THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 13, Number 2/2012, pp. 103–108

TOPOLOGICAL AND NON-TOPOLOGICAL SOLITON SOLUTIONS OF THE BRETHERTON EQUATION

Houria TRIKI [1], Ahmet YILDIRIM [2], T. HAYAT [3], Omar M. ALDOSSAR [4], Anjan BISWAS [5]

[1] Badji Mokhtar University, Department of Physics, 2300 Anaba, ALGERIA
[2] Ege University, Department of Mathematics, 35100 Bornova, Izmir, TURKEY,
University of South Florida, Department of Mathematics and Statistics, Tampa, FL 33620-5700, USA
[3] Qaid-i-Azam University, Department of Mathematics, Islamabad-44000, PAKISTAN,
[4] King Saud University, Department of Physics, P. O. Box 2455, Riyadh-11451, SAUDI ARABIA
[5] Delaware State University, Department of Mathematical Sciences, Dover, DE 19901-2277, USA
E-mail : biswas.anjan@gmail.com, ahmetyildirim80@gmail.com

This paper studies the topological and non-topological 1-soliton solutions to the Bretherton equation. The solitary wave ansatz method, also known as the trial solution method, is used to carry out the integration of the this equation. The 1-soliton solution as well as the shock wave solution is retrieved using this method. The constraint relation between the parameters and coefficients are also obtained for the existence of these kinds of nonlinear waves.

Key words: evolution equations; solitons; integrability.

1. INTRODUCTION

Bretherton equation (BE) is one of the recently studied nonlinear evolution equations (NLEEs) [9]. These NLEEs appear in various fields of applied and nonlinear sciences [1–10]. These include Physics, Biosciences as well as Engineering. There are various types of solutions that are revealed for these NLEEs. These include the cnoidal waves, snoidal waves, shock waves, solitary waves, singular solitary waves, periodic waves as well as double periodic waves. Moreover, there are various techniques of integrating these NLEEs are also available nowdays.

In this paper, the focus will be on obtaining the exact topological as well as non-topological 1-soliton solutions for the following generalized Bretherton equation:

$$u_{tt} - k^2 u_{xx} + a u_{xxxx} + b u^m + c u^n = 0, \tag{1}$$

where k , a , b and c are constants.

To achieve this goal we will use the *solitary wave ansatz method* which has recently been applied successfully to several nonlinear partial differential equations. Here, we will calculate the closed form solutions for any arbitrary values of the exponents n and m with $n > m$ and $n \neq 1$. It needs to be noted that the traveling wave solutions of this NLEE with a positive coefficient of the second order dispersion have recently been constructed by Kudryashov et al [9], for the particular cases $m = 1$ with any n , $m \neq 1$ and $n = 5$ and $m = 2$, $n = 3$. Additionally, Esfahani studied this equation in 2011 [1]. However, in that paper the special case with $m = 1$, $n = 2$ was considered. This paper considers any arbitrary values of m and n .

2. NON-TOPOLOGICAL SOLITON SOLUTION

In order to solve (1), the starting hypothesis is given by

$$u(x,t) = \frac{A}{\cosh^p \tau}, \tag{2}$$

where

$$\tau = B(x - vt) \tag{3}$$

and

$$p > 0 \tag{4}$$

for solitons to exist. Here, in (2) and (3), A is the amplitude of the soliton while v is the velocity of the soliton and B is the inverse width. The exponent p is unknown at this point and its value will fall out in the process of deriving the solution of this equation. From the ansatz (2), one can find that

$$u_{tt} = \frac{p^2 v^2 AB^2}{\cosh^p \tau} - \frac{p(p+1)v^2 AB^2}{\cosh^{p+2}\tau}, \tag{5}$$

$$u_{xx} = \frac{p^2 AB^2}{\cosh^p \tau} - \frac{p(p+1)AB^2}{\cosh^{p+2}\tau}, \tag{6}$$

$$u_{xxxx} = \frac{Ap^4 B^4}{\cosh^p \tau} - \frac{AB^4 p(p+1)\left\{p^2 + (p+2)^2\right\}}{\cosh^{p+2}\tau} + \frac{AB^4 p(p+1)(p+2)(p+3)}{\cosh^{p+4}\tau}, \tag{7}$$

$$u^m = \frac{A^m}{\cosh^{mp}\tau}, \tag{8}$$

$$u^n = \frac{A^n}{\cosh^{np}\tau}. \tag{9}$$

Inserting the expressions (5)-(9) into (1) yields

$$\begin{aligned}&\left\{p^2 AB^2\left(v^2 - k^2\right) + aAp^4 B^4\right\}\frac{1}{\cosh^p \tau} - \\ &-\left\{p(p+1)AB^2\left(v^2 - k^2\right) + aAB^4 p(p+1)\left\{p^2 + (p+2)^2\right\}\right\}\frac{1}{\cosh^{p+2}\tau} + \\ &+\frac{aAB^4 p(p+1)(p+2)(p+3)}{\cosh^{p+4}\tau} + \frac{bA^m}{\cosh^{mp}\tau} + \frac{cA^n}{\cosh^{np}\tau} = 0.\end{aligned} \tag{10}$$

From (10), equating the exponents np and $p+4$ gives

$$np = p + 4, \tag{11}$$

so that

$$p = \frac{4}{n-1}. \tag{12}$$

Again from (10), equating the exponents mp and $p+2$ gives

$$mp = p + 2, \tag{13}$$

that yields

$$p = \frac{2}{m-1}. \tag{14}$$

Now, by equating the two values of p from (12) and (14), gives

$$2m = n + 1. \tag{15}$$

We notice that from (4), (12) and (14) the following restrictions are obtained

$$m > 1, \quad n > 1, \tag{16}$$

with the condition $n > m$.

Now from (10), setting the coefficients of the linearly independent functions $1/\cosh^{p+j}\tau$ to zero, where $j = 0, 2, 4$ gives

$$\begin{aligned} &p^2 AB^2\left(v^2 - k^2\right) + aAp^4B^4 = 0, \\ &p(p+1)AB^2\left(v^2 - k^2\right) + aAB^4 p(p+1)\left\{p^2 + (p+2)^2\right\} - bA^m = 0, \end{aligned} \tag{17}$$

$$aAB^4 p(p+1)(p+2)(p+3) + cA^n = 0. \tag{19}$$

Solving the above system gives

$$A = \left\{-\frac{b(3m-1)}{2mc}\right\}^{\frac{1}{n-m}}, \tag{20}$$

$$B = \frac{m-1}{2}\left\{-\frac{2b}{a(m+1)m^2}\left\{-\frac{b(3m-1)}{2mc}\right\}^{\frac{m-1}{n-m}}\right\}^{\frac{1}{4}} \tag{21}$$

and

$$v = \sqrt{k^2 - a\sqrt{-\frac{2b}{a(m+1)m^2}\left\{-\frac{b(3m-1)}{2mc}\right\}^{\frac{m-1}{(n-m)}}}}. \tag{22}$$

Thus, the 1-soliton solution of the generalized Bretherton equation (1) is given by

$$u(x,t) = \frac{A}{\cosh^{\frac{2}{m-1}}[B(x-vt)]}, \tag{23}$$

where the amplitude A of the soliton is given by (20), the width B is given by (21) while the the velocity v is shown in (22). In view of (22), we clearly see that this solution exists provided that $ab < 0$ and $k > \sqrt{a}$. Also (20) shows that it is necessary to have $bc < 0$ for the non-topological soliton to exist if $n - m$ is an even integer. However, if $n - m$ is an odd integer there is no such restriction but the soliton will be pointing downwards. Finally, we would like to note that the solution (23) exists under the condition (15) with $n > m > 1$.

3. TOPOLOGICAL SOLITON

In this section we will calculate the topological 1-soliton solution of the generalized Bretherton equation (1), for any general values of the exponents n and m with $n > m$, using the *solitary wave ansatz*. To start off, the hypothesis is given by

$$u(x,t) = A\tanh^p\tau, \tag{24}$$

where

$$\tau = B(x - vt) \tag{25}$$

and

$$p > 0 \tag{26}$$

for solitons to exist. Here, in (24) and (25), A and B are free parameters and v is the velocity of the wave. Also, the unknown exponent p will be determined during the course of the derivation of the soliton solution to (1). Therefore from (24),we have

$$u_{tt} = pAB^2v^2\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\}, \tag{27}$$

$$u_{xx} = pAB^2\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\}, \tag{28}$$

$$\begin{aligned} u_{xxxx} = pAB^4 &\left\{(p-1)(p-2)(p-3)\tanh^{p-4}\tau + (p+1)(p+2)(p+3)\tanh^{p+4}\tau - \right.\\ &-2\left\{p^2+(p-2)^2\right\}\{p-1\}\tanh^{p-2}\tau - 2\left\{p^2+(p+2)^2\right\}\{p+1\}\tanh^{p+2}\tau + \\ &\left.+\left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}\tanh^{p}\tau\right\}, \end{aligned} \tag{29}$$

$$u^m = A^m\tanh^{pm}\tau, \tag{30}$$

$$u^n = A^n\tanh^{pn}\tau. \tag{31}$$

Substituting these expressions (27–31) into (1), we obtain

$$\begin{aligned} &pAB^2\left(v^2-k^2\right)\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\} + \\ &+apAB^4\left\{(p-1)(p-2)(p-3)\tanh^{p-4}\tau + (p+1)(p+2)(p+3)\tanh^{p+4}\tau - \right.\\ &-2\left\{p^2+(p-2)^2\right\}\{p-1\}\tanh^{p-2}\tau - 2\left\{p^2+(p+2)^2\right\}\{p+1\}\tanh^{p+2}\tau + \\ &\left.+\left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}\tanh^{p}\tau\right\} + \\ &+bA^m\tanh^{pm}\tau + cA^n\tanh^{pn}\tau = 0. \end{aligned} \tag{32}$$

From (32), equating the exponents np and $p+4$ gives

$$np = p+4, \tag{33}$$

so that

$$p = \frac{4}{n-1}. \tag{34}$$

Again from (32), equating the exponents mp and $p+2$ gives

$$mp = p+2, \tag{35}$$

that yields

$$p = \frac{2}{m-1}. \tag{36}$$

It should be remarked that the topological 1-soliton solution (24) can be obtained when $p > 0$. Hence the conditions $n > 1$, $m > 1$ arises from (34) and (36).

Now from (32) the linearly independent functions are $\tanh^{p+j}\tau$ for $j = 0, \pm 2, \pm 4$. Hence setting their respective coefficients to zero yields a set of algebraic equations:

$$pAB^2\left(v^2-k^2\right)(p-1)-2apAB^4\left\{p^2+(p-2)^2\right\}\left\{p-1\right\}=0, \tag{37}$$

$$-2p^2AB^2\left(v^2-k^2\right)+apAB^4\left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}=0, \tag{38}$$

$$pAB^2\left(v^2-k^2\right)(p+1)-2apAB^4\left\{p^2+(p+2)^2\right\}\left\{p+1\right\}+bA^m=0, \tag{39}$$

$$apAB^4(p+1)(p+2)\left(p+3\right)+cA^n=0, \tag{40}$$

$$apAB^4(p-1)(p-2)\left(p-3\right)=0. \tag{41}$$

To solve (41), we have considered the case $p-1=0$. This yields $p=1$ and therefore $n=5$ and $m=3$ following to (34) and (36), respectively. Further substitution of this value of $p=1$ into (37–40), respectively, gives

$$A=\left\{-\frac{b}{c}\right\}^{\frac{1}{n-m}}, \tag{42}$$

$$B=\frac{1}{2}\left\{\frac{2b}{3a}\left\{-\frac{b}{c}\right\}^{\frac{m-1}{n-m}}\right\}^{\frac{1}{4}}, \tag{43}$$

and

$$v=\sqrt{k^2+2a\sqrt{\frac{2b}{3a}\left\{-\frac{b}{c}\right\}^{\frac{m-1}{n-m}}}}. \tag{44}$$

Since $n=5$ and $m=3$, then the expressions (42–44) will be reduced to

$$A=\sqrt{-\frac{b}{c}}, \tag{45}$$

$$B=\frac{1}{2}\left\{-\frac{2b^2}{3ac}\right\}^{\frac{1}{4}}, \tag{46}$$

and

$$v=\sqrt{k^2+2a\sqrt{-\frac{2b^2}{3ac}}}. \tag{47}$$

Notice that the cases $p-2=0$ and $p-3=0$ are not considered here as it does not give a unique value of v, B and A. Thus, for the generalized Bretherton equation (1) topological solitons will exist only when $n=5$ and $m=3$. This important observation is being made for the first time in this paper.

Hence, finally, the topological 1-soliton solution to the generalized Bretherton equation (1) is given by

$$u(x,t)=A\tanh[B(x-vt)], \tag{48}$$

where the free parameters A and B are given by (45) and (46), and the velocity of the soliton is shown in (47). Note that following to (45–47), we shall have the conditions $bc<0$ and $ac<0$ for toplogical 1-soliton solution to exist.

ACKNOWLEDGMENTS

The third, fourth and fifth authors (TH, OMA & AB) would like to recognize and thankfully appreciate the support from King Saud University (KSU-VPP-117).

REFERENCES

1. A. ESFAHAN, *Traveling wave solutions for generalized Bretherton equation,* Communications in Theoretical Physics, **55**, *3*, pp. 381–386, 2011.
2. M. S. ISMAIL, *Numerical solution of complex modified Korteweg-de Vries equation by collocation method*, Communications in Nonlinear Science and Numerical Simulation, **14**, *3*, pp. 749–759, 2009.
3. A. J. M. JAWAD, M. PETKOVIC, A. BISWAS, *Soliton solutions of Burgers equations and perturbed Burgers equation,* Applied Mathematics and Computation, **216**, *11*, pp. 3370–3377, 2010.
4. L. JIBIN, L. ZHENRONG, *Smooth and non-smooth travelling waves in a nonlinearly dispersive equation,* Applied Mathematical Modelling, **25**, *1*, pp. 41–56, 2000.
5. E. V. KRISHNAN, *Series solutions for a coupled wave equation,* International Journal of Differential equations and Applications, **8**, pp. 13–22, 2003.
6. E. V. KRISHNAN, Y. PENG, *A new solitary wave solution for the new Hamiltonian amplitude equation,* The Journal of Physical Society of Japan, **74**, pp. 896–897, 2005.
7. E. V. KRISHNAN, Y. PENG, *Squared Jacobian elliptic function solutions for the (2+1)-D Korteweg-de Vries equation,* Advances in Theoretical and Applied Mathematics, **1**, pp. 9–25, 2006.
8. E. V. KRISHNAN, Z. Y. YAN, *Jacobian elliptic function solutions using sinh-Gordon equation expansion method,* International Journal of Applied Mathematics and Mechanics, **2**, pp. 1–10, 2006.
9. N. A. KUDRYASHOV, D. I. SINELSHCHIKOV, M. V. DEMINA, *Exact solutions of the generalized Bretherton equation,* Physics Letters A, **375**, pp. 1074–1079, 2011.
10. A. M. WAZWAZ, *A study on KdV and Gardner equations with time-dependent coefficients and forcing terms,* Applied Mathematics and Computation, **217**, *5*, pp. 2277–2281, 2010.

Received December 19, 2011