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AN ENHAMCED RUSSELL MEASURE OF EFFICIENCY IN THE PRESENCE OF NON-DISCRETIONARY FACTORES IN DATA ENVELOPMENT ANALYSIS

Ali ASHRAFI[*], Azmi B JAAFAR[**], Lai Soon LEE[**]

[*] University of Semnan, Faculty of Mathematics, Statistics and Computer Science, Semnan, Iran
[**] Universiti Putra Malaysia, Institute for Mathematical Research, 43400 UPM Serdang, Selangor, Malaysia
E-mail: ashrafiali76@yahoo.com

Standard data envelopment analysis (DEA) models suppose that all inputs and outputs can be varied at the discretion of management of each decision making unit (DMU). In some situations, DMUs can have non-discretionary inputs or outputs that are beyond the control of decision maker, which also need to be considered. This paper proposes an enhanced Russell measure (ERM) model in the presence of non-discretionary factors. The new model is compared with the well-known radial DEA model proposed by Banker and Morey [5]. An empirical data set is used to illustrate the model.

Key words: data envelopment analysis, efficiency, enhanced Russell measure, non-discretionary factor.

1. INTRODUCTION

Data envelopment analysis (DEA) was first introduced by Charnes et al. [1] as a non-parametric technique for evaluating relative efficiencies and performance of a set of related comparable entities, called Decision Making Units or DMUs, in converting inputs into outputs. Each DMU aims to develop performance by reducing inputs or raising outputs. There are two types of measure in DEA; radial measure and non-radial measure. Radial means that a proportional change of inputs and outputs is the main concern and hence it ignores the existence of slacks, whereas non-radial deals with slacks directly and the variations of inputs and outputs are not proportional. Radial measure was first proposed by Charnes et al. [1] (CCR model) and developed by Banker et al. [2] (BCC model). Also, non-radial measure was originally introduced by Färe and Lovell [3] (Russell measure model) and later revisited by Pastor et al. [4] (enhanced Russell measure model). There are many other radial and non-radial models in DEA.

In standard DEA models, it is assumed that all inputs and outputs of DMUs are discretionary, i.e., can be varied at the discretion of decision maker. However, DMUs can have non-discretionary (ND) inputs or outputs that the management of each DMUs cannot control them. For instance, snowfall or weather forecast in measuring the efficiency of maintenance units, the population of the area in measuring the efficiency of a library and age of store in the branch performance of a restaurant chain are not controllable by the user, but they are a part of the production process which needs to be considered. The first DEA model for measuring the efficiency with ND inputs was presented by Banker and Morey [5] and extended by Ruggiero [6], Syrjänen [7] and Muniz et al. [8] among others. In this study, we propose an enhanced Russell measure of efficiency in the presence of ND factors (ERM_{ND}). For this purpose, we first introduce Banker and Morey's model (BM model) and then extend the ERM model that takes into account the influence of ND inputs and outputs.

This paper is organized as follows. Section 2 briefly introduces the BM model. In Section 3, an ERM of efficiency is proposed that takes into account the influence of ND inputs and outputs. Also the relationship with the BM model is described in this section. Section 4 applies the proposed model to the public libraries in Tokyo studied in [9]. Conclusions follow in Section 5.

2. BANKER AND MOREY MODEL (BM MODEL)

Suppose there are n DMUs, each DMU_j (j = 1,…,n) consumes m discretionary inputs x_{ij} (I = 1,…,m) and p ND inputs z_{ij} (I = 1,…,p) to generate s discretionary outputs y_{rj} (r = 1,...,s) and q ND outputs w_{rj} (r = 1,...,q). We assume that all inputs and outputs are positive. Vectors $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^T$ and $z_j = (z_{1j}, z_{2j}, \ldots, z_{pj})^T$ show discretionary and ND input of DMU_j respectively. Also vectors $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T$ and $w_j = (w_{1j}, w_{2j}, \ldots, w_{qj})^T$ show discretionary and ND output of DMU_j respectively. We denote the DMU_j by (x_j, z_j, y_j, w_j) (j = 1,...,n).

The production possibility set P_C in the presence of ND inputs and outputs under the constant returns to scale (CRS) assumption is defined as:

$$P_C = \left\{ (x,z,y,w) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j,\ z \geq \sum_{j=1}^{n} \lambda_j z_j,\ y \leq \sum_{j=1}^{n} \lambda_j y_j,\ w \leq \sum_{j=1}^{n} \lambda_j w_j,\ \lambda \geq 0 \right\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{R}^n$ is the intensity vector.

Banker and Morey [5] proposed the first DEA model to measure the efficiency in the presence of ND inputs and outputs. This is still a well-known and widely applied radial model to handle ND factors. Under the CRS assumption the input-oriented BM model for measuring the efficiency of $\text{DMU}_k\ (k = 1, \ldots, n)$ is indicated as follows:

$$\begin{aligned}
&\theta^*_{BM} = \min\ \theta_{BM} = \theta - \varepsilon \left(\sum_{i \in D_I} s_i^- + \sum_{r \in D_O} s_r^+ \right) \\
&\text{s.t}\ \ \theta x_{ik} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^-,\ \ i \in D_I,\ \ z_{ik} = \sum_{j=1}^{n} \lambda_j z_{ij} + s_i^-,\ \ i \in ND_I, \\
&\quad y_{rk} = \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+,\ \ r \in D_O,\ \ w_{rk} = \sum_{j=1}^{n} \lambda_j w_{rj} - s_r^+,\ \ r \in ND_O, \\
&\quad \lambda_j,\ s_i^-,\ s_r^+ \geq 0,\ \ \forall i,\ r,\ j,\ \ \theta\ \text{unrestricted},
\end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a non-Archimedean infinitesimal constant. Here the symbols D_I, ND_I and D_O, ND_O refer to discretionary and ND, respectively, for the inputs and outputs. Note that the real variable θ is associated only with discretionary inputs. Also, only the discretionary input-slacks and discretionary output-slacks get involved in the objective function.

Definition 2.1. (BM-efficiency). A DMU_k $(k = 1, \ldots, n)$ is BM-efficient if and only if both of the following conditions are satisfied: (i) $\theta^* = 1$; (ii) all slacks in the objective function are zero, i.e., $s_i^- = 0\ (\forall i \in D_I)$ and $s_r^+ = 0\ (\forall r \in D_O)$.

3. ERM MODEL FOR THE CASE OF ND FACTORS

Färe and Lovell [3] introduced a non-radial model, which they called the Russell measure model. Pastor et al. [4] revisited the Russell measure model and proposed a new measure called the enhanced Russell Measure (ERM). In an effort to estimate the efficiency of $\text{DMU}_k = (x_k, z_k, y_k, w_k)$, in the presence of ND factors, we suggest the following DEA model (referring to it as the ERM_{ND} model):

$$\rho_{ND}^{*} = \min \ \rho_{ND} = \frac{\frac{1}{m}\sum_{i\in D_I}\theta_i}{\frac{1}{s}\sum_{r\in D_O}\phi_r}$$

$$\begin{aligned} \text{s.t} \ \ & \theta_i x_{ik} \ge \sum_{j=1}^{n}\mu_j x_{ij}, \quad i\in D_I; \ \ z_{ik} \ge \sum_{j=1}^{n}\mu_j z_{ij}, \quad i\in ND_I; \ \ \phi_r y_{rk} \le \sum_{j=1}^{n}\mu_j y_{rj}, \quad r\in D_O; \\ & w_{rk} \le \sum_{j=1}^{n}\mu_j w_{rj}, \quad r\in ND_O; \quad 0\le\theta_i\le 1, \ i\in D_I; \quad \phi_r\ge 1, \ \ r\in D_O, \ \ \mu=(\mu_1,\ldots,\ \mu_n)\ge 0. \end{aligned} \tag{2}$$

Note that the variables θ_i and ϕ_r are not applied for the ND inputs and the ND outputs, respectively. In the objective function of model (2) the numerator expresses the average efficiency of discretionary inputs and the denominator expresses the average efficiency of discretionary outputs. Therefore, the objective function can be interpreted as the ratio between the average efficiency of discretionary inputs and the average efficiency of discretionary outputs. It can be verified that $0<\rho_{ND}\le 1$. The ERM_{ND} model satisfies such properties as unit invariance and monotone decreasing for any increase in discretionary input usage or any decrease in discretionary output production. Let $(\mu^*,\ \theta_1^*,\ldots,\ \theta_m^*,\ \phi_1^*,\ldots,\ \phi_s^*)$ be an optimal solution of model (2). We define a DMU is ERM_{ND}-efficient as follows.

Definition 3.1. (**ERM_{ND}**-efficiency). A DMU_k $(k=1,\ldots,n)$ is ERM_{ND}-efficient if and only if $\rho_{ND}^*=1$. This condition is equivalent to $\theta_i^*=1$ $(\forall i\in D_I)$ and $\phi_r^*=1$ $(\forall r\in D_O)$ in any discretionary optimal solution.

Note that model (2) is a nonlinear programming problem that can be converted into a linear programming problem by using the approach in [10]. For this purpose, we define a new variable β in such a way that

$$\beta=\left(\frac{1}{s}\sum_{r\in D_O}\phi_r\right)^{-1}.$$

It is obvious that $0<\beta\le 1$ and $\beta\left(\frac{1}{s}\sum_{r\in D_O}\phi_r\right)=1$. We then Multiply the numerator and denominator in the objective of model (2) by this β. Also, we multiply both sides of the constraints in model (2) by this variable. Now, we set

$$\begin{aligned} & u_i=\beta\theta_i, \quad i\in D_I, \\ & v_r=\beta\phi_r, \quad r\in D_O, \\ & t_j=\beta\mu_j, \quad j=1,\ldots,n. \end{aligned}$$

Then, the model (2) becomes the following linear programming problem:

$$\rho_{ND}^{*} = \min \ \frac{1}{m}\sum_{i\in D_I}u_i$$

$$\begin{aligned} \text{s.t} \ \ & \frac{1}{s}\sum_{r\in D_O}v_r=1; \ \ u_i x_{ik}\ge\sum_{j=1}^{n}t_j x_{ij}, \quad i\in D_I; \ \ \beta z_{ik}\ge\sum_{j=1}^{n}t_j z_{ij}, \quad i\in ND_I; \ \ v_r y_{rk}\le\sum_{j=1}^{n}t_j y_{rj}, \quad r\in D_O, \\ & \beta w_{rk}\le\sum_{j=1}^{n}t_j w_{rj}, \quad r\in ND_O; \quad u_i\le\beta\le v_r, \ i\in D_I, r\in D_O; \quad 0\le t_j, \quad 0\le\beta\le 1, \quad j=1,\ldots,\ n. \end{aligned} \tag{3}$$

The relationship between the BM model and the ERM_{ND} model is demonstrated by the two following theorems.

THEOREM 3.1. *The optimal* ρ^*_{ND} *is not greater than the optimal* θ^*_{BM}.

Proof. Suppose $(\lambda^*, \theta^*_{BM})$ is an optimal solution of model (1). We define

$$\hat{\mu} = \lambda^*,$$
$$\hat{\theta}_i = \theta^*_{BM}, \quad i \in D_I,$$
$$\hat{\phi}_r = 1, \qquad r \in D_O.$$

Then $(\hat{\mu}, \hat{\theta}_1, ..., \hat{\theta}_m, \hat{\phi}_1, ..., \hat{\phi}_s)$ is a feasible solution for model (2). Therefore,

$$\rho^*_{ND} \leq \frac{\frac{1}{m}\sum_{i \in D_I} \hat{\theta}_i}{\frac{1}{s}\sum_{r \in D_O} \hat{\phi}_r} = \frac{\frac{1}{m}(m\theta^*_{BM})}{\frac{1}{s}(s)} = \theta^*_{BM}.$$

THEOREM 3.2. *A* DMU_k $(k = 1, ..., n)$ *is BM-efficient if and only if it is* ERM_{ND}*-efficient.*

Proof. Suppose that DMU_k is ERM_{ND}-inefficient and $(\mu^*, \theta^*_1, ..., \theta^*_m, \phi^*_1, ..., \phi^*_s)$ is an optimal solution for model (2) with $\rho^*_{ND} < 1$. Since, $\theta^*_i \leq 1, \forall i \in D_I$ and $\phi^*_r \geq 1, \forall r \in D_O$, we have

$$\exists t \in D_I, \ \theta^*_t < 1, \quad \vee \quad \exists q \in D_O, \ \phi^*_q > 1.$$

Without losing the generality, suppose

$$\exists t \in D_I, \ \theta^*_t < 1.$$

Then, from model (2) we have

$$\sum_{j=1}^{n} \mu^*_j x_{ij} \leq \theta^*_i x_{ik} \leq x_{ik}, \ i \in D_I - \{t\}; \qquad \sum_{j=1}^{n} \mu^*_j x_{tj} \leq \theta^*_t x_{ik} < x_{tk},$$
$$\sum_{j=1}^{n} \mu^*_j z_{ij} \leq z_{ik}, \ i \in ND_I; \qquad \sum_{j=1}^{n} \mu^*_j y_{rj} \geq \phi^*_r y_{rk}, \ r \in D_O; \qquad \sum_{j=1}^{n} \mu^*_j w_{rj} \geq w_{rk}, \ r \in ND_O.$$

Therefore, we have a feasible solution of model (1) with $\theta_{BM} < 1$. Hence, DMU_k is BM-inefficient.

On the other hand, suppose that DMU_k is BM-inefficient and $(\lambda^*, \theta^*_{BM})$ be an optimal solution of model (1) with $\theta^*_{BM} < 1$. Then, from the constraints of model (1) we have

$$\exists t \in D_I, \ \sum_{j=1}^{n} \lambda^*_j x_{tj} < x_{tk}, \quad \vee \quad \exists q \in D_O, \ \sum_{j=1}^{n} \lambda^*_j y_{qj} > y_{qk}.$$

Without losing the generality, suppose that

$$\exists t \in D_I, \ \sum_{j=1}^{n} \lambda^*_j x_{tj} < x_{tk}.$$

We define $\hat{\mu} = \lambda^*, \quad \hat{\theta}_i = 1, \quad i \in D_I - \{t\}, \quad \hat{\theta}_t = \dfrac{\sum_{j=1}^{n} \lambda^*_j x_{tj}}{x_{tk}}, \quad \hat{\phi}_r = 1, \quad r \in D_O.$

Then, $(\hat{\mu}, \hat{\theta}_1, ..., \hat{\theta}_m, \hat{\phi}_1, ..., \hat{\phi}_s)$ is a feasible solution for model (2) and we have

$$\rho_{ND} = \frac{\frac{1}{m}\sum_{i\in D_I}\hat{\theta}_i}{\frac{1}{s}\sum_{r\in D_O}\hat{\phi}_r} = \frac{\frac{1}{m}(\hat{\theta}_t + \sum_{i\in D_{I-\{t\}}} 1)}{\frac{1}{s}\sum_{r\in D_O} 1} = \frac{\frac{1}{m}(\hat{\theta}_t + m - 1)}{1} < 1 \ \ (\text{since } \hat{\theta}_t < 1).$$

Therefore, we have a feasible solution of model (2) with $\rho_{ND} < 1$. Hence, DMU_k is ERM_{ND} -inefficient. □

4. NUMERICAL EXAMPLE

In this section, the proposed model is used for measuring the efficiencies of 23 public libraries in Tokyo provided by Cooper et al. [9]. Data set is presented in Table 1.

Table 1
Data set

Library	Inputs				Outputs	
	Area	Book	Staff	Population	Register	Borrow
L1	2249	163523	26	49196	5561	105321
L2	4617	338671	30	78599	18106	314682
L3	3873	281655	51	176381	16498	524349
L4	5541	400993	78	189397	30810	847872
L5	11381	363116	69	192235	57279	758704
L6	10086	541658	114	194091	66137	1438746
L7	5435	508141	61	228535	35295	839597
L8	7524	338804	74	238691	33188	540821
L9	5077	511467	84	267385	65391	1562274
L10	7029	393815	68	277402	41197	978117
L11	11121	509682	96	330609	47032	930437
L12	7072	527457	92	332609	56064	1345185
L13	9348	601594	127	356504	69536	1164801
L14	7781	528799	96	365844	37467	1348588
L15	6235	394158	77	389894	57727	1100779
L16	10593	515625	101	417513	46160	1071488
L17	10866	566708	118	503914	102967	1707645
L18	6500	467617	74	517318	47236	1223026
L19	11469	768484	103	537746	84510	2299694
L20	10868	669996	107	590601	69576	1901465
L21	10717	844949	120	622550	89401	1909698
L22	19716	1258981	242	660164	97941	3055193
L23	10888	1148863	202	808369	191166	4096300

Source: Cooper et al. (2000)

Each library is associated with four inputs: area, book, staff and population, and two outputs: register and borrow. Note that in this example the population is ND input. Table 2 reports the efficiency scores calculated by the ERM_{ND} model and the BM model, where there are six libraries, namely, L5, L6, L9, L17, L19 and L23 which are fully efficient. As we can see in Table 2, according to Theorem 3.1 the efficiency scores of DMUs calculated by the ERM_{ND} model are not greater than the efficiency scores calculated by BM model. Also, the BM-efficient DMUs remained at the efficient status under ERM_{ND} assessments, as stated by Theorem 3.2.

Table 2

Efficiency Scores

Library	ρ^*_{ND}	θ^*_{BM}	Library	ρ^*_{ND}	θ^*_{BM}
L1	0.1787	0.2242	L13	0.4963	0.6461
L2	0.3530	0.6195	L14	0.4649	0.7125
L3	0.3780	0.5391	L15	0.6905	0.8440
L4	0.4482	0.5920	L16	0.4398	0.5816
L5	1.0000	1.0000	L17	1.0000	1.0000
L6	1.0000	1.0000	L18	0.6185	0.7861
L7	0.4906	0.6423	L19	1.0000	1.0000
L8	0.3784	0.5381	L20	0.6262	0.8470
L9	1.0000	1.0000	L21	0.6318	0.7735
L10	0.5616	0.7045	L22	0.4656	0.6756
L11	0.4202	0.5385	L23	1.0000	1.0000
L12	0.6107	0.7184			

5. CONCLUSIONS

In this paper, we have modified the ERM model of Pastor et al. [4] and proposed a DEA model in the presence of non-discretionary factors (ND). In opposition to the BM model, which is a radial input-oriented (output-oriented) model, the ERM_{ND} model is a non-radial non-oriented model. The new measure can be interpreted as the ratio between the average efficiency of discretionary inputs and the average efficiency of discretionary outputs that are useful to explain the efficiency of the DMU under estimation. We have demonstrated that the efficiency score calculated by the ERM_{ND} model is not greater than the efficiency score by the BM model. Also a DMU is ERM_{ND}-efficient if and only if it is BM-efficient. In addition, an empirical data set from the literature has been used to compare our approach with the BM approach. Finally, the discussion in this paper is based upon the constant returns to scale (CRS) assumption. The ERM_{ND} model can be developed under the assumption of variable returns to scale (VRS) by inserting the convexity constraint, namely $\Sigma_{j=1}^{n}\lambda_j = 1$, into the model.

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