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THE GENERALIZED SOLUTION OF THE BOUNDARY-VALUE PROBLEMS REGARDING THE BENDING OF ELASTIC RODS ON ELASTIC FOUNDATION. II. THE GENERALIZED SOLUTION

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The solution in the distributions space D'_+ for the boundary-value problems regarding the bending of the elastic rods on elastic foundation is given. The expression of the rod deflection for any load of rod is given with the help of the fundamental solution in D'_+ of the operator which describes the rod bending. From the condition that the support of the deflection should be on $[a,b]$ we established four conditions which allow us to determine the constraint concentrated loads and moments as well as the jumps of the deflection and its derivative at the ends of the rod. The obtained results are exemplified for a supported ends rod with concentrated and uniformly distributed loads lying on an elastic foundation.

Key words: Rods theory, Elastic foundation, Jump discontinuities, Distributions theory.

1. INTRODUCTION

This paper is the continuation of [4] in which the complete system of equation of the elastic rod bending on an elastic foundation in the distributions space D'_+ was established.

2. THE GENERALIZED SOLUTION OF THE ELASTIC ROD BENDING ON AN ELASTIC FOUNDATION

In the first part of this paper [4] it is shown that the bending equation of the elastic rods on an elastic foundation is

$$EI\partial_x^4\tilde{v}(x)+k\tilde{v}(x)=q_1+EI\left(\left[\tilde{v}\right]_a\delta'''(x-a)+\left[\tilde{v}\right]_b\delta'''(x-b)+\left[\tilde{\partial}_x\tilde{v}\right]_a\delta''(x-a)+\left[\tilde{\partial}_x\tilde{v}\right]_b\delta''(x-b)\right), \tag{2.1}$$

where $q_1(x)=\tilde{q}(x)+\sum_{i=1}^{n}P_i\delta(x-c_i)+\sum_{i=1}^{n}m_i\delta'(x-c_i)$.

We mention that this equation is written with respect only to the deflection $\tilde{v}$ of the rod in the distributions space D'_+. Denoting by $\omega=\sqrt[4]{\frac{k}{4EI}}$ the equation (2.1) becomes

$$\partial_x^4\tilde{v}(x)+4\omega^4\tilde{v}(x)=F(x), \tag{2.2}$$

where

$$F(x)=\frac{\tilde{q}(x)}{EI}+\frac{1}{EI}\sum_{i=1}^{n}P_i\delta(x-c_i)+\frac{1}{EI}\sum_{i=1}^{n}m_i\delta'(x-c_i)+ \\ +\left[\tilde{v}\right]_a\delta'''(x-a)+\left[\tilde{v}\right]_b\delta'''(x-b)+\left[\tilde{\partial}_x\tilde{v}\right]_a\delta''(x-a)+\left[\tilde{\partial}_x\tilde{v}\right]_b\delta''(x-b). \tag{2.3}$$

The fundamental solution in D'_+ of the operator $\mathcal{P}(\partial_x)=\partial_x^4+4\omega^4$ is $S\in D'_+$ having the expression

$$S(x)=\frac{1}{4\omega^3}H(x)\left(\cosh\omega x\sin\omega x-\sinh\omega x\cos\omega x\right), \tag{2.4}$$

where H represents the Heaviside function.

Indeed, the distribution $S\in D'_+$ satisfies the equation $\partial_x^4S(x)+4\omega^4S(x)=\delta(x)$.

Applying the Laplace transformation in distribution we obtain

$$\hat{S}(p)=L\left[S(x)\right](p),\quad \hat{S}(p)=\frac{1}{p^4+4\omega^4}. \tag{2.5}$$

Applying the inverse Laplace transformation L^{-1} and taking into account [1] we obtain

$$S(x)=L^{-1}\left[\hat{S}(p)\right]=L^{-1}\left[\frac{1}{p^4+4\omega^4}\right]=\frac{1}{4\omega^3}H(x)\left(\cosh\omega x\sin\omega x-\sinh\omega x\cos\omega x\right)\in D'_+. \tag{2.6}$$

We introduce the real-valued functions $u,u_1,u_2,u_3\in C^\infty(\mathbb{R})$ having the expression

$$\begin{aligned} &u(x)=\cosh\omega x\sin\omega x-\sinh\omega x\cos\omega x,\\ &u_1(x)=u'(x)=2\omega\sinh\omega x\sin\omega x,\\ &u_2(x)=u''(x)=2\omega^2\left(\cosh\omega x\sin\omega x+\sinh\omega x\cos\omega x\right),\\ &u_3(x)=u'''(x)=4\omega^3\left(\cosh\omega x\cos\omega x\right).\end{aligned} \tag{2.7}$$

We have

$$u^4(x)=u_3'(x)=-4\omega^4u(x). \tag{2.8}$$

From here results

$$\begin{aligned} u^{(4k)}(x)&=\left(-4\omega^4\right)^k u(x), & u^{(4k+1)}(x)&=\left(-4\omega^4\right)^k u_1(x),\\ u^{(4k+2)}(x)&=\left(-4\omega^4\right)^k u_2(x), & u^{(4k+3)}(x)&=\left(-4\omega^4\right)^k u_3(x).\end{aligned} \tag{2.9}$$

Because any natural number $n\geq 4$ can be written under the form $n=4k+p,\ p=0,1,2,3;\ k\in\mathbb{N},$ we have:

Any $n\geq 4$ order derivative of the function $u\in C^\infty(\mathbb{R})$ given by (2.7) represents a multiple of one of the functions $u,u_1=u',\ u_2=u'',\ u_3=u''',$ namely

$$u^{(n)}(x)=\begin{cases}\left(-4\omega^4\right)^k u(x), & n=4k\\ \left(-4\omega^4\right)^k u_1(x) & n=4k+1\\ \left(-4\omega^4\right)^k u_2(x) & n=4k+2\\ \left(-4\omega^4\right)^k u_3(x) & n=4k+3\end{cases}\qquad k=1,2,3,... \tag{2.10}$$

On the basis of the function expression $u \in C^\infty(\mathbb{R})$ given by (2.7) the fundamental solution $S \in D'_+$ given by (2.4) becomes

$$S(x) = \frac{1}{4\omega^3} H(x)u(x). \tag{2.11}$$

With this the solution of the equation (2.2) in D'_+ for $x \in \mathbb{R}$ is

$$\begin{gathered}\tilde{v}(x) = S(x) * F(x) = \frac{1}{4\omega^3} H(x)u(x) * F(x) = \frac{1}{4EI\omega^3} H(x)u(x) * \tilde{q}(x) + \\ + \frac{1}{4EI\omega^3} \sum_{i=1}^{n} P_i H(x-c_i)u(x-c_i) + \frac{1}{4EI\omega^3} \sum_{i=1}^{n} m_i H(x-c_i)u_1(x-c_i) + \frac{1}{4\omega^3} \Big[[\tilde{v}]_a H(x-a)u_3(x-a) + \\ + [\tilde{v}]_b H(x-b)u_3(x-b) + [\tilde{\partial}_x \tilde{v}]_a H(x-a)u_2(x-a) + [\tilde{\partial}_x \tilde{v}]_b H(x-b)u_2(x-b) \Big],\end{gathered} \tag{2.12}$$

where the symbol " $*$ " represents the convolution product in distribution.

We mention [3] that the following formulas have been applied

$$\begin{gathered} f(x)\delta(x) = f(0)\delta(x), \quad f(x) * \delta^{(p)}(x - x_0) = f^{(p)}(x - x_0), \quad p = 0,1,2,..., \\ \big(H(x)u(x)\big)' = H(x)u_1(x), \ \big(H(x)u(x)\big)'' = H(x)u_2(x), \ \big(H(x)u(x)\big)''' = H(x)u_3(x), \\ u(0) = 0, \ u_1(0) = 0, \ u_2(0) = 0, \ u_3(0) = 4\omega^3. \end{gathered} \tag{2.13}$$

Let be the functions $u \in C^\infty(\mathbb{R})$ and $\tilde{q}(x) = \begin{cases} q(x), & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$, where q is an integrable function on $[a,b]$. Then we have

$$H(x)u(x) * \tilde{q}(x) = \begin{cases} 0, & x < a \\ \int_a^x q(t)u(x-t)\mathrm{d}t, & x \in [a,b] \\ \int_a^b q(t)u(x-t)\mathrm{d}t, & x \geq b \end{cases} \tag{2.14}$$

Definition 2.1. By generalized boundary value problem for the bending elastic rods on an elastic foundation, we mean the determination of the distribution $\tilde{v} \in D'(\mathbb{R})$ which is null for $x \notin [a,b]$ and for $f \in D'(\mathbb{R})$ satisfy the equation

$$\partial_x^4 \tilde{v}(x) + 4\omega^4 \tilde{v}(x) = f(x). \tag{2.15}$$

Proposition 2.1. *The distribution* $\tilde{v} \in D'(\mathbb{R})$ *with* $\operatorname{supp} \tilde{v} = [a,b]$ *having the expression*

$$\tilde{v}(x) = \begin{cases} 0, & x \notin [a,b] \\ \frac{1}{4EI\omega^3} \int_a^x q(t)u(x-t)\mathrm{d}t + \frac{1}{4EI\omega^3} \sum_{i=1}^{n} P_i H(x-c_i)u(x-c_i) + & \\ + \frac{1}{4EI\omega^3} \sum_{i=1}^{n} m_i H(x-c_i)u_1(x-c_i) + & x \in [a,b] \\ + \frac{1}{4\omega^3} \Big[[\tilde{v}]_a H(x-a)u_3(x-a) + [\tilde{\partial}_x \tilde{v}]_a H(x-a)u_2(x-a) \Big], & \end{cases} \tag{2.16}$$

represents the unique solution of the generalized boundary value problem for the bending elastic rods on an elastic foundation (2.2), if the following conditions are satisfied

$$\int_a^b q(t)u(b-t)\mathrm{d}t+\sum_{i=1}^{n}P_i u(b-c_i)+\sum_{i=1}^{n}m_i u_1(b-c_i)+EI\Big[\left[\tilde v\right]_a u_3(b-a)+4\omega^3\left[\tilde v\right]_b+\left[\tilde\partial_x\tilde v\right]_a u_2(b-a)\Big]=0\,, \tag{2.17}$$

$$\begin{aligned}&\int_a^b q(t)u_1(b-t)\mathrm{d}t+\sum_{i=1}^{n}P_i u_1(b-c_i)+\sum_{i=1}^{n}m_i u_2(b-c_i)+\\&+EI\Big[-4\omega^4\left[\tilde v\right]_a u(b-a)+\left[\tilde\partial_x\tilde v\right]_a u_3(b-a)+4\omega^3\left[\tilde\partial_x\tilde v\right]_b\Big]=0,\end{aligned} \tag{2.18}$$

$$\int_a^b q(t)u_2(b-t)\mathrm{d}t+\sum_{i=1}^{n}P_i u_2(b-c_i)+\sum_{i=1}^{n}m_i u_3(b-c_i)+-4\omega^4 EI\Big[\left[\tilde v\right]_a u_1(b-a)+\left[\tilde\partial_x\tilde v\right]_a u(b-a)\Big]=0\,, \tag{2.19}$$

$$\int_a^b q(t)u_3(b-t)\mathrm{d}t+\sum_{i=1}^{n}P_i u_3(b-c_i)-4\omega^4\sum_{i=1}^{n}m_i u(b-c_i)-4\omega^4 EI\Big[\left[\tilde v\right]_a u_2(b-a)+\left[\tilde\partial_x\tilde v\right]_a u_1(b-a)\Big]=0\,. \tag{2.20}$$

Indeed, to have $\operatorname{supp}\tilde v=[a,b]$, we shall impose that $\tilde v(x)=0$ for $x>b$. We shall show that this condition is always possible, which will lead to the conditions (2.17)-(2.20).

According to (2.12) and taking into account (2.14), $\tilde v(x)$ for $x>b$ has the expression

$$\begin{aligned}\tilde v(x)&=\frac{1}{4EI\omega^3}\int_a^b q(t)u(x-t)\mathrm{d}t+\frac{1}{4EI\omega^3}\sum_{i=1}^{n}P_i u(x-c_i)+\frac{1}{4EI\omega^3}\sum_{i=1}^{n}m_i u_1(x-c_i)+\\&+\frac{1}{4\omega^3}\Big[\left[\tilde v\right]_a u_3(x-a)+\left[\tilde v\right]_b u_3(x-b)+\left[\tilde\partial_x\tilde v\right]_a u_2(x-a)+\left[\tilde\partial_x\tilde v\right]_b u_2(x-b)\Big],\quad x>b\end{aligned} \tag{2.21}$$

this is because $H(x-c_i)=1, i=\overline{1,n}$ for $x>b$.

We observe that $\tilde v(x)$ for $x>b$ can be expanded in Taylor's series according to the power of $x-b$ and we have

$$\tilde v(x)=\tilde v(b+0)+\frac{1}{1!}\tilde\partial_x\tilde v(b+0)(x-b)+\frac{1}{2!}\tilde\partial_x^2\tilde v(b+0)(x-b)^2+\frac{1}{3!}\tilde\partial_x^3\tilde v(b+0)(x-b)^3+\ldots, \tag{2.22}$$

To have $\tilde v(x)=0$ for $x>b$ it is necessary and sufficient to satisfy the conditions

$$\tilde v(b+0)=0,\quad \tilde\partial_x\tilde v(b+0)=0,\quad \tilde\partial_x^2\tilde v(b+0)=0,\quad \tilde\partial_x^3\tilde v(b+0)=0\,. \tag{2.23}$$

Indeed, on the basis of formula (2.21) for $x>b$, the deflection $\tilde v(x)$ is represented with the help of functions (2.7). Taking into account the formula (2.10), it results that the derivatives $\tilde\partial_x^n\tilde v(x)$, $n\geq 4$ will be express only with the help of a multiple $\left(-4\omega^4\right)^k$ of the functions $u,\ u_1=u',\ u_2=u'',\ u_3=u'''$.

Hence we deduce that the four conditions given by (2.23) imply the relations

$$\tilde\partial_x^n\tilde v(b+0)=0,\qquad n\geq 4\,. \tag{2.24}$$

Therefore, from the conditions (2.23) it results $\tilde v(x)=0$ for $x>b$.

Next, we shall explicit the conditions (2.23).

I. From $\tilde v(b+0)=0$ by virtue of (2.21) and taking into account $u(0)=0$, $u_1(0)=0$, $u_2(0)=0$, $u_3(0)=4\omega^3$ we have

$$\frac{1}{4EI\omega^3}\int_a^b q(t)u(b-t)\mathrm{d}t+\frac{1}{4EI\omega^3}\sum_{i=1}^n P_i u(b-c_i)+\frac{1}{4EI\omega^3}\sum_{i=1}^n m_i u_1(b-c_i)+$$
$$+\frac{1}{4\omega^3}\Big[\big[\tilde v\big]_a u_3(b-a)+\big[\tilde v\big]_b u_3(0)+\big[\tilde\partial_x\tilde v\big]_a u_2(b-a)+\big[\tilde\partial_x\tilde v\big]_b u_2(0)\Big]=0, \tag{2.25}$$

namely the condition (2.17).

II. Because $\tilde\partial_x\tilde v(b+0)=0$, by virtue of (2.21) and taking into account the formulas (2.10) we obtain

$$\frac{1}{4EI\omega^3}\int_a^b q(t)u_1(b-t)\mathrm{d}t+\frac{1}{4EI\omega^3}\sum_{i=1}^n P_i u_1(b-c_i)+\frac{1}{4EI\omega^3}\sum_{i=1}^n m_i u_2(b-c_i)+$$
$$+\frac{1}{4\omega^3}\Big[\big[\tilde v\big]_a\left(-4\omega^4\right)u(b-a)+\big[\tilde v\big]_b\left(-4\omega^4\right)u(0)+\big[\tilde\partial_x\tilde v\big]_a u_3(b-a)+\big[\tilde\partial_x\tilde v\big]_b u_3(0)\Big]=0, \tag{2.26}$$

namely the condition (2.18).

III. From (2.21) the condition $\tilde\partial_x^2\tilde v(b+0)=0$ implies the relation

$$\frac{1}{4EI\omega^3}\int_a^b q(t)u_2(b-t)\mathrm{d}t+\frac{1}{4EI\omega^3}\sum_{i=1}^n P_i u_2(b-c_i)+\frac{1}{4EI\omega^3}\sum_{i=1}^n m_i u_3(b-c_i)+$$
$$+\frac{1}{4\omega^3}\Big[\big[\tilde v\big]_a\left(-4\omega^4\right)u_1(b-a)+\big[\tilde v\big]_b\left(-4\omega^4\right)u_1(0)+$$
$$+\big[\tilde\partial_x\tilde v\big]_a\left(-4\omega^4\right)u(b-a)+\big[\tilde\partial_x\tilde v\big]_b\left(-4\omega^4\right)u(0)\Big]=0, \tag{2.27}$$

namely the condition (2.19).

IV. From (2.21) the condition $\tilde\partial_x^3\tilde v(b+0)=0$ implies the relation

$$\frac{1}{4EI\omega^3}\int_a^b q(t)u_3(b-t)\mathrm{d}t+\frac{1}{4EI\omega^3}\sum_{i=1}^n P_i u_3(b-c_i)+\frac{1}{4EI\omega^3}\sum_{i=1}^n m_i\left(-4\omega^4\right)u(b-c_i)+$$
$$+\frac{1}{4\omega^3}\Big[\big[\tilde v\big]_a\left(-4\omega^4\right)u_2(b-a)+\big[\tilde v\big]_b\left(-4\omega^4\right)u_2(0)+$$
$$+\big[\tilde\partial_x\tilde v\big]_a\left(-4\omega^4\right)u_1(b-a)+\big[\tilde\partial_x\tilde v\big]_b\left(-4\omega^4\right)u_1(0)\Big]=0, \tag{2.28}$$

namely the condition (2.20).

Because the convolution algebra D'_+ is without divisors of zero, it results that the distribution $\tilde v\in D'_+$ given by (2.16), represents the unique solution of the equation (2.2) with $\operatorname{supp}\tilde v=[a,b]$ owing to the conditions (2.17)–(2.20). With this the proposition is proved.

3. EXAMPLE

Let be OA (Fig. 3.1) an elastic rod having the length ℓ, a constant cross-section, supported in the points O and A, which lies on an elastic foundation.

We admit that on the rod act uniform distributed loads of intensity q, as well as a concentrated load of value P applied in the point $c\in(0,\ell)$.

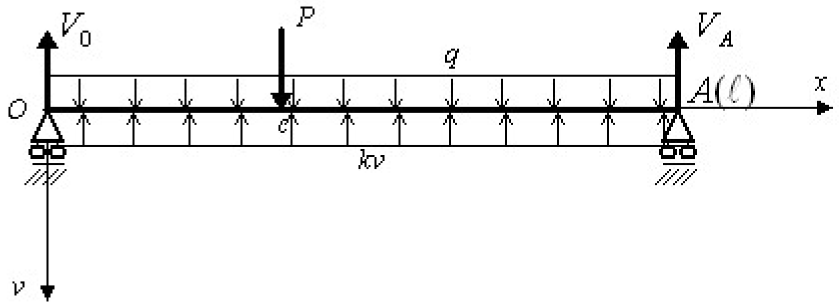

Fig. 1 – Elastic rod supported on an elastic foundation.

We shall apply the formula (2.16) to determine the deflection $\tilde{v}(x)$ in a point $x \in \mathbb{R}$. Due to the way in which the rod is fixed and because $c_1 = a = 0, c_n = b = \ell$ the boundary conditions are

$$\begin{gathered}\tilde{v}(0+0)=0,\ \tilde{v}(0-0)=0,\ \tilde{v}(\ell+0)=0,\ \tilde{v}(\ell-0)=0,\\ \tilde{\partial}_x^2\tilde{v}(0+0)=0,\ \tilde{\partial}_x^2\tilde{v}(0-0)=0,\ \tilde{\partial}_x^2\tilde{v}(\ell+0)=0,\ \tilde{\partial}_x^2\tilde{v}(\ell-0)=0.\end{gathered} \tag{3.1}$$

The boundary conditions (3.1) can be written under the form

$$\left[\tilde{v}\right]_0 = \tilde{v}(0+0)-\tilde{v}(0-0)=0,\ \left[\tilde{v}\right]_\ell = \tilde{v}(\ell+0)-\tilde{v}(\ell-0)=0,\ \left[\tilde{\partial}_x^2\tilde{v}\right]_0=0,\ \left[\tilde{\partial}_x^2\tilde{v}\right]_\ell=0. \tag{3.2}$$

Because we have

$$\left[\tilde{\partial}_x\tilde{v}\right]_0 = \tilde{v}'(0+0)-\tilde{v}'(0-0)=\tilde{v}'(0+0),\ \left[\tilde{\partial}_x\tilde{v}\right]_\ell = \tilde{v}'(\ell+0)-\tilde{v}'(\ell-0) = -\tilde{v}'(\ell-0), \tag{3.3}$$

due to the formula (2.16) we obtain for the deflection $\tilde{v}$ the expression

$$\tilde{v}(x)=\begin{cases} 0, & x\notin[0,\ell) \\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(x-t)\mathrm{d}t-\dfrac{V_0}{4EI\omega^3}H(x)u(x)+\dfrac{P}{4EI\omega^3}H(x-c)u(x-c) & \\ -\dfrac{V_A}{4EI\omega^3}H(x-\ell)u(x-\ell)+\dfrac{1}{4\omega^3}\tilde{v}'(0+0)H(x)u_2(x). & x\in[0,\ell]\end{cases} \tag{3.4}$$

Taking into account (3.3) and the formula $\int_0^x f(x-t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t$ we obtain

$$\tilde{v}(x)=\begin{cases} 0, & x\notin[0,\ell) \\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(t)\mathrm{d}t-\dfrac{V_0u(x)}{4EI\omega^3}+\dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3}, & x\in[0,c) \\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(t)\mathrm{d}t-\dfrac{V_0u(x)}{4EI\omega^3}+\dfrac{Pu(x-c)}{4EI\omega^3}+\dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3}, & x\in[c,\ell]\end{cases} \tag{3.5}$$

We observe that in this relation of the deflection $\tilde{v}$ appear only two unknowns, namely: the reaction V_0 in O and the rotation of rod to the right in the point O, $\tilde{v}'(0+0)$. These unknowns as well as the unknowns $V_A, \tilde{v}'(\ell-0)$ representing the reaction in the point A and the rotation of rod to the left in the point A, respectively, will be determined from the conditions (2.17)–(2.20).

Using the formula $\int_0^x f(x-t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t$ and the relation $u(0)=0, u_1(0)=0, u_2(0)=0, u_3(0)=4\omega^3$, the conditions (2.17)–(2.20) become

$$q\int_0^\ell u(t)\mathrm{d}t - V_0 u(\ell) + Pu(\ell-c) + EI\tilde{v}'(0+0)u_2(\ell) = 0\,, \tag{3.6}$$

$$q\int_0^\ell u_1(t)\mathrm{d}t - V_0 u_1(\ell) + Pu_1(\ell-c) + EI\left[\tilde{v}'(0+0)u_2(\ell) - 4\omega^3\tilde{v}'(\ell-0)\right] = 0\,, \tag{3.7}$$

$$q\int_0^\ell u_2(t)\mathrm{d}t - V_0 u_2(\ell) + Pu_2(\ell-c) - 4\omega^4 EI\tilde{v}'(0+0)u(\ell) = 0\,, \tag{3.8}$$

$$q\int_0^\ell u_3(t)\mathrm{d}t - V_0 u_3(\ell) + Pu_3(\ell-c) - 4\omega^3 V_A - 4\omega^4 EI\tilde{v}'(0+0)u_1(\ell) = 0\,. \tag{3.9}$$

We denote

$$\begin{aligned} u_1^*(x) &= \frac{u_1(x)}{2\omega} = \sinh\omega x\sin\omega x,\\ u_2^*(x) &= \frac{u_2(x)}{2\omega^2} = \cosh\omega x\sin\omega x + \sinh\omega x\cos\omega x,\\ u_3^*(x) &= \frac{u_3(x)}{4\omega^3} = \cosh\omega x\cos\omega x, \end{aligned} \tag{3.10}$$

and taking into account [2], p.242 we can write the relations

$$\begin{aligned} I &= \int_0^x u(t)\mathrm{d}t = \frac{1}{\omega}\left[1-\cos\omega x\cosh\omega x\right] = \frac{1}{\omega}\left[1-u_3^*(x)\right], \quad I_1 = \int_0^\ell u_1^*(t)\mathrm{d}t = \frac{u(\ell)}{2\omega},\\ I_2 &= \int_0^\ell u_2^*(t)\mathrm{d}t = \omega\sinh\omega\ell\sin\omega\ell = \omega u_1^*(\ell), \quad I_3 = \int_0^\ell u_3^*(t)\mathrm{d}t = \frac{u_2^*(\ell)}{2\omega}. \end{aligned} \tag{3.11}$$

From the equations (3.6) and (3.8) we shall obtain for the unknowns V_0 and $\tilde{v}'(0+0)$ the expressions

$$\begin{aligned} &\left[u^2(\ell) + u_2^{*2}(\ell)\right]V_0 = P\left[u(\ell)u(\ell-c) + u_2^*(\ell)u_2^*(\ell-c)\right] + q\left[u(\ell)\int_0^\ell u(t)\mathrm{d}t + u_2^*(\ell)\int_0^\ell u_2^*(t)\mathrm{d}t\right],\\ &2EI\omega^2\left[u^2(\ell) + u_2^{*2}(\ell)\right]\tilde{v}'(0+0) = P\left[u(\ell)u_2^*(\ell-c) - u_2^*(\ell)u(\ell-c)\right] + q\left[u(\ell)\int_0^\ell u_2^*(t)\mathrm{d}t - u_2^*(\ell)\int_0^\ell u(t)\mathrm{d}t\right]. \end{aligned} \tag{3.12}$$

From the equations (3.7) and (3.9) we shall obtain for the unknowns V_A and $\tilde{v}'(\ell-0)$ the expressions

$$V_A = q\int_0^\ell u_3^*(t)\mathrm{d}t - u_3^*(\ell)V_0 + Pu_3^*(\ell-c) - 2EI\omega^2 u_1^*(\ell)\tilde{v}'(0+0)\,, \tag{3.13}$$

$$2EI\omega^2\tilde{v}'(\ell-0) = q\int_0^\ell u_1^*(t)\mathrm{d}t - u_1^*(\ell)V_0 + Pu_1^*(\ell-c) + 2EI\omega^2 u_3^*(\ell)\tilde{v}'(0+0)\,. \tag{3.14}$$

Consequently, the four unknowns $V_0,\ V_A,\ \tilde{v}'(0+0),\ \tilde{v}'(\ell-0)$ are determined.

On the basis of the relations (3.11), the formula (3.5) of the deflection becomes

$$\tilde{v}(x)=\begin{cases} 0, & x\notin[0,\ell) \\ \dfrac{q}{4EI\omega^4}(1-u_3^*(x))-\dfrac{u(x)V_0}{4EI\omega^3}+\dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3}, & x\in[0,c) \\ \dfrac{q}{4EI\omega^4}(1-u_3^*(x))-\dfrac{u(x)V_0}{4EI\omega^3}+\dfrac{Pu(x-c)}{4EI\omega^3}+\dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3}, & x\in[c,\ell] \end{cases} \tag{3.15}$$

and the relations (3.12), (3.13) and (3.14) can be written under the form

$$\left[u^2(\ell)+u_2^{*2}(\ell)\right]V_0=P\left[u(\ell)u(\ell-c)+u_2^*(\ell)u_2^*(\ell-c)\right]+q\left[\frac{u(\ell)}{\omega}(1-u_3^*(\ell))+\omega u_1^*(\ell)u_2^*(\ell)\right], \tag{3.16}$$

$$2EI\omega^2\left[u^2(\ell)+u_2^{*2}(\ell)\right]\tilde{v}'(0+0)=P\left[u(\ell)u_2^*(\ell-c)-u_2^*(\ell)u(\ell-c)\right]+q\left[\omega u(\ell)u_1^*(\ell)-\frac{u_2^*(\ell)}{\omega}(1-u_3^*(\ell))\right], \tag{3.17}$$

$$V_A=\frac{q}{2\omega}u_2^*(\ell)-u_3^*(\ell)V_0+Pu_3^*(\ell-c)-2\omega^2EIu_1^*(\ell)\tilde{v}'(0+0), \tag{3.18}$$

$$2\omega^2EI\tilde{v}'(\ell-0)=\frac{q}{2\omega}u(\ell)-u_1^*(\ell)V_0+Pu_1^*(\ell-c)+2\omega^2EIu_3^*(\ell)\tilde{v}'(0+0). \tag{3.19}$$

Substituting the values V_0 and $\tilde{v}'(0+0)$ given by (3.16), (3.17) in (3.15) we shall obtain the expression of the deflection $\tilde{v}(x),\ x\in[0,\ell]$ for the elastic rod on the elastic foundation.

Consequently, the reactions V_0 and V_A in the points O and A, as well as the deflection of the rod on the elastic foundation have been determined.

4. CONCLUSIONS

We give a general method of solving the boundary value problem for the bending elastic rods on an elastic foundation. The conditions (2.17)–(2.20) ensure the rod equilibrium and allow the determination of all unknowns as well as the constraint concentrated moments and forces which appear in the deflection expression. The solution is given for any sort of loading of the rod such as distributed and concentrated loads.

All these show the efficiency of the convolution algebra D'_+ in solving the bending problems of elastic rods on an elastic foundation.

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