-Weir duality formulation [3].

Let \(FD\) denote the set of all feasible solutions of (MWDP).

3. INVEXITY AND WEAK DUALITY

**Definition 3.1.** ([4]) There is weak duality between the problems (CNP) and (MWDP) if

\[
\phi(x) \geq \psi(y, \lambda)
\]

for all \(x \in F\) and all \((y, \lambda) \in FD\).
Definition 3.2. ([4]) The (CNP) problem is said to be invex if there exists a function \( \eta : V \times V \times [0,T] \rightarrow \mathbb{R}^n \) such that \( t \mapsto \eta(x(t),y(t),t) \in L^\infty_{\mathcal{T}}[0,T] \) and
\[
\varphi(x) - \varphi(y) \geq \int_0^T \nabla f(y(t),t) \eta(x(t),y(t),t) dt,
\]
\[
g(x(t),t) - g(y(t),t) \geq \nabla g_i(\eta(x(t),y(t),t)) \text{ a.e. in } [0,T], i \in I,
\]
for all \( x \in F \) and \( y \in X \).

Theorem 3.1. The invexity of (CNP) implies the weak duality between (CNP) and (MWDP).

We note that the omission of the terms \( g_i(x(t),t), i \in I \) and \( g_i(y(t),t), i \notin I_0 \) in the last inequality from Definition 3.2 does not affect the conclusion of Theorem 3.1. Thus, this makes it possible to use a generalized WD-invexity for (CNP).

4. GENERALIZED WD-INVEXITY AND WEAK DUALITY

In this section we introduce a generalized WD-invexity and a generalized constraint qualification. Then we prove the equivalence of weak duality for (CNP) and generalized WD-invexity defined below.

Definition 4.1. We say that the (CNP) problem is generalized weak duality invex (generalized WD-invex) if there exists a function \( \eta : V \times V \times [0,T] \rightarrow \mathbb{R}^n \) such that \( t \mapsto \eta(x(t),y(t),t) \in L^\infty_{\mathcal{T}}[0,T] \) and
\[
\varphi(x) - \varphi(y) \geq \int_0^T \nabla f(y(t),t) \eta(x(t),y(t),t) dt,
\]
\[
-g_i(y(t),t) \geq \nabla g_i(\eta(x(t),y(t),t)) \text{ a.e. in } [0,T], i \in I_0,
\]
\[
0 \geq \nabla g_i(\eta(x(t),y(t),t)) \text{ a.e. in } [0,T], i \notin I_0,
\]
for all \( x \in F \) and \( y \in X \).

Remark 4.1. For \( \nu = 0 \) generalized WD-invexity reduces to WD-invexity defined in [4].

Definition 4.2. We say that \( g \) satisfies the generalized constraint qualification (GCQ) if there is no \( v_i \in L^\infty_{\mathcal{T}}[0,T], v_i(t) \geq 0 \text{ a.e. in } [0,T], i \in I_0, \) not all zero, such that
\[
\int_0^T \sum_{i \in I_0} v_i(t) g_i(x(t),t) dt \geq 0 \text{ for all } x \in X.
\]

Remark 4.2. For \( I_k = \Phi, k = 1, \nu \), (GCQ) becomes (CQ2) introduced in [4].

Theorem 4.1. Under (GCQ), weak duality holds between (CNP) and (GMWDP) if and only if (CNP) is generalized WD-invex.

5. CONCLUSION

In this note we proved that in the context of continuous-time nonlinear programming problem, weak duality is attained if the general Mond-Weir dual of the problem previously mentioned has the WD-invexity property. On account of the importance and accuracy of the results from [4], we think that it is interesting and useful to establish corresponding formulations for both the multiobjective continuous case and the case where invexity is replaced by, for example, \( \rho \)-invexity.
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