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THE ARITHMETIC MEAN ITERATIVE METHODS FOR SOLVING DENSE LINEAR SYSTEMS ARISE FROM FIRST KIND LINEAR FREDHOLM INTEGRAL EQUATIONS

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In the previous studies, the effectiveness of the Arithmetic Mean (AM) iterative method and its variants for solving various scientific problems has been investigated. Consequently, in this paper, the implementation and performance one of the AM method variants i.e. Quarter-Sweep Arithmetic Mean (QSAM) method for solving dense linear system associated with the numerical solution of first kind linear Fredholm integral equations are considered. The details of the method are discussed. Some numerical analyses were also conducted to verify the efficiency of the method.

Key words: first kind Fredholm integral equations, Newton-Cotes quadrature, dense linear system, arithmetic mean.

1. INTRODUCTION

Many scientific problems lead to the necessity to solve linear systems as part of the computations. It is well recognized that iterative methods are applied widely in large scale computations for linear systems problems. Among the existing iterative methods, Arithmetic Mean (AM) method and its variants have been extensively applied for solving various types of linear systems. In a series of papers, the effectiveness of the AM methods were studied and tested on linear and nonlinear systems. For instance, researches by Benzi and Dayar [1], Galligani [3], Galligani and Ruggiero [4], Galligani and Ruggiero [6], Hasan *et al*. [7], Muthuvalu and Sulaiman [10-12], Ruggiero and Galligani [16] and, Sulaiman *et al*. [17, 18]. Moreover, the AM method is also have been successfully applied as a preconditioner with Conjugate Gradient (CG) method for solving symmetric positive definite linear system [5].

Recently, the standard AM method [16] and its variant known as Half-Sweep Arithmetic Mean (HSAM) [17] method were employed to solve approximation equations generated from the first and second kind linear Fredholm integral equations [12]. Consequently, in this paper, performance of another AM variants i.e. Quarter-Sweep Arithmetic Mean (QSAM) [18] method for solving dense linear system generated from the discretization of first kind linear Fredholm integral equations with semi-smooth kernel is investigated. Actually, the QSAM method is derived by combining the standard AM method with quarter-sweep iteration concept [14]. The performance of the QSAM method will be compared with the Full-Sweep Gauss-Seidel (FSGS), Half-Sweep Gauss-Seidel (HSGS), Quarter-Sweep Gauss-Seidel (QSGS), FSAM and HSAM iterative methods.

The remainder of this paper is organized in following way. In Section 2, the formulation of the full-, half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations will be elaborated. The latter section of this paper will discusses the formulations of the FSAM, HSAM and QSAM methods, and some numerical results will be shown in fourth section to assert the performance of the tested methods. Besides that, analysis on computational complexity for FSAM, HSAM and QSAM methods is discussed in Section 5 and the concluding remarks are given in final section.

2. CLOSED COMPOSITE NEWTON-COTES QUADRATURE APPROXIMATION EQUATIONS

Consider first kind linear Fredholm integral equations with the semi-smooth kernel defined as follows

$$\int_{\alpha}^{\beta} K(y,t)\,x(t)\,\mathrm{d}t = b(y), \tag{1}$$

where the kernel K and function b are given, and x is the unknown function to be determined. If operator κ is denoted by

$$\kappa : S \to T \quad \kappa(x(t)) = \int_{\alpha}^{\beta} K(y,t)\,x(t)\,\mathrm{d}t, \tag{2}$$

then the following definition is satisfied.

Definition 1 [8]. A kernel $K(y,t)$ is called q - semi-smooth if

$$K(y,t) = \begin{cases} K_1(y,t) & \text{if } \alpha \le t \le y \\ K_2(y,t) & \text{if } y \le t \le \beta, \end{cases} \tag{3}$$

where $K_{1,2}(y,t) \in C^q_{[\alpha,\beta]\times[\alpha,\beta]}$ for some $q > 1$.

Definition 2 [9]. Let $\kappa : S \to T$ be an operator from normed space S into a normed space T, the equation $\kappa x = b$ is called well-posed if κ is onto, one to one and the inverse operator $\kappa^{-1} : T \to S$ is continuous. Otherwise the equation is called ill-posed.

As a matter of fact, some valid numerical methods for discretizing (1) have been developed in recent years. In this paper, a discretization scheme under category of closed composite Newton-Cotes quadrature method was utilized in order to construct approximation equations for problem (1) with semi-smooth kernel. Let interval divided uniformly into n subintervals and the discrete set of points of y and t given by $y_i = \alpha + ih$ and $t_j = \alpha + jh$ where the constant step size, h is defined as follows

$$h = \frac{\beta - \alpha}{n}. \tag{4}$$

Before further explanation, the following notation will be used for simplicity

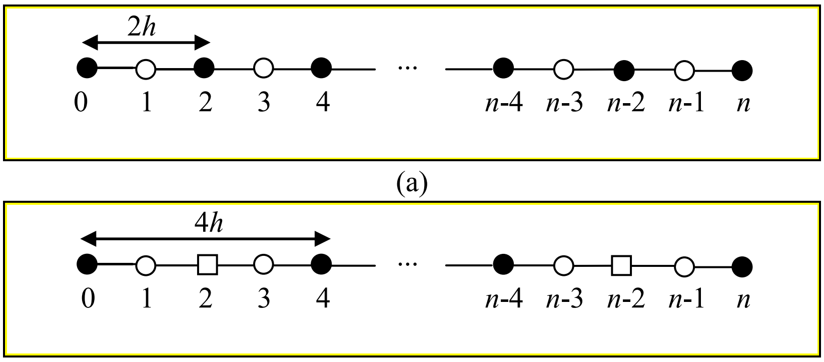
$$\begin{cases} K_{i,j} = K(y_i, t_j) \\ x_j = x(t_j) \\ b_i = b(y_i) \end{cases}. \tag{5}$$

As discussed in [12], application of the closed composite Newton-Cotes quadrature method reduced problem (1) to

$$\sum_{j=0}^{n} w_j K_{i,j}\, x_j = b_i,\ i = 0,\ 1,\ 2, \cdots, n-2,\ n-1,\ n, \tag{6}$$

where w_j is the quadrature weights. The standard closed composite Newton-Cotes quadrature approximation equations as defined in Eq. (6) can also be referred as full-sweep closed composite Newton-Cotes quadrature approximation equations. To formulate the half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations for problem (1), consider interval that divided uniformly as shown in Fig. 1.

Based on Fig. 1, the half- and quarter-sweep iterative methods will compute approximate values onto node points of type ● only until the convergence criterion is reached. Then, approximate solutions for the remaining points (points of the different types) can be computed directly.



(b)

Fig. 1 – Distribution of uniform node points for the: a) half- and b) quarter-sweep cases respectively.

By applying the half- and quarter-sweep iteration concepts, the generalized full-, half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations is

$$\sum_{j=0,p,2p}^{n} w_j K_{i,j}\, x_j = b_i \,, \tag{7}$$

for $i = 0, p, 2p, \cdots, n-2p, n-p, n$. The value of p, which corresponds to 1, 2 and 4 represents the full-, half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations respectively. Moreover, Eq. (7) can be represented in matrix form as

$$Ax = b \,, \tag{8}$$

where

$$A = \begin{bmatrix} w_0 K_{0,0} & w_p K_{0,p} & w_{2p} K_{0,2p} & \cdots & w_{n-2p} K_{0,n-2p} & w_{n-p} K_{0,n-p} & w_n K_{0,n} \\ w_0 K_{p,0} & w_p K_{p,p} & w_{2p} K_{p,2p} & \cdots & w_{n-2p} K_{p,n-2p} & w_{n-p} K_{p,n-p} & w_n K_{p,n} \\ w_0 K_{2p,0} & w_p K_{2p,p} & w_{2p} K_{2p,2p} & \cdots & w_{n-2p} K_{2p,n-2p} & w_{n-p} K_{2p,n-p} & w_n K_{2p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_0 K_{n-2p,0} & w_p K_{n-2p,p} & w_{2p} K_{n-2p,2p} & \cdots & w_{n-2p} K_{n-2p,n-2p} & w_{n-p} K_{n-2p,n-p} & w_n K_{n-2p,n} \\ w_0 K_{n-p,0} & w_p K_{n-p,p} & w_{2p} K_{n-p,2p} & \cdots & w_{n-2p} K_{n-p,n-2p} & w_{n-p} K_{n-p,n-p} & w_n K_{n-p,n} \\ w_0 K_{n,0} & w_p K_{n,p} & w_{2p} K_{n,2p} & \cdots & w_{n-2p} K_{n,n-2p} & w_{n-p} K_{n,n-p} & w_n K_{n,n} \end{bmatrix}_{\left(\frac{n}{p}+1\right)\times\left(\frac{n}{p}+1\right)} \tag{9}$$

$$x = \begin{bmatrix} x_0 & x_p & x_{2p} & \cdots & x_{n-2p} & x_{n-p} & x_n \end{bmatrix}^T , \tag{10}$$

and

$$b = \begin{bmatrix} b_0 & b_p & b_{2p} & \cdots & b_{n-2p} & b_{n-p} & b_n \end{bmatrix}^T . \tag{11}$$

From Eq. (8), it is obvious that, applications of the half- and quarter-sweep iteration concepts reduced the size of the original matrix $(N+1)\times(N+1)$ to $\left(\frac{N}{2}+1\right)\times\left(\frac{N}{2}+1\right)$ and $\left(\frac{N}{4}+1\right)\times\left(\frac{N}{4}+1\right)$ respectively.

In order to facilitate the formulation of full-, half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations for problem (1), further discussion will be restricted onto first order Newton-Cotes quadrature method i.e. composite trapezoidal (CT) scheme. Based on CT scheme, weights w_j will satisfy the following relation

$$w_j = \begin{cases} \dfrac{1}{2} ph, & j = 0, n \\ ph, & \text{otherwise.} \end{cases} \tag{12}$$

3. ARITHMETIC MEAN ITERATIVE METHODS

As afore-mentioned, FSAM, HSAM and QSAM methods will be utilized to solve the corresponding full-, half- and quarter-sweep closed composite Newton-Cotes quadrature approximation equations based on CT scheme. Fundamentally, iteration process for the FSAM, HSAM and QSAM involves of solving two independent systems i.e. x^1 and x^2. To develop the formulation for AM methods, let the coefficient matrix A in Eq. (8) be split as

$$A = D - L - U \,, \tag{13}$$

where

$$D = \begin{bmatrix} w_0 K_{0,0} & & & & & & \\ & w_p K_{p,p} & & & & & \\ & & w_{2p} K_{2p,2p} & & 0 & & \\ & & & \ddots & & & \\ & & 0 & & w_{n-2p} K_{n-2p,n-2p} & & \\ & & & & & w_{n-p} K_{n-p,n-p} & \\ & & & \cdots & & & w_n K_{n,n} \end{bmatrix} \tag{14}$$

$$-L = \begin{bmatrix} & & & & & & \\ w_0 K_{p,0} & & & & & & \\ w_0 K_{2p,0} & w_p K_{2p,p} & & & 0 & & \\ \vdots & \vdots & \ddots & & & & \\ w_0 K_{n-2p,0} & w_p K_{n-2p,p} & \cdots & w_{n-3p} K_{n-2p,n-3p} & & & \\ w_0 K_{n-p,0} & w_p K_{n-p,p} & \cdots & w_{n-3p} K_{n-p,n-3p} & w_{n-2p} K_{n-p,n-2p} & & \\ w_0 K_{n,0} & w_p K_{n,p} & \cdots & w_{n-3p} K_{n,n-3p} & w_{n-2p} K_{n,n-2p} & w_{n-p} K_{n,n-p} & \end{bmatrix} \tag{15}$$

and

$$-U = \begin{bmatrix} & w_p K_{0,p} & w_{2p} K_{0,2p} & w_{3p} K_{0,3p} & \cdots & w_{n-p} K_{0,n-p} & w_n K_{0,n} \\ & & w_{2p} K_{p,2p} & w_{3p} K_{p,3p} & \cdots & w_{n-p} K_{p,n-p} & w_n K_{p,n} \\ & & & w_{3p} K_{2p,3p} & \cdots & w_{n-p} K_{2p,n-p} & w_n K_{2p,n} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & w_{n-p} K_{n-2p,n-p} & w_n K_{n-2p,n} \\ & & & & & & w_n K_{n-p,n} \\ & & & & & & \end{bmatrix} \tag{16}$$

respectively.

Thus, for nonsingular $(D - \omega L)$ and $(D - \omega U)$ matrices, the general formulation for all three AM methods is defined as follows

$$\begin{cases} (D-\omega L)\, x^1 = \left[\left(1-\omega\right) D + \omega U \right] x^{(k)} + \omega b \\ (D-\omega U)\, x^2 = \left[\left(1-\omega\right) D + \omega L \right] x^{(k)} + \omega b. \\ x^{(k+1)} = \frac{1}{2}\left(x^1 + x^2 \right) \end{cases} \tag{17}$$

Clearly, the iteration matrix for AM methods is defined as

$$T_{AM} = \frac{1}{2}\left[\left(D-\omega L\right)^{-1} \left(\left(1-\omega\right) D + \omega U \right) + \left(D-\omega U\right)^{-1} \left(\left(1-\omega\right) D + \omega L \right) \right] \tag{18}$$

and it is already noted that the AM methods converges if and only if spectral radius of the iteration matrix is less than one, $\rho\left(T_{AM}\right) < 1$ [2]. The AM methods as explained in (17) are characterized by having within its overall mathematical structure certain well-defined substructures that can be executed simultaneously. This feature makes the AM methods preferably suitable for implementation on a multiprocessor system. By assuming $x_i = x_j$, the algorithm for FSAM, HSAM and QSAM methods associated with full-, half and quarter-sweep closed composite Newton-Cotes quadrature approximation equations respectively to solve problem (1) would be described in Algorithm 1.

Algorithm 1. FSAM, HSAM and QSAM schemes

i. Initializing all the parameters and set $x^{(0)} = x^1 = x^2$ and ε

ii. Iteration cycle

  a. Stage 1

   1. Level 1

    for $i = 0, p, 2p, \cdots, n-2p, n-p, n$

    Compute $x_i^1 \leftarrow \left(1-\omega\right) x_i^{(k)} + \dfrac{\omega}{w_i K_{i,i}} \left(b_i - \sum\limits_{j=0,p,2p}^{i-p} w_j K_{i,j} x_j^1 - \sum\limits_{j=i+p,i+2p,i+3p}^{n} w_j K_{i,j} x_j^{(k)} \right)$

  b. Stage 2

   1. Level 2

    for $i = n, n-p, n-2p \cdots, 2p, p, 0$

    Compute $x_i^2 \leftarrow \left(1-\omega\right) x_i^{(k)} + \dfrac{\omega}{w_i K_{i,i}} \left(b_i - \sum\limits_{j=0,p,2p}^{i-p} w_j K_{i,j} x_j^{(k)} - \sum\limits_{j=i+p,i+2p,i+3p}^{n} w_j K_{i,j} x_j^2 \right)$

   2. **for** $i = 0, p, 2p, \cdots, n-2p, n-p, n$

    Compute $x_i^{(k+1)} \leftarrow \frac{1}{2}\left(x_i^1 + x_i^2 \right)$

iii. Check the convergence. If the converge criterion is satisfied, go to Step (iv), otherwise, repeat the iteration cycle (i.e., go to Step (ii))

iv. Stop

Based on Algorithm 1, the AM algorithms are explicitly performed by using all equations at Levels 1 and 2 alternately until the solution satisfied a specified convergence criterion i.e. the maximum norm $\left\| x^{(k+1)} - x^{(k)} \right\| \le \varepsilon$, where ε is the convergence criterion.

After the iteration process, additional calculation is required for HSAM and QSAM methods to calculate the remaining points. In this paper, second order Lagrange [13] technique will be applied to compute the remaining points. The formulations to calculate remaining points using second order Lagrange interpolation for HSAM and QSAM are defined in (19) and (20) respectively as follows

$$x_i = \begin{cases} \frac{3}{8}x_{i-1} + \frac{3}{4}x_{i+1} - \frac{1}{8}x_{i+3}\,, & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4}x_{i-1} + \frac{3}{8}x_{i+1} - \frac{1}{8}x_{i-3}\,, & i = n-1 \end{cases} \tag{19}$$

and

$$x_i = \begin{cases} \frac{3}{8}x_{i-2} + \frac{3}{4}x_{i+2} - \frac{1}{8}x_{i+6}\,, & i = 2,6,10,\cdots,n-6 \\ \frac{3}{4}x_{i-2} + \frac{3}{8}x_{i+2} - \frac{1}{8}x_{i-6}\,, & i = n-2 \\ \frac{3}{8}x_{i-1} + \frac{3}{4}x_{i+1} - \frac{1}{8}x_{i+3}\,, & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4}x_{i-1} + \frac{3}{8}x_{i+1} - \frac{1}{8}x_{i-3}\,, & i = n-1. \end{cases} \tag{20}$$

4. NUMERICAL TESTS

In order to compare the performances of the iterative methods described in the previous section, several tests were carried out on the following two first kind linear Fredholm integral equations with semi-smooth kernel which will generates dense matrix A.

Test Problem 1 [14]. Consider the linear Fredholm integral equations of the first kind

$$\int_0^1 K(y,t)\,x(t)\,\mathrm{d}t = \frac{1}{6}(y^3 - y)\,,\ 0 < y < 1\,, \tag{21}$$

with kernel

$$K(y,t) = \begin{cases} t(y-1), & t < y \\ y(t-1), & y \le t \end{cases}. \tag{22}$$

The exact solution of the problem (21) is

$$x(y) = y\,. \tag{23}$$

Test Problem 2 [14]. Consider the following first kind linear Fredholm integral equations

$$\int_0^1 K(y,t)\,x(t)\,\mathrm{d}t = e^y + (1-e)y - 1,\,,\ 0 < y < 1 \tag{24}$$

with kernel

$$K(y,t) = \begin{cases} t(y-1), & t \le y \\ y(t-1), & y < t \end{cases} \tag{25}$$

and the exact solution is given by

$$x(y) = e^y\,. \tag{26}$$

For the numerical tests, three parameters i.e. number of iterations, CPU time (in seconds) and maximum absolute error will be measured. All the simulations were implemented by a computer with processor Intel(R) Core(TM) 2 CPU 1.66GHz and algorithm codes were written in C programming language. Throughout the simulations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$ and carried out on several different values of n. Meanwhile, the experimental values of ω were obtained within ± 0.01 by running the program for different values of ω and choosing the one(s) that gives the minimum number of iterations.The numerical results of the tested iterative methods for test problems 1 and 2 are tabulated in Tables 1 and 2 respectively. Meanwhile, reduction percentages in terms of number of iterations and CPU time for the HSGS, QSGS, FSAM, HSAM and QSAM methods compared with FSGS method have been summarized in Table 3.

Table 1

Numerical results of the tested iterative methods with CT scheme for test problem 1

Methods	Number of iterations					
	n					
	240	**480**	**960**	**1920**	**3840**	**7680**
FSGS	303	375	451	540	637	743
HSGS	237	303	375	451	540	637
QSGS	179	237	303	375	451	540
FSAM	135	136	139	140	158	215
HSAM	132	135	136	139	140	158
QSAM	125	132	135	136	139	140
Methods	**CPU time (in seconds)**					
	n					
	240	**480**	**960**	**1920**	**3840**	**7680**
FSGS	4.29	13.39	52.31	183.39	595.79	1974.28
HSGS	2.97	5.14	17.18	68.83	225.64	694.11
QSGS	2.08	3.21	6.24	21.19	87.94	274.33
FSAM	3.25	10.75	37.19	127.33	364.12	1120.38
HSAM	2.44	3.92	13.51	52.27	173.17	557.73
QSAM	1.28	2.73	4.93	17.36	66.39	212.75
Methods	**Maximum absolute error**					
	n					
	240	**480**	**960**	**1920**	**3840**	**7680**
FSGS	6.847871E-10	7.368467E-10	9.065918E-10	8.796745E-10	9.369449E-10	9.687818E-10
HSGS	6.493864E-10	6.847871E-10	7.368467E-10	9.065918E-10	9.148253E-10	9.369449E-10
QSGS	5.976255E-10	6.493864E-10	6.847871E-10	7.368467E-10	9.065918E-10	9.252085E-10
FSAM	8.699241E-10	1.346541E-09	8.040829E-10	7.487605E-10	1.106130E-09	1.649299E-09
HSAM	9.066320E-10	8.707608E-10	1.374749E-09	8.040829E-10	7.487605E-10	1.106130E-09
QSAM	1.014292E-09	9.066320E-10	8.707608E-10	1.414400E-09	8.040829E-10	7.487605E-10

5. COMPUTATIONAL COMPLEXITY ANALYSIS

In order to measure the computational complexity of the FSAM, HSAM and QSAM methods, an estimation amount of the computational work required for iterative methods have been conducted. The computational work is estimated by considering the arithmetic operations performed per iteration. To estimate the computational work for AM methods, the value for kernel K, function b and quadrature weights w_j are store beforehand.

Based on Algorithm 1, it can be observed that the number of arithmetic operations required (excluding the convergence test) in computing a value for each node point in the solution domain for FSAM, HSAM and QSAM methods are $\frac{2n}{p}+5$ addition/subtraction (ADD/SUB) and $\frac{4n}{p}+9$ multiplication/division (MUL/DIV) operations. For HSAM and QSAM methods, the iteration processes is carried out only on $\frac{n}{2}+1$ and $\frac{n}{4}+1$ mesh points respectively. Thus, additional two ADD/SUB and six MUL/DIV operations are involve to calculate a mesh point for the remaining points after convergence by using second order Lagrange interpolation. Hence, the total arithmetic operations involved for FSAM, HSAM and QSAM methods are summarized in Table 4.

Table 2

Numerical results of the tested iterative methods with CT scheme for test problem 2

Methods	Number of iterations					
	n					
	240	480	960	1920	3840	7680
FSGS	315	388	470	559	657	769
HSGS	247	315	388	470	559	657
QSGS	188	247	315	388	470	559
FSAM	141	144	145	145	164	224
HSAM	137	141	144	145	145	164
QSAM	130	137	141	144	145	145
Methods	**CPU time (in seconds)**					
	n					
	240	480	960	1920	3840	7680
FSGS	4.44	14.83	58.32	195.66	633.92	2159.60
HSGS	3.13	5.68	19.24	76.42	238.07	757.13
QSGS	2.28	3.81	7.21	26.39	98.41	293.16
FSAM	3.65	12.19	39.74	132.61	370.42	1163.35
HSAM	2.70	3.97	14.61	56.50	182.69	578.60
QSAM	1.42	2.85	5.13	19.25	72.23	223.41
Methods	**Maximum absolute error**					
	n					
	240	480	960	1920	3840	7680
FSGS	3.916332E-06	9.810628E-07	2.453851E-07	6.290920E-08	2.602358E-08	4.977974E-08
HSGS	1.740438E-05	4.387550E-06	1.101409E-06	2.758089E-07	6.904009E-08	2.602358E-08
QSGS	6.846369E-05	1.740438E-05	4.387550E-06	1.101409E-06	2.758089E-07	6.904009E-08
FSAM	3.915194E-06	9.806430E-07	2.449272E-07	6.115767E-08	1.747092E-08	7.573154E-09
HSAM	1.740456E-05	4.386528E-06	1.101024E-06	2.753261E-07	6.842856E-08	1.866109E-08
QSAM	6.846439E-05	1.740456E-05	4.386528E-06	1.101024E-06	2.753261E-07	6.842856E-08

6. CONCLUDING REMARKS

In this paper, an application of the AM iterative methods for solving dense nonsymmetric matrices arising from the first kind linear Fredholm integral equations with semi-smooth kernel is examined. Through the numerical results obtained, it clearly shows that applications of the AM methods reduce number of iterations and execution time compared to the GS methods. Meanwhile, among the AM methods, QSAM method has the least number of iterations and compute with the fastest time for all mesh sizes. In terms of accuracy, approximate solutions for all the three AM methods are in good agreement compared to the GS method. Finally, it can be concluded that the QSAM method is better than other two AM (FSAM and HSAM) and GS methods in the sense of number of iterations and execution time. This mainly because of the reduction in terms of computational complexity; since the QSAM method will only consider approximately quarter of all interior node points in a solution domain during iteration process.

Table 3

Reduction percentages of the HSGS, QSGS, FSAM, HSAM and QSAM methods compared with FSGS method

Methods	Number of iterations	
	Test Problem 1 (%)	Test Problem 2 (%)
HSGS	14.26 - 21.79	14.56 - 21.59
QSGS	27.32 - 40.93	27.30 - 40.32
FSAM	55.44 - 75.20	55.23 - 75.04
HSAM	56.43 - 78.74	56.50 - 78.68
QSAM	58.74 - 81.16	58.73 - 81.15
Methods	**CPU time**	
	Test Problem 1 (%)	**Test Problem 2 (%)**
HSGS	30.76 - 67.16	29.50 - 67.01
QSGS	51.51 - 88.45	48.64 - 87.64
FSAM	19.71 - 43.26	17.79 - 46.14
HSAM	43.12 - 74.18	39.18 - 74.95
QSAM	70.16 - 90.58	68.01 - 91.21

Table 4

Total computing operations involved for the FSAM, HSAM and QSAM methods

Methods	Arithmetic Operations			
	Per iteration		After Convergence	
	ADD/SUB	MUL/DIV	ADD/SUB	MUL/DIV
FSAM	$2n^2+7n+5$	$4n^2+13n+9$	-	-
HSAM	$\frac{n^2}{2}+\frac{7n}{2}+5$	$n^2+\frac{13n}{2}+9$	n	$3n$
QSAM	$\frac{n^2}{8}+\frac{7n}{4}+5$	$\frac{n^2}{4}+\frac{13n}{4}+9$	$\frac{3n}{2}$	$\frac{9n}{2}$

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