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THREE-DIMENSIONAL GINZBURG-LANDAU DISSIPATIVE SOLITONS SUPPORTED BY A TWO-DIMENSIONAL TRANSVERSE GRATING

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In a recent brief report (D. Mihalache *et al*., Phys. Rev. A **81**, 025801 (2010)) families of spatiotemporal dissipative solitons in a Ginzburg-Landau dynamical model of bulk optical media including a combination of gain, saturable absorption, and two-dimensional transverse grating were reported. In the present work I give a comprehensive study of stable fundamental (vorticityless) dissipative solitons and quasistable vortex solitons in the model based on the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity and a two-dimensional periodic potential representing the transverse grating.

Key words: Spatiotemporal solitons; Dissipative solitons; Vortex solitons; Optical lattices.

1. INTRODUCTION AND THE FORMULATION OF THE THREE-DIMENSIONAL DYNAMICAL MODEL

The complex Ginzburg-Landau (CGL) equation [1–3] and the complex Swift-Hohenberg equation [4] constitute two generic partial differential equations modeling nonlinear dynamics in situations close to pattern-forming transitions in various dissipative media, such as optical cavities, viscous fluid flows and thermal convection, reaction-diffusion mixtures, etc. The specific feature of the CGL and Swift-Hohenberg equations in situations when they are supposed to produce stable localized pulses (alias dissipative solitons, DSs [5–10]) is the cubic-quintic (CQ) combination of nonlinear gain and loss terms, which are added to the common linear loss term [1]. In optical media, this combination corresponds to the interplay of linear gain and saturable absorption [2]. A challenging issue is the search for stable two-dimensional (2D) and three-dimensional (3D) DSs, because of the possibility of the critical or supercritical collapse, in the 2D and 3D physical settings, respectively, under the action of the self-focusing cubic nonlinearity. Multidimensional localized states with intrinsic vorticity, i.e., vortex solitons, are prone to the additional instability against azimuthal perturbations splitting them into sets of fundamental (vorticityless) solitons [11–15]. A promising physical setting for the generation of stable fundamental and vortical 3D solitons is the use of confining 2D and 3D optical lattices [16–21].

Stable 2D dissipative vortex solitons in the form of "spiral solitons", with topological charge (intrinsic vorticity) S = 1 and 2, were found in the framework of the CQ CGL equation in Refs. [22–23]. Recently, stable vortex solitons in the Ginzburg-Landau model of a two-dimensional lasing medium with a transverse grating were found [24]. The vortices were chiefly considered in the onsite (rhombic) form, but the stabilization of offsite vortices (square-shaped ones) and quadrupoles was demonstrated too [24]. Stable fundamental (S = 0) DSs were found in 3D optical models based on CQ CGL equations [25–30]. Dissipative spatiotemporal solitons (alias "light bullets") in optical parametric oscillator models were recently introduced in Ref. [31]; thus it was shown that when chromatic dispersion operates together with 2D diffraction, the degenerate optical parametric oscillator exhibits 3D localized structures in a regime devoid of Turing or modulational instabilities [31]. Three-dimensional double soliton complexes, including rotating ones [32], have been found too. Recently, 3D stable vortex solitons with values of the "spin" S = 1, 2, and 3 have been reported as stationary solutions to the respective CQ CGL equation [28–30]. Collisions between 3D vortex solitons in the framework of the CQ CGL equation were investigated too [33–35]. In the present work I give a comprehensive study of the existence and stability of both fundamental and vortical 3D spatiotemporal DSs in the dissipative bulk medium with a 2D transverse grating and anomalous group-velocity dispersion (GVD); see Ref. [36] for a brief report of these studies. The CQ CGL equation is

$$\mathrm{i}U_z + (1/2)(U_{xx} + U_{yy} + DU_{tt}) + \{\mathrm{i}\delta + (1-\mathrm{i}\varepsilon)\,|U|^2 - (\nu - \mathrm{i}\mu)\,|U|^4 + p[\cos(2x) + \cos(2y)]\}U = 0, \quad (1)$$

where (x, y) are the transverse coordinates, and t is the reduced temporal variable [36]. The coefficients which are scaled to be 1/2 and 1 account, respectively, for the diffraction in the transverse plane and self-focusing Kerr nonlinearity, ν taking into account the quintic nonlinearity, that may compete with the cubic term, and the coefficient D in Eq. (1) is the GVD. In the following I consider only anomalous GVD and I take $D = 1$. In the dissipative part of the equation, real constants δ, ε and μ represent, respectively, the linear loss, cubic gain, and quintic loss, which are the basic ingredients of the CQ CGL equation. The parameter p is the strength of the square grating in the transverse plane, whose period is scaled to be π.

2. FAMILIES OF STATIONARY 3D DISSIPATIVE SOLITONS AND THEIR STABILITY ANALYSIS

Stationary DSs were generated as attractors by direct simulations of Eq. (1). Thus found objects are single-peaked fundamental solitons (with $S = 0$), and rhombus-shaped vortical solitons (alias onsite vortices), built as compound objects, consisting of four separate peaks of the local intensity, set at local potential minima of the lattice potential, with an empty site in the middle, see Fig. 5 below. The topological charge (intrinsic vorticity) of the complex patterns is provided by the phase shift $\pi/2$ between adjacent peaks, which corresponds to the total phase circulation of 2π around the core of the pattern, as it should be in vortices with topological charge $S = 1$. The numerical results are presented below for a fixed set of three parameters in Eq. 1: $\mu = 1$, $\nu = 0.1$, and $\delta = 0.4$, which adequately represents the generic case, while cubic gain ε and strength p of the transverse periodic potential are varied. After a particular stationary solution was found by the direct integration of Eq. (1), it was then used as the initial configuration for a new run of simulations, with slightly modified values of the parameters, with the aim to find an attractor corresponding to these new parameters. When the simulations converged to stationary localized modes, their stability was additionally tested by adding to them a random (white) noise at the amplitude level of up to 10%, and running the subsequent simulations typically, up to $z = 300$ (see Figs. 4 and 6 below). In the course of the stability test, the evolution of amplitude $\max|U(x, y, t)|$ of the established pattern and the total energy (alias norm) E of the solution were monitorized,

$$E = \iiint |U(x,y,z,t)|^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}t. \quad (2)$$

Figure 1 illustrates the soliton families by plotting the corresponding soliton amplitudes versus ε for two characteristics values of strength p of the transverse grating. In Fig. 2 we show the total energy E, as function of cubic gain ε for families of the fundamental solitons with $S = 0$ and rhombic vortices with $S = 1$. The shapes of the fundamental solitons for different values of the grating's strength, p, are displayed in Fig. 3 by means of isosurface plots of total field intensity. The families of fundamental solitons shown in Fig. 3 are entirely *stable*, i.e., the solitons feature efficient self-healing, restoring their stationary shape after the addition of random perturbations, see Fig. 4. The stability of fundamental solitons in the present 3D model is not surprising, as they are stable in both the 2D [22–23] and 3D [28–30] versions of Eq. (1) without the lattice potential ($p = 0$), unlike localized vortices, which cannot be stable in either case. In Fig. 5 we show the transverse distribution of rhombus-shaped vortices for three representative values of parameter pairs (p, ε). The temporal trajectories of the four constituents of unstable $S = 1$ rhombic vortices whose stationary profiles are displayed in Fig. 5 are shown in Fig. 6 for two representative values of the parameters p and ε. Fig. 6 clearly demonstrates *quasistabilization* of the vortex solitons with the increase of the grating's strength p. Indeed, from Fig. 6 the splitting distance for $p = 1$, $z \approx 100$ may be estimated as tantamount to ~10 soliton's dispersion lengths, with the temporal width of each constituent being T ~2.

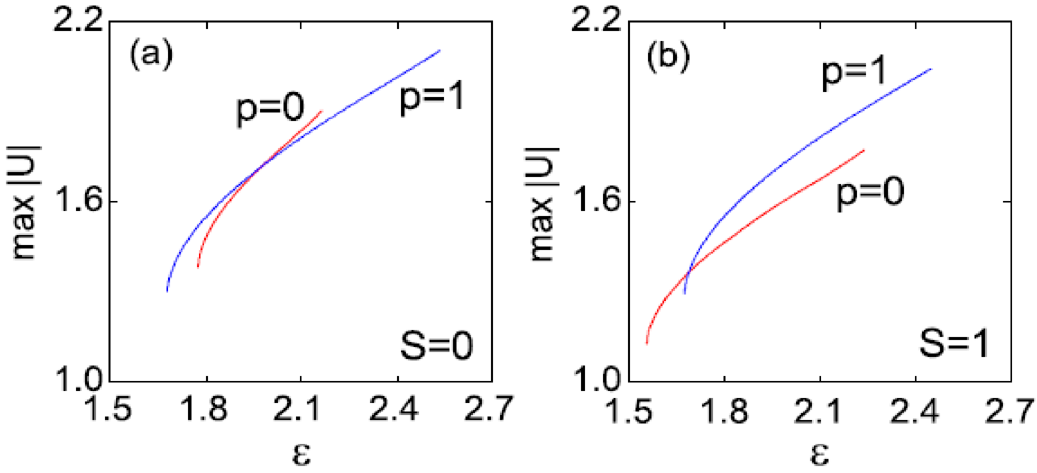


Fig. 1 – The amplitude max$|U(x, y, t)|$ as function of cubic gain ε for fundamental solitons (S = 0) (a), and for rhombic vortex solitons with S = 1 (b).

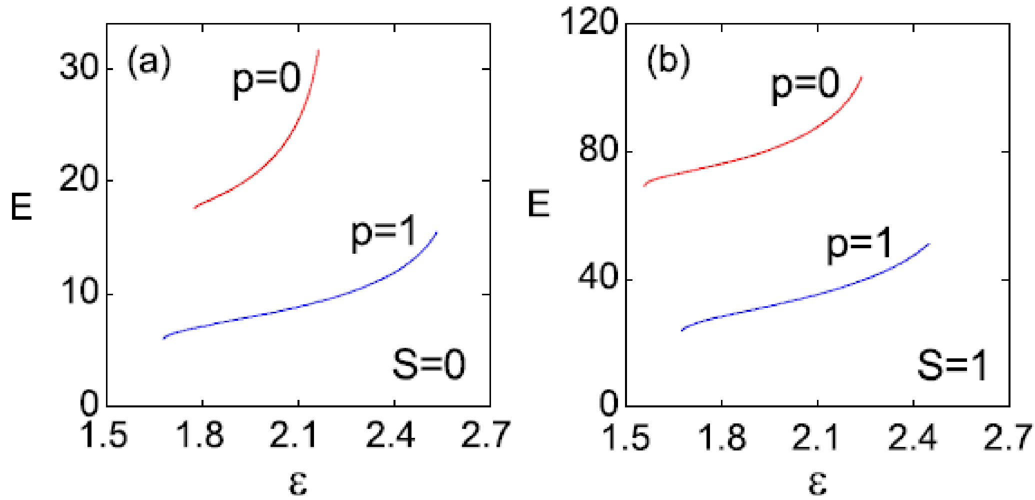


Fig. 2 – The total energy (norm) E versus cubic gain ε for fundamental solitons (a), and for rhombus-shaped vortex solitons with S = 1 (b), at two different values of strength p of the lattice potential (see Ref. [36]).

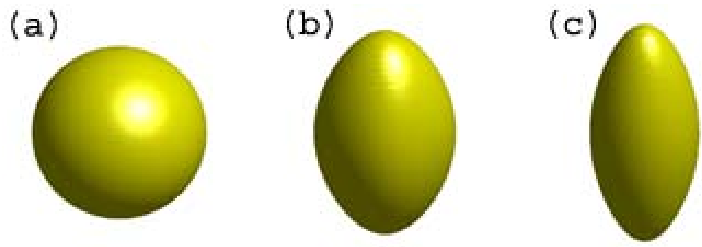


Fig. 3 – Isosurface plots of total field intensity showing typical stable fundamental solitons: a) p = 0, b) p = 1, and c) p = 4. Here the cubic gain parameter is ε = 2. The anisotropy of the soliton's shape in panels (b) and (c) is explained by the action of the transverse grating with strength p (see Ref. [36]).

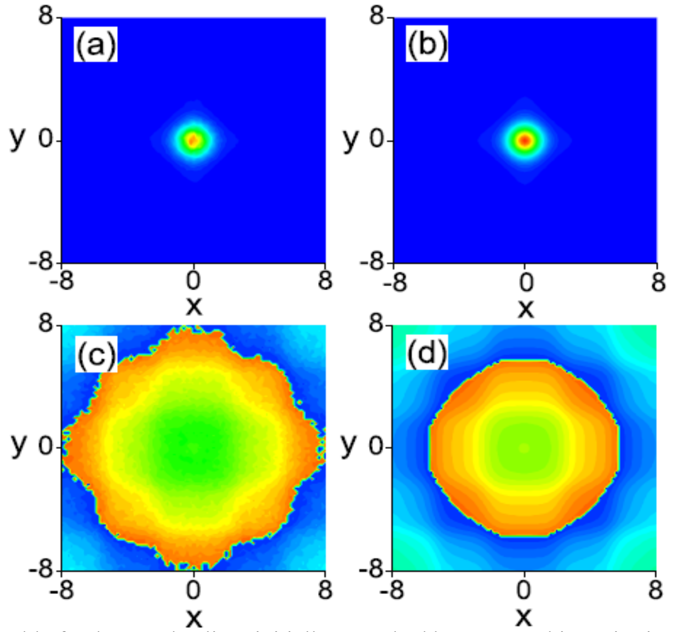


Fig. 4 – The self-healing of a stable fundamental soliton initially perturbed by a 10% white noise is shown by the plots of the local amplitude, $|U(x, y, t = 0)|$ and the respective phase in the (x, y) cross-section: a) the perturbed amplitude distribution at $z = 0$; b) the self-cleaned amplitude field at $z = 200$. Panels (c) and (d) show the same as (a) and (b), but for the soliton phase. The parameters are $p = 1$ and $\varepsilon = 2$.

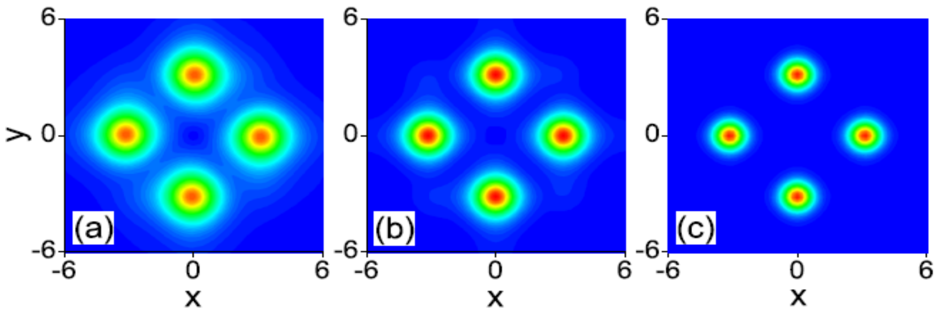
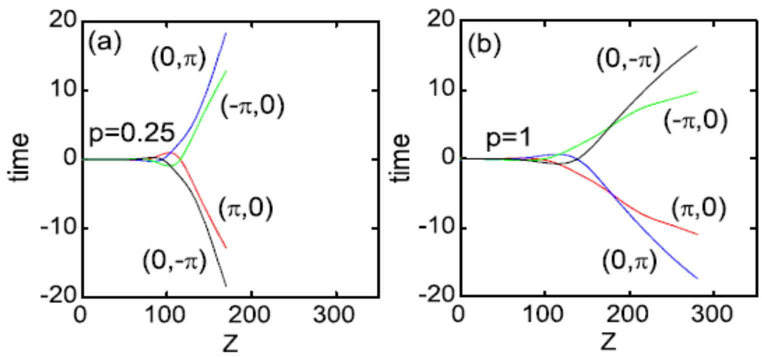


Fig. 5 – The transverse distribution of the local field amplitude $|U(x, y, 0)|$ in the rhombic vortex solitons with vorticity $S = 1$. The parameters are: a) $\varepsilon = 1.9$, $p = 0.25$; b) $\varepsilon = 1.7$, $p = 1$; c) $\varepsilon = 1.8$, $p = 4$.

Additional numerical simulations show that with the increase of the lattice strength to $p = 4$, the distance of the stable transmission increases to $z \approx 1500$, which exceeds a meter in physical units (i.e., the 3D vortex will be a completely stable object in the experiment). Thus a sufficiently strong transverse lattice potential (a 2D grating) makes the quasistable 3D vortex solitons practically stable physical objects.



a) b)

Fig. 6 – Temporal trajectories of the four constituents of quasistable $S = 1$ rhombic vortices whose stationary profiles are displayed in Fig. 5 (panels (a) and (b)), showing their separation in the temporal direction: a) $p = 0.25$, $\varepsilon = 1.9$; b) $p = 1$, $\varepsilon = 1.7$.

3. CONCLUSIONS

In summary, I performed a comprehensive study of families of both fundamental (vorticityless) and vortical spatiotemporal dissipative solitons in the framework of the three-dimensional complex Ginzburg-Landau equation with the cubic-quintic nonlinearity, periodic potential in the transverse plane, and anomalous group-velocity dispersion in the temporal direction [36]. The dynamical model introduced in Ref. [36] applies to the description of bulk optical media with the combination of gain and saturable absorption. Solitons of both types were readily found, as robust attractors, by direct simulations of the cubic-quintic complex Ginzburg-Landau equation. However, only the fundamental solitons are *completely stable* against strong random (white noise) perturbations, while the vortices, built as rhombus-shaped complexes of four fundamental solitons with appropriate phase shifts between them, are *quasistable* physical objects; they may be split by random perturbations into the constituents separating in the free (temporal) direction. Nevertheless, a sufficiently strong two-dimensional grating makes the three-dimensional vortices practically stable objects.

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