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ANALYTIC SOLUTIONS FOR AXISYMMETRIC INCOMPRESSIBLE FLOWS WITH WALL INJECTION AND REGRESSION

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One gives analytical solutions for axisymmetric flows, considering wall injection and regression. Except a narrow region near wall, the potential flow or some particular rotational flows are possible solutions for this outer flow. The wall regression causes unsteady effects. The analytical solutions are found as linear combinations of eigenfunctions for steady and unsteady outer flow. For laminar viscous flow the partial differential Navier-Stokes equations are reduced to an ordinary differential equation solved numerically.

Key words: Axisymmetric, injection, analytical solution.

1. INTRODUCTION

The possibility to obtain simpler solutions for incompressible flow with injection was, after our knowledge, not thoroughly studied. Several papers [1,2] take into account a vortex solution generated by the injection itself, although the superposition of eigenfunctions does not satisfy the nonlinear vortex equation of Helmholtz [4]. Of course much attention is paid to wall injection in connection with boundary layer control [5,6] as well as in connection with modern rocket engines. An important domain of application is the flow in rocket motors [1–3,8,9]. The injection and regression are caused by solid fuel consumption.

In order to find analytical solutions for the partial differential system of equations of the axi-symmetric flow with fluid injection and wall regression in a channel, a step by step strategy is adopted: 1) first one solves the nonviscid stationary (no regression) flow with constant injection a closed analytical solution being obtained; 2) then one solves the problem of variable injection velocity, adding to a solution of type 1) for the average velocity a series in eigenfunctions, an approach that becomes possible with this authors strategy; 3) the unsteady problem is solved for constant regression speed but variable injection; analytical expressions are obtained too; 4) the steady viscous flow is solved in a closed form for laminar regime. Finally numerical results are given and commented.

2. MODEL FORMULATION

One considers an incompressible flow trough a cylindrical channel of inner diameter *a* and length *L* An injection of same fluid takes place trough the wall at a speed $\bar{u}_w$. The wall can have a regression at the constant speed $\bar{u}_r$ the radius varying with time, $\bar{t}$. *One uses bars for dimensional variables; bars will be dropped for the corresponding dimensionless magnitudes.*

The governing system of equations for incompressible flow, in vectorial form, are the continuity equation, $\bar{\nabla}\bar{\boldsymbol{V}} = 0$ and the momentum equation:

$$\bar{\nabla}\bar{\boldsymbol{V}} = 0\,;\quad \frac{\partial \bar{\boldsymbol{V}}}{\partial \bar{t}} + \bar{\nabla}\left(\frac{\bar{V}^2}{2}\right) + \bar{\omega}\times\bar{\boldsymbol{V}} = -\frac{1}{\bar{\rho}}\bar{\nabla}\bar{p} + \bar{\nu}\bar{\nabla}^2\bar{\boldsymbol{V}}\,,\quad \bar{\rho} = \text{const.}, \tag{1}$$

where $\bar{\boldsymbol{V}}\,(\bar{v}_z, \bar{v}_r, \bar{v}_\varphi)$ is the velocity vector, $\bar{p}$ is the static pressure, $\bar{\rho}$ is the density and $\bar{\nu}$ the cinematic viscosity, $\bar{\nabla}\cdot$ stands for the vectorial differential operator and $\bar{\omega}$ is the vorticity: $\bar{\omega} = \nabla \times \bar{\boldsymbol{V}}$. If one uses cylindrical coordinates ($\bar{z}$, $\bar{r}$, φ), for axi-symmetrical flow, one has $\bar{v}_\varphi = 0$, $\bar{V}^2 = \bar{v}_z^2 + \bar{v}_r^2$, $\bar{\omega} = \bar{\omega}\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\varphi$ being the unit vector of the φ coordinate.

By introducing the stream function $\bar{\psi}(\bar{z}, \bar{r}, \varphi)$ defined by:

$$\bar{v}_z = \frac{1}{\bar{r}}\frac{\partial\bar{\psi}}{\partial\bar{r}}, \quad \bar{v}_r = -\frac{1}{\bar{r}}\frac{\partial\bar{\psi}}{\partial\bar{z}}, \tag{2}$$

the continuity equation is identically satisfied and the vorticity $\bar{\omega}$ has the expression:

$$\bar{\mathcal{L}}(\bar{\psi}) = -\frac{\bar{\omega}}{\bar{r}}, \quad \bar{\mathcal{L}}. = \frac{1}{\bar{r}^2}\frac{\partial^2 .}{\partial\bar{z}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial .}{\partial\bar{r}}\right), \quad \bar{\omega} = \frac{\partial\bar{v}_r}{\partial\bar{z}} - \frac{\partial\bar{v}_z}{\partial\bar{r}}, \tag{3}$$

Applying the curl operator to the momentum equation – the second equation (1) – the pressure is eliminated, to yield the equation for the vorticity transport:

$$\frac{\partial}{\partial\bar{t}}\left(\bar{\mathcal{L}}(\bar{\psi})\right) + \bar{v}_z\frac{\partial}{\partial\bar{z}}\left(\bar{\mathcal{L}}(\bar{\psi})\right) + \bar{v}_r\frac{\partial}{\partial\bar{r}}\left(\bar{\mathcal{L}}(\bar{\psi})\right) = \bar{\nu}\bar{\mathcal{L}}\left(\bar{r}^2\bar{\mathcal{L}}(\bar{\psi})\right). \tag{4}$$

which represents the Helmholtz equation written in a convenient form by authors.

The equation (4) should be solved for the following boundary conditions:

$$\bar{z} = 0, \ \bar{v}_z(0, \bar{r}, \bar{t}) = \bar{v}_{z0}(\bar{r}, \bar{t}), \tag{5}$$

$$\bar{r} = a, \ \bar{v}_r(\bar{z}, a, \bar{t}) = -\bar{u}_w(\bar{z}, \bar{t}), \ \bar{v}_z = 0, \tag{6}$$

$$\bar{r} = 0, \ 0 \le \bar{z} \le L, \ \bar{v}_r(z, 0, \bar{t}) = 0, \tag{7}$$

where $\bar{v}_{z0}(\bar{r}, \bar{t})$ and $\bar{u}_w(\bar{z}, \bar{t})$ are given quantities.

2.1. Dimensionless quantities

One defines dimensionless coordinates z, r and t *by simply dropping bars from the dimensional ones*: $\bar{z} = Lz$, $\bar{r} = a_0 r$, $\bar{t} = ta_0/\bar{U}_{ref}$, $\bar{U}_{ref}$ = const.being a reference velocity. In particular, it can be the entrance or exit velocity on the axis or an average velocity conveniently selected. The flow dimensionless quantities are v_z, v_r, p and ψ defined as follows $\bar{v}_z = \bar{U}_{ref}v_z$, $\bar{v}_r = \bar{U}_{ref}v_r$, $\bar{\psi} = a_0^2\bar{U}_{ref}\psi$, $\bar{p} = \bar{\rho}\bar{U}_{ref}^2 p$, $\bar{\omega} = \omega\bar{U}_{ref}/a_0$.

The equations (2), (3) and (4) become, in dimensionless form:

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial\bar{r}}, \ v_r = -\frac{1}{r}\frac{a_0}{L}\frac{\partial\psi}{\partial z}, \ -\frac{\omega}{r} = \mathcal{L}(\psi), \tag{8}$$

$$\frac{\partial}{\partial t}\left(\mathcal{L}(\psi)\right) + v_z\frac{\partial}{\partial z}\left(\mathcal{L}(\psi)\right) + v_r\frac{\partial}{\partial\bar{r}}\left(\mathcal{L}(\psi)\right) = \frac{1}{\mathrm{Re}}\mathcal{L}\left(r^2\mathcal{L}(\psi)\right), \tag{9}$$

$\mathrm{Re} = a_0\bar{U}_{ref}/\bar{\nu}$ being the reference Reynolds number and $\mathcal{L}\cdot$ the dimensionless operator:

$$\mathcal{L}\cdot = \frac{a_0^2}{L^2}\frac{1}{r^2}\frac{\partial^2\cdot}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\cdot}{\partial r}\right). \tag{10}$$

3. INVISCID STEADY SOLUTIONS

The equation (9) has a particular solution $\mathcal{L}(\psi) = C$, C being an arbitrary constant. For $C = 0$ one obtains a potential flow: $\omega = 0$.

3.1. An exact steady solution for constant injection velocity

For u_w = const. and C = const one has the particular solution:

$$\psi_p = \frac{r^2}{2}\left(\alpha_0 + \alpha_1 z\right) + \frac{C}{8} r^4 . \tag{11}$$

The velocity components result as bellow:

$$v_{zp}(z,r) = \frac{1}{r}\frac{\partial \psi}{\partial r} = \alpha_0 + \alpha_1 z + \frac{C}{2} r^2 \,,\; v_{rp}(z,r) = -\frac{1}{r}\frac{a_0}{L}\frac{\partial \psi_p}{\partial z} = -\frac{r}{2}\frac{a_0}{L}\alpha_1 . \tag{12}$$

Considering a constant injection velocity u_w = const. from the second equation (12) and the boundary condition (6) yield $\alpha_1 = 2\frac{L}{a_0}u_w$, $v_{zp}(z,r) = \alpha_0 + 2\frac{L}{a_0}u_w z + \frac{C}{2}r^2$ and $v_{rp} = -r u_w$. The relation between the constants α_0 , C and the mean entrance velocity v_{0av} is:

$$v_{0av} = 2\int_0^1 \left(\alpha_0 + \frac{C}{2}r^2\right) r \mathrm{d}r = \left(\alpha_0 + \frac{C}{4}\right). \tag{13}$$

Two of quantities v_{0av}, α_0 or C should be specified. It turns out that a vorticity $\omega = -rC$ can be introduced at the entrance for $\alpha_0 \neq v_{0av}$. In particular, for $u_w = 0$ (no injection) and for $C = -2\alpha_0$ one obtains a Poiseuille – like flow:

$$\psi_p = \alpha_0\left(\frac{r^2}{2} - \frac{r^4}{4}\right),\; v_{zp} = \alpha_0(1 - r^2),\; v_{rp} = 0 , \tag{14}$$

Because in this case the fluid is assumed to be inviscid, the pressure losses are due the vortex.

3.2. Extension of potential solution for variable injection velocity

In order to solve the equation $\mathcal{L}(\psi) = C$ for variable injection velocity one uses the method of separation of variables. First, one introduces the stream function $\psi_1(z,r)$, defined by:

$$\psi = \psi_p + \psi_1 ,\; \mathcal{L}(\psi_p) = C ,\; \mathcal{L}(\psi_1) = 0 , \tag{15}$$

where ψ_p is given by (11), where u_w is replaced by the average injection velocity u_{wav}:

$$u_{wav} = \frac{a_0}{L}\int_0^1 u_w(z)\mathrm{d}z . \tag{16}$$

Obviously, the new unknown, ψ_1, satisfies the equation $\mathcal{L}(\psi_1) = 0$. The boundary conditions for this equation are:

$$z = 0 ,\; 0 < r \le 1 ,\quad \frac{1}{r}\frac{\partial \psi_1}{\partial r} = 0 ,\; r = 0 ,\; 0 < z < 1 ,\quad \frac{1}{r}\frac{\partial \psi_1}{\partial z} = 0 , \tag{17}$$

$$r = 1 ,\quad 0 < z < 1 ,\quad -\frac{1}{r}\frac{\partial \psi_1}{\partial z} = -\left(u_{wav} - u_w\right) ,\quad z = 1 ,\quad 0 < r < 1 ,\quad \frac{1}{r}\frac{\partial \psi_1}{\partial r} = 0 . \tag{18}$$

Taking:

$$\psi_1(z,r) = Z(z)R(r) , \tag{19}$$

introducing in $\mathcal{L}(\psi_1) = 0$ and separating variables, one obtains:

$$\left(\frac{a_0}{L}\right)^2 \frac{Z''}{Z} = -\frac{r}{R}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R'}{r}\right)\right] = -\lambda^2 , \tag{20}$$

λ being an arbitrary real constant. From the above relation, one obtains two ordinary differential equations. The general solution of the equation in the variable z is:

$$Z(z) = A_1 \cos\left(\lambda \frac{L}{a_0} z\right) + A_2 \sin\left(\lambda \frac{L}{a_0} z\right), \tag{21}$$

where A_1 and A_2 are constants. For the equation in the variable r one yields the solution:

$$R(r) = r\,(B_1 I_1(\lambda r) + B_2 K_1(\lambda r)), \tag{22}$$

where I_1 and K_1 are the modified Bessel functions, B_1 and B_2 being constants. Because $\lim_{r\to 0} K_1(r) = \infty$, one takes $B_2 = 0$. Imposing the entrance condition (17) one gives $A_1 = 0$ and relations (17) yield:

$$\psi(z,r) = \psi_p(z,r) + Ar \sin\left(\lambda \frac{L}{a_0} z\right) I_1(\lambda r), \tag{23}$$

A being a constant. *Taking the advantage that introducing the mean injection velocity* u_{wav} *all the flow rate at the exit* ($z = 1$) *is taken over by the velocity component* v_{zp} one obtains an equation in eigenvalues, $\sin(\lambda L / a_0) = 0$, wherefrom $\lambda_n L / a_0 = \pi n$, $n = 1, 2, \ldots$ .Therefore, a solution of the equation $\mathcal{L}(\psi) = C$ and obviously a solution of the general equation (9) is:

$$\psi(z,r) = \psi_p(z,r) + \sum_{n=1}^{\infty} A_n r I_1(\lambda_n r) \sin(\Lambda_n z),\ \lambda_n = \frac{\pi a_0}{L} n,\ \Lambda_n = \lambda_n \frac{L}{a_0} = n\pi, \tag{24}$$

$$v_z(z,r) = \alpha_0 + 2\frac{L}{a_0} u_{wav} z + 2(v_{0av} - \alpha_0) r^2 + \sum_{n=1}^{\infty} A_n \lambda_n I_0(\lambda_n r) \sin(\Lambda_n z), \tag{25}$$

$$v_r(z,r) = -r u_{wav} - \sum_{n=1}^{\infty} A_n r I_1(\lambda_n r) \cos(\Lambda_n z). \tag{26}$$

The coefficients A_n are determined from the injection condition $v_r(z,1) = -u_w(z)$ leading to the equation:

$$\sum_{n=1}^{\infty} A_n r I_1(\lambda_n r) \cos(\Lambda_n z) = u_w - u_{wav};\quad \frac{1}{2}\frac{1}{u_{wav}} A_n \lambda_n I_1(\lambda_n) = \int_0^1 (\beta_w - 1)\cos(\Lambda_n z)\,\mathrm{d}z,\quad \beta_w = \frac{u_w}{u_{wav}}, \tag{27}$$

where the property of orthogonal trigonometric functions $\cos(\Lambda_n z)$ was applied. In Table 1 the coefficients A_n are given for four cases of injection (column 2).

Table 1

Series coefficients A_n and $A_n^{(t)}$ for steady and unsteady flows

β_w	Steady	Unsteady, regression rate $u_r = \text{const.}$
	$\frac{1}{u_{wav}} A_n \lambda_n I_1(\lambda_n)$	$\frac{1}{u_{wav}(1+u_r t)} A_n \lambda_n I_1(\lambda_n (1+u_r t))$
1	0	0
$2z$	$-\frac{4}{\Lambda_n^2}\left[1-(-1)^n\right]$	$-\frac{4}{\Lambda_n^2}\left[1-(-1)^n\right]$
$2(1-z)$	$\frac{4}{\Lambda_n^2}\left[1-(-1)^n\right]$	$\frac{4}{\Lambda_n^2}\left[1-(-1)^n\right]$
$\frac{3}{2}(1-z^2)$	$-\frac{6}{\Lambda_n^2}(-1)^n$	$-\frac{6}{\Lambda_n^2}(-1)^n$

4. UNSTEADY INVISCID FLOW DUE TO THE WALL REGRESSION

One considers a regressing wall with constant velocity (due, for example, to the wall consumption), such that the distance a is:

$$a = a_0(1+u_r t)\,,\ u_r = \frac{\overline{u}_r}{\overline{U}_{ref}} = \text{const.}\,,\ |u_r| << 1\,. \tag{28}$$

First, one introduces a new variable, r_1, defined by: $r_1 = r/(1+u_r t)$, $r_1 \in [0,1]$, *the advantage being constant limits for* r_1. Then the operator $\mathcal{L}$ is denoted by $\mathcal{L}_1$ in the new variables:

$$\mathcal{L}_1 \cdot = \frac{1}{(1+u_r t)^4}\left[\frac{a^2}{L^2}\frac{1}{r_1^2}\frac{\partial^2 \cdot}{\partial z^2}\bigg|_{(r_1,t)} + \frac{1}{r_1}\frac{\partial}{\partial r_1}\left(\frac{1}{r_1}\frac{\partial \cdot}{\partial r_1}\right)\bigg|_{(z,t)}\right]. \tag{29}$$

If we denote the stream function for unsteady case by $\psi^{(t)}(z,r,t) \equiv \varphi(z,r_1,t)$, the equation (9) with v_z, v_r defined by (8) becomes:

$$\frac{\partial}{\partial t}\left(\mathcal{L}_1(\varphi)\right) - \frac{r_1}{1+u_r t}\frac{\partial}{\partial r_1}\left(\mathcal{L}_1(\varphi)\right) + v_z \frac{a_0}{L}\frac{\partial}{\partial z}\left(\mathcal{L}_1(\varphi)\right) = \frac{1}{\text{Re}}\mathcal{L}_1\left(r_1^2(1+u_r t)^2\mathcal{L}_1(\varphi)\right). \tag{30}$$

4.1. Wall regression for constant injection velocity

In this particular case, one looks for a solution φ_p, satisfying the equation $\mathcal{L}_1(\varphi_p) = C = \text{const.}$ The equation (29) for non viscous flow ($\text{Re} \to \infty$) is then satisfied. In comparison with the steady case (the second equation (15)), we have only to replace a_0 by a and C by $C^{(t)}$ given by $C^{(t)} = C(1+u_r t)^4$. Therefore, one obtains, in coordinates (z, r_1, t):

$$\varphi_p(z,r_1,t) = \frac{r_1^2}{2}(\alpha_0^{(t)} + \alpha_1^{(t)} z) + \frac{C^{(t)}}{8}r_1^4\,, \quad \text{or} \quad \psi_p^{(t)} = \frac{r^2(\alpha_0^{(t)} + \alpha_1^{(t)} z)}{2(1+u_r t)^2} + \frac{C}{8}r^4\,. \tag{31}$$

Thus the rotational term remains unmodified, but r has a time variable boundary. The velocities, in case of $u_w = const.$, are:

$$v_{zp}^{(t)} = \frac{\alpha_0^{(t)} + \alpha_1^{(t)} z}{(1+u_r t)^2} + \frac{C}{2}r^2\,,\ v_{rp}^{(t)} = -\frac{r}{2}\frac{a_0}{L}\frac{\alpha_1^{(t)}}{(1+u_r t)^2}\,. \tag{32}$$

The boundary condition $v_{rp}^{(t)} = -u_w$, for $r = 1+u_r t$, yields:

$$\alpha_1^{(t)} = 2\frac{L}{a_0}u_w(1+u_r t)\,,\ v_{rp}^{(t)} = -\frac{u_w r}{1+u_r t}\,, v_{zp}^{(t)} = \frac{\alpha_0^{(t)}}{(1+u_r t)^2} + 2\frac{L}{a_0}u_w\frac{z}{1+u_r t} + \frac{C}{2}r^2\,. \tag{33}$$

The mean entrance velocity $v_{0av}^{(t)}$ is:

$$v_{0av}^{(t)} = 2\int_0^1 v_z^{(t)} r\,\mathrm{d}r = \frac{\alpha_0^{(t)}}{(1+u_r t)^2} + \frac{C}{4}\,, \tag{34}$$

two of quantities $v_{0av}^{(t)}$, $\alpha_0^{(t)}$ and C being given.

4.2. Unsteady flow with constant wall regression due to the variable injection

As for steady flow, we consider first the injection solution for average injection velocity, u_{wav}, to use results of &3.1. Then one looks for a function $\psi_1^{(t)}(z,r,t)$ such that the general solution is the sum: $\psi^{(t)}(z,r,t) = \psi_p^{(t)}(z,r,t) + \psi_1^{(t)}(z,r,t)$. So one obtains $\mathcal{L}(\psi_1^{(t)}) = 0$.

For convenience, one introduces the function $\varphi_1(z,r_1 t)$, as $\psi_1^{(t)}(z,r,t) = \varphi(z,r_1,t)$ that should satisfy the equation (29), for $\varphi = \varphi_1$. If φ_1 fulfils the equation $\mathcal{L}_1(\varphi_1) = 0$, one refinds the equation $\mathcal{L}(\psi_1) = 0$ with r_1 and a instead of r and a_0. Therefore the solution has the form (24):

$$\psi^{(t)} = \psi_p^{(t)} + \sum_{n=1}^{\infty} A_n^{(t)} r_1 I_1(\lambda_n^{(t)} r_1)\sin(\Lambda_n z),\ \Lambda_n = \pi n,\ \lambda_n^{(t)} = \Lambda_n \frac{a_0(1+u_r t)}{L}. \tag{35}$$

Written in the initial variables, the function $\psi^{(t)}$ is:

$$\psi^{(t)} = \psi_p^{(t)} + \sum_{n=1}^{\infty} \frac{A_n^{(t)}}{1+u_r t} r I_1(\lambda_n r)\sin(\Lambda_n z), \tag{36}$$

λ_n, Λ_n being the same quantities as for the steady case.

Finally, besides division by $(1+u_r t)$, the solution coefficients $A_n^{(t)}$ contain $I_0(\lambda_n(1+u_r t))$ (Table 1, column 3). The velocities are:

$$v_z^{(t)} = v_{zp}^{(t)} + \sum_{n=1}^{\infty} \frac{A_n^{(t)}}{1+u_r t} \lambda_n I_0(\lambda_n r)\sin(\Lambda_n z),\ v_r^{(t)} = v_{rp}^{(t)} + \sum_{n=1}^{\infty} \frac{A_n^{(t)}}{1+u_r t} \lambda_n I_1(\lambda_n r)\cos(\Lambda_n z) \tag{37}$$

$$v_{zp}^{(t)} = \frac{\alpha_0}{(1+u_r t)^2} + 2\frac{L}{a_0} u_{wav} \frac{z}{1+u_r t} + \frac{C}{2} r^2,\ v_{rp}^{(t)} = -\frac{u_{wav} r}{1+u_r t},\ v_{0av}^{(t)} = \frac{\alpha_0}{(1+u_r t)^2} + \frac{C}{4}. \tag{38}$$

Here $\Lambda_n = \pi n$ and $\lambda_n = \Lambda_n \dfrac{a_0}{L}$. In (36), **(**38**)** given are two of the parameters α_0, C, $v_{0av}^{(t)}$, the last one representing the average velocity at entrance ($z = 0$). $A_n^{(t)}$ are given in Table 1.

5. VISCOUS SOLUTION FOR STEADY FLOW WITH CONSTANT INJECTION VELOCITY

By comparison with the flow in a long pipe ($L / a_0 >> 1$) where more information is available [1,2,9], one expects different solutions for laminar and turbulent regimes. We consider here the laminar case only.

5.1. The uniform steady injection in laminar flow

We assume as before a long channel ($L / a >> 1$) in order to neglect the ends effects. The equation to be solved is (9) with $\partial / \partial t = 0$ (steady flow).

One looks for a solution of the form:

$$\psi = (\alpha_0 + \alpha_1 z)\Phi(x),\ x \equiv r^2/2, \tag{39}$$

$$v_z = (\alpha_0 + \alpha_1 z)\Phi'(x),\ \alpha_0 = v_{zav0},\quad \alpha_1 = 2\frac{L}{a_0} u_w,\ v_r = -u_w \sqrt{2/x}\,\Phi(x), \tag{40}$$

where a new variable $x = r^2/2$ is introduced for convenience. By using the expressions (39), (40) the equation for steady flow becomes:

$$\mathrm{Re}_w \left(\Phi'\Phi'' - 2\Phi\Phi'''\right) = 2\Phi''' + x\Phi^{IV},\ \mathrm{Re}_w = \bar{u}_w a_0 / \bar{\nu} = u_w / \nu\ \left(\alpha_0 + \alpha_1 z\right) \neq 0, \tag{41}$$

the upper scripts indicating derivatives with respect to *x*. The equation (41) is a nonlinear ordinary differential equation of the fourth order. It will be integrated numerically with a Runge-Kutta method [7].

Remark. The multiplication by *x* of the highest order derivative in (41) introduces a difficulty in the numerical calculation at $x = 0$ (axis). However, the axis can be approached as much as necessary.

5.2. The boundary conditions

The choice of a particular form (41) of the solution implies a certain form of the boundary conditions. It is understood that at entrance ($z=0$) and at the exit ($z=1$), conditions are to be satisfied in connection with the expressions (41). These are then related to the conditions required for solving the differential equation (41), i.e. for the function $\Phi(x)$. These boundary conditions are:

$$r=1\ ,\ x=1/2\,,\ \Phi'(1/2)=0\ ,\ v_z(z,1)=0\,,\ r=1\ ,\ x=1/2\,,\ v_r(z,1)=-u_w\,,\ \Phi(1/2)=1/2\,, \tag{42}$$

$$r=0\,,\ x=0\ ,\ \frac{\partial v_z}{\partial r}=0\,,\frac{\partial v_z}{\partial x}=0\,,\ \Phi''(0)=0\,,\ r=0\,,\ x=0\ ,\ v_r=0\ ,\ \Phi(0)=0\,. \tag{43}$$

The second derivative $\Phi''(x)$ is important for calculation of the shear stress.

5.3. The method of solving

The solution of the differential equation (41) is searched in general, numerically, by using a fourth order Runge-Kutta method for an equivalent system of four differential equations. This is a *boundary value problem with two parameters,* as the four boundary conditions are equally shared between wall ($r=1$, $x=1/2$) and axis ($r=0$, $x=0$).

One can verify that even for $\mathrm{Re}_w = 0$, a singular term $x\ln x$ exists; it can be however removed in the limit $x\to+0$. In the numerical calculation, we shall construct the solution from two parts:

a) in an interval $r\in[r_\varepsilon\,,1]$, $x\in[\varepsilon,1/2]$, $\varepsilon=r_\varepsilon^2/2<<1$, one solves numerically the equation (41) for the boundary conditions (42), and for conditions near the axes that make connection with an inviscid solution $\Phi_{inv}(x)=\beta x$, β being a coupling constant. The conditions are:

$$r=r_\varepsilon\ ,\ x=\varepsilon=r_\varepsilon^2/2\ ,\ \Phi_a(\varepsilon)=\beta\varepsilon\,, r=r_\varepsilon\ ,\ x=\varepsilon=r_\varepsilon^2/2\ ,\ v_z=\beta(\alpha_0+\alpha_1 x)\,,\ \Phi_a'(\varepsilon)=\beta\,, \tag{44}$$

$$r=r_\varepsilon\ ,\ x=\varepsilon=r_\varepsilon^2/2\,,\ \frac{\partial v_z}{\partial r}=0\ ,\ \frac{\partial v_z}{\partial x}=0\ ,\ \Phi_a''(\varepsilon)=0\,. \tag{45}$$

b) in the interval $r\in[0,r_\varepsilon]$, $x\in[0,\varepsilon]$ the solution is $\Phi_a''(\varepsilon)=0$, $x\in[0,\varepsilon]$, $r\in\left[0;\sqrt{2\varepsilon}\right]$.

5.4. Numerical results

One takes a small value for $\varepsilon, \varepsilon=10^{-4}$, $r_\varepsilon=10^{-2}\sqrt{2}$ and one solves the differential equation for the five conditions (42) and (44). The corresponding five unknowns are the four arbitrary constants of the fourth order differential equation (41) and the coupling constant β.

The results are given in Table 2 for five Reynolds numbers Rew, including the Poiseuille flow (Rew = 0) for comparison. *As one can see, there is a natural variation of the parameters with Re_w.*

Table 2

Results for laminar flow

Re_w	$\beta=\Phi''(\varepsilon)$	$\Phi''(1/2)$	$\Phi'''(\varepsilon)$	$\varphi'''(r_\varepsilon)$
0; ε = 0	2.0000	–4.0000	0.0000	0.0000
0; ε = 10^{-4}	1.9970	–4.0003	–42138.0	-0.08428
1.0	1.9080	–4.2507	–33910.0	-0.06748
10.0	1.6880	–4.8063	–13214.0	-0.02650
100.0	1.5850	–4.9309	–2142.0	$-4.282\cdot10^{-3}$
1000.0	1.5743	–4.9402	–342.0	$-6.840\cdot10^{-4}$

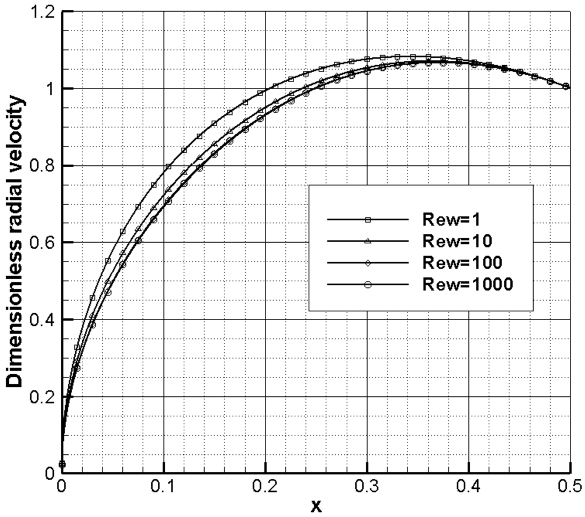

Fig. 1 – Radial velocity function of injection Reynolds number.

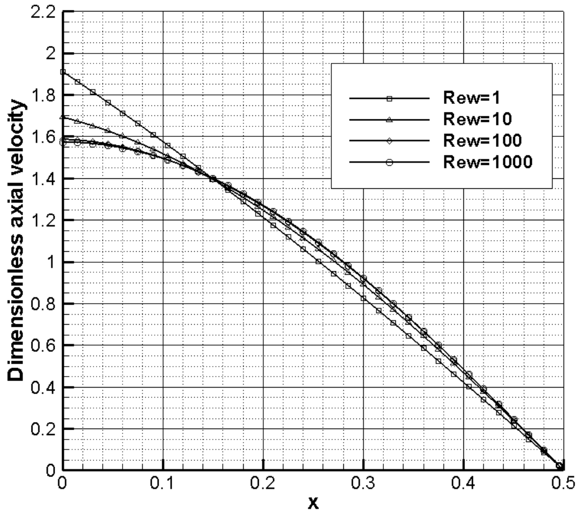

Fig. 2 – Axial velocity function of injection Reynolds number.

In Figures 1, 2 the variations of the dimensionless velocities are given. *The examples of calculations suggest not only qualitative but as well a reasonable order of magnitude for velocities[1;2].* For example, the injection Reynolds numbers, Re_w , are of the order $10^2 - 10^3$. The proposed method holds even for larger values.The increase of the axial velocity depends mainly on injection, in agreemen with our formulas.

6. CONCLUSIONS

The given solutions are important and useful both for theory and applications. From theoretical point of view, finding analytical solutions for variable injection speed, with possibility to include rotational effects in case of the non viscid flow and to reduce the system of partial differential Navier-Stokes equations for viscous flow to only one ordinary differential equation (41) gives an attractive useful model of calculation. One remarks the compactness of the proposed model based on a unique equation (9) able to yield a variety of solutions. As regards the applications, these are related, for example, to flow in rocket motors either solid or hybrid. A lot of experiments are done at present and simple effective methods of estimation of the possible interval of parameter variations are still looked for [1,2,3,8,9] ; a simple velocity field is useful for further estimations as the heat transfer. The incompressibility assumption is justified by the small Mach numbers reported experimentally [1,8,9]; anyhow a constant density solution is useful. The regression (do to the solid fuel consumption) is small (1–5 mm/sec according to [9]) Also, the used interval of Reynolds numbers makes the laminar solution interesting. Further analysis not presented here from space reasons proves that the proposed method permits extensions to turbulent regimes.

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