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CONVOLUTION ON WEIGHTED L^p-SPACES OF LOCALLY COMPACT GROUPS

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Let G be a locally compact group and $2 < p < \infty$. We have recently considered the property that convolutions of functions in the L^p-space of G exist, and have shown that this is equivalent to compactness of G. Here, we study this property on the weighted L^p-space of G; as the main result, we prove that G is σ-compact if convolutions of functions in the weighted L^p-space of G exist.

Key words: convolution, L^p-space, locally compact group, weight function.

1. INTRODUCTION

Throughout the paper, let G be a locally compact group with a fixed left Haar measure λ, and let ω be a weight function on G; that is, a measurable real valued function on G such that $\omega(x) > 0$ and $\omega(xy) \le \omega(x)\ \omega(y)$ for all $x, y \in G$. For $1 \le p < \infty$, let $L^p(G,\omega)$ denote the Banach space of all functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [8]; we denote this space by $\ell^p(G,\omega)$ when G is discrete. For measurable functions f and g on G, the convolution

$$f * g = \int_G f(y) g(y^{-1}x) \mathrm{d}\lambda(y)$$

is defined at each point $x \in G$ for which this makes sense; i.e., the function $y \mapsto f(y)g(y^{-1}x)$ is λ-integrable. Then $f * g$ is said to exist as a function if $f * g(x)$ exists for almost all $x \in G$. Convolution has applications in various fields such as statistics, computer vision, numerical analysis, numerical linear algebra, signal processing, electrical engineering, and differential equations. The convolution $f * g$ does not necessarily exist for all measurable functions f and g. So, it would be interesting to know when does $f * g$ exist for all functions f and g in a space X of measurable functions on G. If this is the case, then it is desirable to study the closeness of X under the convolution. Several authors have been studied the existence of convolution on certain function spaces; see for example the authors [1]. It is well-known that $L^1(G)$ is always closed under the convolution. Saeki [20] proved that, for $1 < p < \infty$, the space $L^p(G)$ is closed under the convolution if and only if G is compact; see also Crombez [4–5], Johnson [9], Kunze [11], Lohoue [12], Milnes [13], Rajagopalan [14–17], Rickert [18–19], Urbanik [21], Zelazko [22–24], for some special cases, and Kinani, Benazzouz [7] and the authors [2] and [3] for the more general case of weighted L^p-space; see also Kitada and Yang [10]. But the convolution of elements in even does not exist in general. In fact, we have proved in [1] that, for $2 < p < \infty$, the convolution $f * g$ exists for all $f, g \in L^p(G)$ if and only if G is compact. In this paper, we investigate this property for the weighted space $L^p(G, \omega)$ and give some necessary or sufficient conditions for that the property holds.

2. CONVOLUTION ON L^p (G, Ω)

A weight function ω on G is called symmetric if $\omega = \tilde{\omega}$, where $\tilde{\omega}(x) = \omega(x^{-1})$, for all $x \in G$; note that the weight function $\omega^* = \omega\tilde{\omega}$ is symmetric. Our first result shows that if there is a weight function ω such that $f * g$ exists for all $f, g \in L^p(G, \omega)$, then there is a symmetric weight function on G with the same property.

LEMMA 2.1. *Let G be a locally compact group, ω be a weight function on G, and $1 < p < \infty$. If $f * g$ exists for all $f, g \in L^p(G, \omega)$, then $f * g$ exists for all $f, g \in L^p(G, \omega^*)$.*

Proof. Let $f, g \in L^p(G, \omega^*)$ be positive. Then $f\tilde{\omega}, g\tilde{\omega} \in L^p(G, \omega)$ and $f\tilde{\omega} * g\tilde{\omega} \geq (f * g)\tilde{\omega}$, almost everywhere. It follows that $f * g$ exists.

Our next result is indeed the main result of the paper.

THEOREM 2.2. *Let G be a locally compact group, ω be a symmetric weight function on G* and $2 < p < \infty$. *If $f * g$ exists for all $f, g \in L^p(G, \omega)$, then $\omega^{-1}(F)$ is contained in a compact subset of G for all compact subset F* of $[1, \infty)$.

Proof. We only need to prove that $\omega^{-1}([1, m])$ is contained in a compact subset of G for all natural numbers $m \geq 2$. To that end, suppose toward a contradiction that there is $m_0 \geq 2$ such that $\omega^{-1}([1, m_0])$ is not contained in any compact subset of G. Fix a compact symmetric neighborhood U of the identity element e of G, and find an element s_1 of $\omega^{-1}([1, m_0])$ with $s_1 \notin s_0 \mathrm{U}^4$, where $s_0 = e$. Since ω is symmetric, we can assume $\Delta(s_1) \leq 1$, where Δ is the modular function of G. We therefore may find a sequence $(s_k)_{k \geq 1}$ in $\omega^{-1}([1, m_0])$ such that $\Delta(s_k) \leq 1$ and

$$s_k \notin s_1 \mathrm{U}^4 \cup \ldots \cup s_{k-1} \mathrm{U}^4 \qquad (k \geq 2).$$

For each $x \in G$, set

$$f(x) = \frac{1}{k^{1/2}} \Delta(x^{-1})^{1/p} \omega(x)^{-1},$$

if $x \in \mathrm{U} s_k^{-1}$ for some $k \geq 1$ and $f(x) = 0$ otherwise. It is not hard to see that the sets $\mathrm{U} s_1^{-1}, \mathrm{U} s_2^{-1}, \ldots$ are pairwise disjoint, hence this formula defines a function f on G. We show that $f \in L^p(G, \omega)$. To see this, we note that

$$\int_G |f(x)|^p \omega(x)^p \, d\lambda(x) = \sum_{k=1}^{\infty} \frac{1}{k^{p/2}} \int_G \Delta(x^{-1}) \chi_U(x s_k) \, d\lambda(x) = \sum_{k=1}^{\infty} \frac{1}{k^{p/2}} \Delta(s_k^{-1}) \int_G \Delta(s_k x^{-1}) \chi_U(x) \, d\lambda(x)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^{p/2}} \int_U \Delta(x^{-1}) \, d\lambda(x).$$

Since U is compact and Δ is continuous, it follows that $f \in L^p(G, \omega)$. A similar argument implies that if for each $x \in G$ we set

$$g(x)=\frac{1}{k^{1/2}}\,\omega(x)^{-1},$$

when $x\in s_k\mathrm{U}^2$ for some $k\geq 1$ and $g(x)=0$ otherwise, then $g\in L^p(G,\omega)$. Furthermore, U is compact and so by [6, Proposition 1.16], there is a constant $\mathrm{M}>0$ such that $\omega(x)\leq\mathrm{M}$ for all $x\in\mathrm{U}$. We thus conclude that for every $x\in\mathrm{U}$,

$$(f*g)(x)=\int_G f(y)g(y^{-1}x)\mathrm{d}\lambda(y)=\sum_{k=1}^{\infty}\frac{1}{k}\int_{Us_k^{-1}}\Delta(y^{-1})^{1/p}\,\omega(y)^{-1}\,\omega(y^{-1}x)^{-1}\,\mathrm{d}\lambda(y)\geq$$

$$\geq\omega(x)^{-1}\sum_{k=1}^{\infty}\frac{1}{k}\int_{Us_k^{-1}}\Delta(y^{-1})^{1/p}\,\omega(y)^{-2}\,\mathrm{d}\lambda(y)\geq\frac{1}{m_0M}\,\omega(x)^{-1}\sum_{k=1}^{\infty}\frac{1}{k}\int_{Us_k^{-1}}\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y).$$

Consequently

$$\int_{Us_k^{-1}}\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y)=\Delta(s_k^{-1})\int_U\Delta(s_ky^{-1})^{1/p}\,\mathrm{d}\lambda(y)=\Delta(s_k^{-1})\int_U\Delta(s_k)^{1/p}\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y)=$$

$$=\Delta(s_k^{-1})^{1-1/p}\int_U\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y)\geq\int_U\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y),$$

where the last inequality follows from $\Delta(s_k)\leq 1$ and $1-\frac{1}{p}>0$ and hence $\Delta(s_k^{-1})^{1-1/p}\geq 1$ for all $k\geq 1$. These all imply that for every $x\in\mathrm{U}$,

$$(f*g)(x)\geq\frac{1}{M^2}\,\omega(x)^{-1}\sum_{k=1}^{\infty}\frac{1}{k}\int_{Us_k^{-1}}\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y)\geq\frac{1}{M^2}\,\omega(x)^{-1}\sum_{k=1}^{\infty}\frac{1}{k}\int_U\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y).$$

Now, since the interior of U is nonempty, we get

$$\int_U\Delta(y^{-1})^{1/p}\,\mathrm{d}\lambda(y)>0,$$

and thus $(f*g)(x)=\infty$ for all $x\in\mathrm{U}$; that is, $f*g$ does not exist, a contradiction.

As two consequences of Theorem 2.2, we have the following corollaries.

COROLLARY 2.3. *Let G be a locally compact group, ω be a symmetric continuous weight function on G and $2<p<\infty$. If $f*g$ exists for all $f,g\in L^p(G,\omega)$, then $\omega^{-1}([1,m])$ is compact, for all $m\in\mathbb{N}$.*

Proof. Since $[1,m]$ is a compact subset of $[1,\infty)$, then the result follows by Theorem 2.2.

COROLLARY 2.4. *Let G be a locally compact group, ω be a weight function on G such that ω^* is bounded from above, and $2<p<\infty$. Then $f*g$ exists for all $f,g\in L^p(G,\omega)$ if and only if G is compact.*

Proof. Lemma 2.1 implies that $f*g$ exists for all $f,g\in L^p(\mathrm{G},\omega^*)$. Since ω^* is bounded, it follows that $L^p(G,\omega^*)=L^p(G)$. Now the result is concluded by [1, Theorem 1.1].

As the main consequence of Lemma 2.1 and Theorem 2.2, we have the following result.

THEOREM 2.5. *Let G be a locally compact group, ω be a weight function on G and $2<p<\infty$. If $f*g$ exists for all $f,g\in L^p(\mathrm{G},\omega)$, then G is σ-compact.*

Proof. By Lemma 2.1, $f*g$ exists for all $f,g\in L^p(G,\omega^*)$. Because ω^* is a symmetric weight function on G, then for each $m\in\mathbb{N}$, $(\omega^*)^{-1}([1,m])$, is a compact subset of G by Corollary 2.3. Since $G=\bigcup_{m=1}^{\infty}(\omega^*)^{-1}([1,m])$, then the result is obtained.

Remark 2.6. (a) Let us recall that Theorem 2.5 does not remain true for $1<p<\infty$. In fact, if G is an arbitrary discrete group, ω is a weight function on G and $1<p\le 2$, then $f*g$ exists for all $f,g\in\ell^p(G,\omega)$.

(b) The converse of Theorem 2.5 is not valid even for discrete groups. For example, consider the additive group $\mathbb{Z}$ and define the weight function

$$\omega(n)=(1+|n|)^{1/4}\qquad (n\in\mathbb{Z}).$$

Consider the space $\ell^p(\mathbb{Z},\omega)$ for $4<p<\infty$ and the function $f\in\ell^p(\mathbb{Z},\omega)$ on $\mathbb{Z}$ defined by $f(n)=(1+|n|)^{-1/2}$ for all $n\in\mathbb{Z}$, and note that $(f*f)(0)$ does not exist.

(c) Let G be a locally compact group and $2<p<\infty$. We have recently shown that $f*g$ exists for all $f,g\in L^p(G)$ if and only if G is compact. However, this result is not true for the weighted case in general; indeed, if ω is the weight function on the discrete group $\mathbb{Z}$ defined by

$$\omega(n)=(1+|n|)^{2/q}\qquad (n\in\mathbb{Z}),$$

then $f*g$ exists for all $f,g\in\ell^p(\mathbb{Z},\omega)$, where $q=\frac{p}{p-1}$ is the exponential conjugate of p. This means that σ-compactness in Theorem 2.5 can not be replaced by compactness.

PROPOSITION 2.7. *Let G be a locally compact group and ω be a weight function G with $\omega^{-1}\in L^q(G)$, where $q=\frac{p}{p-1}$. Then $f*g$ exists for all $f,g\in L^p(G,\omega)$.*

Proof. It follows from the Holder inequality that $L^p(G,\omega)\subseteq L^1(G)$ if $\omega^{-1}\in L^q(G)$. So, the result follows from the fact that $f*g$ exists for all $f,g\in L^1(G)$.

Remark 2.8. (a) Let G be a locally compact group and ω be a weight function on G. Clearly, the topology of G plays an important role in the study of convolution on $L^p(G,\omega)$. For example, $f*g$ exists for all $f,g\in L^p(\mathbb{R},\omega_\alpha)$, where $\alpha>\frac{1}{q}$ and $\omega_\alpha(x)=(n+1)^\alpha$ for $x\in[n-1,n]\cup[-n,-n+1]$ and $n\ge 1$; indeed, $\omega_\alpha^{-1}\in L^q(\mathbb{R})$ whereas $f*g$ does not exist for some $f,g\in\ell^p(\mathbb{R},\omega_\alpha)$.

(b) The converse of Proposition 2.7 is not valid. For example consider the weighted space $\ell^4(\mathbb{Z},\omega)$, where ω is a weight function on $\mathbb{Z}$ defined by $\omega(n)=(1+|n|)^{1/2}$ for all $n\in\mathbb{Z}$. Then $f*g$ exists for all $f,g\in\ell^4(\mathbb{Z},\omega)$ whereas $\omega^{-1}\notin\ell^{4/3}(\mathbb{Z})$.

In the following result, for $g:G\mapsto\mathbb{C}$ we set $\tilde{g}(x)=g(x^{-1})$ for all $x\in G$.

PROPOSITION 2.9. *Let G be a discrete group, ω be a weight function on G and $2<p<\infty$. Then $f*g$ exists for all $f,g\in\ell^p(G,\omega)$ if and only if $\ell^p(G,\omega)\,\ell^p(G,\tilde{\omega})\subseteq\ell^2(G)$.*

Proof. We only need to note that

$$(f*g)(x)=\sum_{y\in G} f(xy)\tilde{g}(y),$$

for all $f,g\in \ell^p(G,\omega)$ and $x\in G$.

COROLLARY 2.10. *Let* G *be a discrete group,* $2<p<\infty$ *and* ω *be a symmetric weight function on* G*. Then* $f*g$ *exists for all* $f,g\in \ell^p(G,\omega)$ *if and only if* $\ell^p(G,\omega)\subseteq \ell^2(G)$

Our observations in Theorem 2.5 and Remark 2.6 lead us to the following questions.

Question 1. Let G be a locally compact group, ω be a weight function on G and $1<p\le 2$. Does $f*g$ exists for all $f,g\in L^p(G,\omega)$?

Question 2. Let $2<p<\infty$. For which σ-compact groups and weight functions ω on G, $f*g$ exists for all $f,g\in L^p(G,\omega)$?

It was also pointed out to us by the referee that a Lie group (in fact, any smooth manifold) is σ-compact if and only if has countably many connected components. So in particular Theorem 2.5 says nothing in the case when G is a connected Lie group, for instance additive group $\mathbb{R}^n$. Therefore the following question arises naturally.

Question 3. Let $1<p<\infty$. For which connected Lie groups and weight functions ω on G, $f*g$ exists for all $f,g\in L^p(G,\omega)$?

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