

A MATRIX CAUCHY-SCHWARZ TYPE INEQUALITY AND THE BOOLEAN PRODUCT OF OPERATOR-VALUED LINEAR MAPS DEFINED ON INVOLUTIVE ALGEBRAS

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We extend the proof of the main result in [7] to completely positive maps defined on involutive algebras. This extension uses an adequate matrix Cauchy-Schwarz type inequality derived from a F. Boca's result [4] permitting in fact two proofs of the mentioned statement.

Key words: universal free product (*-,C*-)algebra, Boolean product of linear maps, complete positivity, Stinespring dilation, GNS representation, Cauchy-Schwarz type inequality.

1. INTRODUCTION

The Boolean product of linear functionals on algebras, and the involved independence are fundamental in the so-called Boolean quantum probability theory and related topics (see, e.g., [14, 11, 2, 1, 6, 8]). This theory is one of the three noncommutative probability theories (the others being R.L.Hudson's Boson or Fermion probability theory and D.V. Voiculescu's free probability theory) emerged from an associative product which does not depend on the order of its factors and fulfills a universal rule for mixed moments (see R. Speicher's answer [13] to M. Schürmann's conjecture [12] on the universal products of *-algebraic probability spaces).

In [7], we considered the Boolean product for linear maps between algebras and directly showed (adapting Boca's ideas from [3]), it preserves the complete positivity in C^* -algebraic setting. Then, in [10] we gave a new proof of this fact.

In the present Note, we extend the proof of the main result in [7] to completely positive maps defined on involutive algebras and valued into C^* -algebras, via an appropriate matrix Cauchy-Schwarz type inequality. Actually, this inequality permits us two proofs for the cited main result in [7] (the second one through the way in [10]).

2. COMPLETE POSITIVITY AND THE BOOLEAN PRODUCT OF LINEAR MAPS

Let A be a (complex) *-algebra (i.e., a complex algebra endowed with a conjugate linear involution $*$, which is an anti-isomorphism). We consider the cone A_+ of positive elements in A consisting of finite sums $\sum a_i^* a_i$, with $a_i \in A$. Thus, A_+ determines a preorder structure on the real linear subspace of self-adjoint elements in A .

For any positive integer n , let $M_n(A)$ be the *-algebra of $n \times n$ matrices $[a_{ij}]$ with entries from A . When A is a C^* -algebra, A_+ determines an order structure on the real linear subspace of self-adjoint elements in A , and $M_n(A)$ becomes a C^* -algebra.

Let B be another *-algebra and $Q : A \longrightarrow B$ be a linear map. We say Q is positive if $Q(A_+) \subset B_+$.

For any positive integer n , let $Q_n : M_n(A) \longrightarrow M_n(B)$ be the inflation map given by $Q_n([a_{ij}]) = [Q(a_{ij})]$, for $[a_{ij}] \in M_n(A)$. Then Q is called n -positive if the map Q_n induced by Q is positive. The map Q is completely positive if it is n -positive, for all positive integer n .

The celebrated Stinespring dilation theorem characterizes the completely positive maps defined on C^* -algebras, and extends the GNS theorem concerning the (positive functionals, i.e.,) states on this kind of C^* -algebras (see, e.g., [15]).

F. Boca noticed in [3] that the classical Stinespring dilation theorem is still true for unital completely positive maps defined on unital C^* -algebras A verifying the Combes axiom (i.e., for every $a \in A$, there exists a scalar $\lambda(a) > 0$ with $x^* a^* a x \leq \lambda(a) x^* x$, for all $x \in A$). Explicitly :

THEOREM 2.1. *Let A be a unital C^* -algebra satisfying the Combes axiom, $L(H)$ be the bounded linear operators on a Hilbert space H , and $Q : A \longrightarrow L(H)$ be a unital completely positive map.*

Then there exists a unique (up to a unitary equivalence) Stinespring dilation (K, π) of Q , where $K \supset H$ is a Hilbert space, and $\pi : A \longrightarrow L(K)$ is a unital C^ -representation, such that $Q(a) = P_H^K \pi(a)|_H$, for $a \in A$; and $\overline{\text{sp}} \pi(A)H = K$.*

Moreover, in his investigations on the method of constructing irreducible finite index subfactors of Popa [4], Boca deduced the following Cauchy-Schwarz type inequality for completely positive maps defined on C^* -algebras, via the Kolmogorov decomposition theorem for operator-valued positive definite kernels (see Lemma 3.5 in [4]).

LEMMA 2.2. *Let A be a unital C^* -algebra, $L(H)$ be the bounded linear operators on a Hilbert space H , and $Q : A \longrightarrow L(H)$ be a unital completely positive map.*

Then Q is a Schwarz map, i.e.,

$$Q(a^* a) \geq Q(a)^* Q(a), \text{ for all } a \in A.$$

The next fact was stated by M.-D. Choi and E.G. Effros via a C. Lance's non-unital version of Stinespring dilation theorem in their famous paper on the completely positive lifting problem for C^* -algebras (see Lemma 3.9 in [5]).

LEMMA 2.3. *Let A and B be C^* -algebras, and $Q : A \longrightarrow B$ be a contractive, completely positive map. Let \tilde{A} and \tilde{B} be the unitizations of A and B , and $\tilde{Q} : \tilde{A} \longrightarrow \tilde{B}$ be the unitization of Q given by $\tilde{Q}(a \oplus \lambda 1) = Q(a) \oplus \lambda 1$ ($a \in A, \lambda \in \mathbb{C}$).*

Then \tilde{Q} is completely positive.

The universal free product (C^* -algebra) is the direct sum in the category of (complex) (C^* -algebras, non necessary unital [3, 7, 12, 16].

As linear space, a realization of the universal free product corresponding to a family of (C^* -algebras $(A_i)_{i \in I}$ is

$$A = \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \dots \neq i_n} A_{i_1} \otimes \dots \otimes A_{i_n}.$$

By natural operations, A is organized as a (C^* -algebra).

In particular, if A_i are C^* -algebras, A satisfies the Combes axiom.

Let B and A_i be (complex) algebras, and $Q_i : A_i \longrightarrow B$ be linear maps; $i \in I$.

In [7], we considered the Boolean product $Q = \bullet Q_i$ as the unique linear map defined on the universal free product A of the algebras A_i , $i \in I$, such that

$$Q(a_1 \dots a_n) = Q_{i_1}(a_1) \dots Q_{i_n}(a_n),$$

for all $n \geq 1$, with $i_1 \neq \dots \neq i_n$, and $a_k \in A_{i_k}$, if $k = 1, \dots, n$; with respect to the natural embeddings of A_i into A arising from the free product construction.

When B and A_i are (complex) $*$ -algebras, and Q_i are Hermitian maps (in particular, e.g., when A_i are unital, and Q_i are unital positive maps), Q naturally becomes Hermitian.

3. THE MATRIX CAUCHY-SCHWARZ TYPE INEQUALITY AND THE BOOLEAN PRODUCT OF LINEAR MAPS DEFINED ON $*$ -ALGEBRAS

We naturally derive from Boca's Lemma 2.2 the next matrix Cauchy-Schwarz type inequality, which is adequate to our actual purposes.

LEMMA 3.1. *Let A be a unital $*$ -algebra, B be a unital C^* -algebra, and $Q : A \longrightarrow B$ be a unital completely positive map.*

Then

$$[Q(a_i * a_j)]_{i,j=1,\dots,n} \geq [Q(a_i) * Q(a_j)]_{i,j=1,\dots,n} \text{ in } M_n(B),$$

for all $n \geq 1$, and all $a_1, \dots, a_n \in A$.

Proof. By the GNS theorem, we may assume that $B \subseteq L(H)$, with a Hilbert space H . The map Q being completely positive, every inflation map Q_n , induced by Q , is completely positive, too.

So, it is enough to apply Boca's Lemma 2.2 to the element $[a_{ij}]_{i,j=1,\dots,n} \in M_n(A)$, having the first row with the entries $a_{1j} := a_j$, for $j = 1, \dots, n$, and the other entries null.

By the previous Lemma, we can extend the main result in [7] under the following form.

THEOREM 3.2. *Let A_i be $*$ -algebras, B be a C^* -algebra, and $Q_i : A_i \longrightarrow B$ be linear maps, such that their unitizations are completely positive; $i \in I$.*

Let $Q = \bullet Q_i$ be the Boolean product of $(Q_i)_{i \in I}$ defined on the $$ -algebraic free product A of $(A_i)_{i \in I}$.*

Then [the unitization of] Q is completely positive.

Proof. Let \tilde{A}_i, \tilde{A} and \tilde{B} be the unitizations of A_i, A and B [by \mathbb{C} , the field of complex numbers]; let $\tilde{Q}_i : \tilde{A}_i \longrightarrow \tilde{B}$, and $\tilde{Q} : \tilde{A} \longrightarrow \tilde{B}$ be the unitizations of Q_i and Q .

By the GNS theorem, we may consider that $B \subseteq \tilde{B} \subseteq L(H)$, with a Hilbert space H .

Let $W = \{1\} \cup \{a_1 \dots a_n; n \geq 1, a_j \in A_{i_j}, i_1 \neq \dots \neq i_n\}$ be the set of reduced words in \tilde{A} .

For $w = a_1 \dots a_n \in W$, $w \neq 1$, call n the length of w and denote $\tilde{w} := \{1, a_1, a_1 a_2, \dots, a_1 a_2 \dots a_n\}$. The length of the empty word 1 is zero and $\tilde{1} = \{1\}$.

Call a subset of W complete if it contains 1 and it includes \tilde{w} , whenever it contains a word w .

We have to show that

$$\sum_{i,j=1}^n \langle h_i, \tilde{Q}(x_i^* x_j) h_j \rangle \geq 0, \text{ for all } n \geq 1, \text{ all } x_1, \dots, x_n \in \tilde{A}, \text{ and all } h_1, \dots, h_n \in H.$$

As in [3], it suffices to prove that

$$S_X^f := \sum_{x,y \in X} \langle f(x), \tilde{Q}(x^* y) f(y) \rangle \geq 0,$$

for all complete finite sets $X \subset W$ and all maps $f : X \rightarrow H$; because every $x_i \in \tilde{A}$, as before, for $i = 1, \dots, n$, may be expressed as $x_i = \sum_{s=1}^N \lambda_{is} a_s$, with the same N , the same $a_s \in W$, and some scalars $\lambda_{is} \in \mathbf{C}$; and every finite set $X \subset W$ is contained in the (complete finite) set $\cup_{w \in X} \tilde{w} \subset W$.

We can observe the next fact.

Remark 3.3. Consider a complete finite set $X_1 \subset W$ such that $S_{X_1}^g \geq 0$, for all maps $g : X_1 \rightarrow H$.

*Then there exists $V_x \in L(H, H^{\oplus n})$ such that $\tilde{Q}(x^*y) = V_x^* V_y$, if $x, y \in X_1$; and, $[V_x^* V_y]_{x, y \in X_1} \geq [V_x^* V_1 V_1^* V_y]_{x, y \in X_1}$ in $M_n(L(H)) = L(H^{\oplus n})$; denoting by n the cardinal of X_1 . \square*

Let X be a complete finite set having k words, and $f : X \rightarrow H$ be a map.

If $k \geq 3$, choose a word $a_1 \cdots a_m$ of maximum length in X , with $a_j \in A_{i_j}$, for $j = 1, \dots, m$;

and $i_1 \neq \dots \neq i_m$.

Let $X_2 := X \cap \{c_1 \cdots c_m; c_j \in A_{i_j}, j = 1, \dots, m\}$ and $X_1 := X \setminus X_2$; then X_1 is complete, consists of, say, n words, and $a_1 \cdots a_m \notin X_1$.

1) If $m = 1$, i.e. $X_2 = X \cap A_i$, for some $i \in I$, then (as in [7]) $\tilde{Q}(x^*y) = \tilde{Q}(x)^* Q_i(y)$, for $x \in X_1$, and $y \in X_2$, by the very definition of the Boolean product.

Thus,

$$S_X^f = S_{X_1}^f + 2 \operatorname{Re} \left\langle \sum_{x \in X_1} \tilde{Q}(x) f(x), \sum_{x \in X_2} Q_i(x) f(x) \right\rangle + S_{X_2}^f.$$

If $S_{X_1}^g \geq 0$, for all maps $g : X_1 \rightarrow H$, there exists $V_x \in L(H, H^{\oplus n})$ such that $\tilde{Q}(x^*y) = V_x^* V_y$, if $x, y \in X_1$, by the Remark 3.3; and the above considerations imply

$$\begin{aligned} S_{X_1}^f &= \sum_{x, y \in X_1} \langle f(x), V_x^* V_y f(y) \rangle \geq \sum_{x, y \in X_1} \langle f(x), V_x^* V_1 V_1^* V_y f(y) \rangle = \\ &= \sum_{x, y \in X_1} \langle f(x), \tilde{Q}(x)^* \tilde{Q}(y) f(y) \rangle = \left\| \sum_{x \in X_1} \tilde{Q}(x) f(x) \right\|^2. \end{aligned}$$

Moreover, we infer through the matrix Cauchy-Schwarz type inequality in Lemma 3.1, due to the complete positivity of \tilde{Q}_i :

$$S_{X_2}^f = \sum_{x, y \in X_2} \langle f(x), Q_i(x^*y) f(y) \rangle \geq \sum_{x, y \in X_2} \langle f(x), Q_i(x)^* Q_i(y) f(y) \rangle = \left\| \sum_{x \in X_2} Q_i(x) f(x) \right\|^2.$$

Consequently,

$$\begin{aligned} S_X^f &\geq \left\| \sum_{x \in X_1} \tilde{Q}(x) f(x) \right\|^2 + 2 \operatorname{Re} \left\langle \sum_{x \in X_1} \tilde{Q}(x) f(x), \sum_{x \in X_2} Q_i(x) f(x) \right\rangle + \left\| \sum_{x \in X_2} Q_i(x) f(x) \right\|^2 = \\ &= \left\| \sum_{x \in X_1} \tilde{Q}(x) f(x) + \sum_{x \in X_2} Q_i(x) f(x) \right\|^2, \end{aligned}$$

whenever $S_{X_1}^g \geq 0$, for all $g : X_1 \rightarrow H$.

2) If $m \geq 2$, every word $x \in X_2$ may be written as $x_0 a$ with $x_0 \in X_1$, $x_0 \neq 1$, and $a \in A_{i_m}$.

Thus, the very definition of the Boolean product yields (as in [7]) the following factorizations :

$$\tilde{Q}(x^*y) = Q(x^*x_o) Q_{i_m}(a), \text{ if } x \in X_1, \text{ and } y = x_o a \in X_2;$$

$$\tilde{Q}(x^*y) = Q_{i_m}(a') * Q(x'_o x_o) Q_{i_m}(a), \text{ if } x = x'_o a' \in X_2, \text{ and } y =$$

If $S_{X_1}^g \geq 0$, for all maps $g : X_1 \rightarrow H$, there exists $V_x \in L(H, H^{\oplus n})$ such that $\tilde{Q}(x^*y) = V_x^* V_y$, if $x, y \in X_1$, by the Remark 3.3, of course.

In consequence, we infer in the light of this hypothesis :

$$S_{X_1}^f = \sum_{x,y \in X_1} \langle f(x), V_x^* V_y f(y) \rangle = \left\| \sum_{x \in X_1} V_x f(x) \right\|^2,$$

$$\sum_{\substack{x \in X_1 \\ y = x_o a \in X_2}} \langle f(x), \tilde{Q}(x^*y) f(y) \rangle = \sum_{\substack{x \in X_1 \\ y = x_o a \in X_2}} \langle f(x), V_x^* V_{x_o} Q_{i_m}(a) f(y) \rangle,$$

and

$$S_{X_2}^f = \sum_{\substack{y = x_o a \in X_2 \\ x = x'_o a' \in X_2}} \langle V_{x_o} Q_{i_m}(a') f(x), V_{x_o} Q_{i_m}(a) f(y) \rangle = \left\| \sum_{x = x_o a \in X_2} V_{x_o} Q_{i_m}(a) f(x) \right\|^2.$$

Therefore,

$$\begin{aligned} S_X^f &= \left\| \sum_{x \in X_1} V_x f(x) \right\|^2 + 2\text{Re} \langle \sum_{x \in X_1} V_x f(x), \sum_{y = x_o a \in X_2} V_{x_o} Q_{i_m}(a) f(y) \rangle + \left\| \sum_{x = x_o a \in X_2} V_{x_o} Q_{i_m}(a) f(x) \right\|^2 = \\ &= \left\| \sum_{x \in X_1} V_x f(x) + \sum_{x = x_o a \in X_2} V_{x_o} Q_{i_m}(a) f(x) \right\|^2, \end{aligned}$$

whenever $S_{X_1}^g \geq 0$, for all $g : X_1 \rightarrow H$.

In conclusion, the proof completes by induction, since

$$S_X^f = \langle f(1), \tilde{Q}(1) f(1) \rangle = \|f(1)\|^2, \text{ when } k = 1 \text{ (i.e. } X = \{1\}),$$

and

$$S_X^f = \left\langle \begin{bmatrix} f(1) \\ f(a) \end{bmatrix}, \begin{bmatrix} 1 & Q_i(a) \\ Q_i(a)^* & Q_i(a^*a) \end{bmatrix} \cdot \begin{bmatrix} f(1) \\ f(a) \end{bmatrix} \right\rangle \geq 0,$$

by the 2-positivity of \tilde{Q}_i ; because $\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix} \in M_2(\tilde{A}_i)_+$, when $k = 2$ (i.e. $X = \{1, a\}$, with $a \in A_i$, for some $i \in I$).

In particular, we obtain the following fact (see, e.g., [12]).

COROLLARY 3.4 *Let φ_i be linear functionals defined on (complex)*-algebras, such that their unitizations are positive; $i \in I$.*

Then the Boolean product $\varphi = \bullet \varphi_i$ is positive, too.

By Choi-Effros' Lemma 2.3, the previous theorem extends Lemma 2 in [7].

COROLLARY 3.5. *Let A_i and B be C^* -algebras, and $Q_i : A_i \longrightarrow B$ be contractive, completely positive maps; $i \in I$. Let A be the $*$ -algebraic free product of $(A_i)_{i \in I}$ and $Q : A \longrightarrow B$ be the Boolean product of $(Q_i)_{i \in I}$.*

Then [the unitization of] Q is completely positive.

We denote by $*_0 A_i$ the non-unital universal (or full) free product C^* -algebra corresponding to a family $(A_i)_{i \in I}$ of C^* -algebras.

After separation and completion of the corresponding universal free product $*$ -algebra A in its enveloping C^* -seminorm

$$\|a\| = \sup \{ \|\pi(a)\|; \pi \text{ } * \text{-representation of } A \text{ as bounded operators on a Hilbert space} \},$$

one can realize the universal (or full) free product $*_0 A_i$ in the category of C^* -algebras.

Therefore Theorem 3.2 implies the main result in [7], via Boca's Theorem 2.1.

COROLLARY 3.6. *Let A_i and B be C^* -algebras, and $Q_i : A_i \longrightarrow B$ be contractive, completely positive maps; $i \in I$.*

Then the Boolean product $Q = \bullet Q_i$ defined on the $$ -algebraic free product of the algebras A_i , $i \in I$, is contractive, with respect to the enveloping C^* -seminorm, and completely positive.*

*Therefore, Q extends to a unique map $Q : *_0 A_i \longrightarrow B$ which is contractive and completely positive.*

Moreover, Lemma 3.1 reformulates the complete positivity of $Q = \bullet Q_i$ in Theorem 3.2.

COROLLARY 3.7. *Let A_i be $*$ -algebras, B be a C^* -algebra, and $Q_i : A_i \longrightarrow B$ be linear maps, such that their unitizations are completely positive; $i \in I$.*

Let $Q = \bullet Q_i$ be the Boolean product of $(Q_i)_{i \in I}$ defined on the $$ -algebraic free product A of $(A_i)_{i \in I}$.*

Then [the unitization of] Q is completely positive.

Therefore

$$[Q(a_i * a_j)]_{i,j=1,\overline{n}} \geq [Q(a_i) * Q(a_j)]_{i,j=1,\overline{n}} \text{ in } M_n(B),$$

for all $n \geq 1$, and all $a_1, \dots, a_n \in A$.

Thus, via Choi-Effros' Lemma 2.3, we also deduce Lemma 3.1 in [10]; and we obtain for the second time the main result in [7] (see Th. 3.2 in [10] or the above Corollary 3.6), by the same Boca's Theorem 2.1.

COROLLARY 3.8. *Let A_i and B be C^* -algebras, and $Q_i : A_i \longrightarrow B$ be contractive, completely positive maps; $i \in I$. Let A be the $*$ -algebraic free product of $(A_i)_{i \in I}$ and $Q : A \longrightarrow B$ be the Boolean product of $(Q_i)_{i \in I}$.*

Then

$$[Q(a_i * a_j)]_{i,j=1,\overline{n}} \geq [Q(a_i) * Q(a_j)]_{i,j=1,\overline{n}} \text{ in } M_n(B),$$

for all $n \geq 1$, and all $a_1, \dots, a_n \in A$.

In particular, Q is completely positive. \square

In the same way, one can prove that the amalgamated Boolean (or, moreover, conditionally free) product of linear maps defined on $*$ -algebras and valued into C^* -algebras preserves the complete positivity (for these and other extensions, see, e.g., [8, 9]).

ACKNOWLEDGEMENTS

I am deeply grateful to Professor Ioan Cuculescu and to Professor Marius Iosifescu for many valuable discussions on these and related topics and their essential support, which has made this work possible. I am deeply grateful to Professor Florin Boca for his interest concerning my work, many stimulating ideas, and moral support. I am deeply indebted to Professor Marius Radulescu for his valuable advises concerning this Note and moral support. I am deeply indebted to Professor Ioan Stancu-Minasian and to Professor Gheorghita Zbaganu for many stimulating conversations and moral support.

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Received November 14, 2011