HIGH-ACCURACY ALGORITHMS TO THE SOLUTION OF THE OPTIMAL OUTPUT FEEDBACK PROBLEM FOR THE LINEAR SYSTEMS[*]

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On the basis of Levine-Athans algorithm the high accuracy algorithm for the construction of the output optimal linear regulator is proposed. Initial approximation is chosen by applying the penalty function method that allows developing algorithms which produce high accuracy solution using symbolic calculations. Results are extended for the discrete case and illustrated on the examples.

Key words: output feedback, numerical methods, algebraic Lyapunov equation, symbolic calculations.

1. INTRODUCTION

One of the important branches of the optimal control theory is a construction of the optimal regulator over all phase coordinates. For the solution of this problem there exist various calculation algorithms basing on the finding of the non-negatively defined solution of the matrix algebraic Riccati equations (MARE) [1, 2]. Another important branch of these investigations is a construction of the optimal output regulator [3-9]. To solve this problem in [8] the convex analysis method, in [5] Newton method, in [3, 4] penalty function method had been applied. In [3, 4, 8] the algorithms are proposed which don't require choice of the initial conditions. But they don't allow obtaining a solution with enough accuracy.

However there exists a class of problems requiring the construction of the regulator over a part of the phase coordinates. Solution of these problems is reduced to the solution of two essentially nonlinear equations [5]. Recently, in connection with developing of the Symbolic Toolbox of the package Matlab allowing to carry out calculation with high accuracy, the algorithms are developed for the solution of MARE both in stationary [2] and periodic [1] cases. The similar situation appears in the solution of the optimal output regulator problem, where Newton method is applicable.

In the present paper on the base of given in [5] method a calculation algorithm is proposed. This algorithm allows one to solve the considered problem with high accuracy using Symbolic Toolbox of package Matlab. The initial condition is chosen using the penalty function method [3]. An essential difference of accuracy of the solution of the problem obtained by the previous algorithms which use calculations by usual procedures of Matlab is illustrated on the examples.

2. STATEMENT OF THE PROBLEM

Let the movement of object be described by the following system of the linear differential equations with constant coefficients

$$\dot{x}(t) = Fx(t) + Gu(t), \quad x(0) = x_0 \tag{1}$$

$$y = Cx(t). \tag{2}$$

It needs to minimize the functional

$$J = < \int_0^\infty (x'Qx + u'Ru)\mathrm{d}t >, \tag{3}$$

with the regulation law

$$u(t) = Ky(t), \tag{4}$$

under the condition that the closed-loop system (1–4) be asymptotically stable, i.e. the condition $\mathrm{Re}(\lambda \ (F + GKC)) < 0$ has to be satisfied. Here K is sought $(m \times l)$ dimensional, C-given $(l \times n)$ dimensional constant matrices, x is a state vector of the system, u-vector of controlling influences, y-output vector (measurement), x_0 is a random vector of the initial conditions with $< x_0 >= 0$, $X_0 = < x_0 \ x_0' >$ – covariance matrix, $< \cdot >$ is a sign of mathematical expectation, $F, G, Q = Q' \ge 0$; $R = R' > 0$ – constant matrices of the corresponding dimensions, a prime hereinafter means operation of transpose.

It is known [5], that the solution of the problem (1–4) may be reduced to finding

$$K = -R^{-1}G'SUC'(CUC')^{-1}, \tag{5}$$

where $S = S' > 0$ and $U = U' > 0$ are the solutions of the following nonlinear matrix algebraic equations

$$(F + GKC)'S + S(F + GKC) + Q + C'K'RKC = 0, \tag{6}$$

$$(F + GKC)U + U(F + GKC)' + X_0 = 0. \tag{7}$$

The sought matrix K may be determined by solving the equations (5–7).

Note that, using the results of [1] one can carry out the sensitivity analysis of the equations (6–7). It will allow one the use the problem (1–4) for the solution of some applied problems.

3. MAIN RESULTS

The solution of the stated above problem may be found by the following iterative scheme [5].

Algorithm 1. 1. The matrices $F, G, Q = Q' > 0, \quad R = R' > 0, \ C$ are given.

2. Choose initial approximation K_0 that provides asymptotical stability of the matrix $(F + GK_{n-1}C)$. Suppose $(n-1)$ iterations have been done.

3. Solving algebraic Lyapunov equations

$$S_n(F + GK_{n-1}C) + (F + K_{n-1}C)'S_n + Q + C'K_{n-1}'RK_{n-1}C = 0,$$
$$U_n(F + GK_{n-1}C)' + (F + GK_{n-1}C)U_n + X_0 = 0,$$

find $S_n > 0, U_n > 0$. This step may be solved by the help of the procedure alyap.m. The algorithm and realization of this procedure is given below (see algorithm 3).

4. Check up the criteria $Sp(S_n) \le Sp(S_{n-1})$, where $Sp(\cdot)$ is a trace of the matrix. If it is satisfied, the iteration is stopped, otherwise:

5. Calculate

$$K_n = -R^{-1}G'S_nU_nC'(CU_nC')^{-1},$$

take $K_{n-1} = K_n$ and go to step 3.

The most difficult procedure here is the finding the initial approximation. For the solution of this problem, we offer below the method that uses penalty function [3]. It stimulated by the fact that the other methods [4, 8] do not admit the use of Symbolic calculations technique.

Let the controlling influence be searched as

$$u = Lx, \tag{8}$$

i.e. let's consider an optimal stabilization problem over all phase coordinates. It means that, the optimal regulator is sought as a function of all phase coordinates.

It is known [11, 12] that the solution of the problem (1), (3), (8) indeed is

$$L = -R^{-1}G'S, \tag{9}$$

where $S = S' > 0$ is a solution of the matrix algebraic Riccati equation (ARE)

$$F'S + SF - SGR^{-1}G'S + Q = 0. \tag{10}$$

It is not difficult to prove, that the problem (1), (3), (8) may be reduced to the problem (1–4) with the help of Singular Value Decomposition (SVD) of the matrix C.

The procedure MATLAB $[V, D, U] = svd(C)$ produces a diagonal matrix D of the same dimension as C that has a form $D = [\sigma_e\ 0]$, with nonnegative diagonal elements σ_e in decreasing order, and unitary matrices V and U such that [13]

$$C = VDU', \quad V'V = E, \quad U'U = E, \tag{11}$$

where E -unit matrix of the corresponding dimension.

After some simple transformations the problem (1), (3), (8) is reduced to the solution of the following output optimal control problem

$$\dot{z} = \overline{F}z + \overline{G}u,$$

$$J = \int_0^\infty (z'(t)\overline{Q}z(t) + u'(t)Ru(t))\mathrm{d}t \to \min,$$

where

$$U'x = z, \quad \overline{F} = U'FU, \quad \overline{G} = U'G, \quad \overline{Q} = U'QU, \quad h = \sigma_e^{-1}V'y, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_1 = h, \quad u = K\sigma_e^{-1}V'y.$$

The specific character of this problem is that here only part of the coordinates is measured, i.e. $z = z(z_1)$.

Let us represent the matrix L in the form $L = [L_1, L_2]$, where L_1 is $m \times l$ dimensional, $L_2 - m \times (n - l)$ dimensional matrices. If to the problem (1), (3), (8) to add the additional condition

$$L_2 = 0, \tag{12}$$

then

$$u = Lx = [L_1\ \ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = L_1 x_1.$$

Thus, in this case $K = L_1$.

For fulfillment the condition (12) to the functional (3) we add the term $[0\ \ L_2]'[0\ \ L_2]$ with scalar weight $\alpha > 0$. Then replacing the functional (3) by

$$I = \int_0^\infty x'(t)(Q + L'RL + \alpha[0\ \ \ L_2]'[0\ \ \ \ L_2])x(t)\mathrm{d}t, \tag{13}$$

one may easily prove that $L_2 \to 0$, *as* $\alpha \to \infty$ and the functional (13) reaches its minimum on
$$u = L_1 x_1, \text{ i.e. } J(u) \to I(u).$$
It is known [11, 12] that the value of the functional (13) on the trajectory
$$\dot{x} = (F + GL)x(t),$$
is calculated as
$$J = Sp(S(\alpha, L), X_0),$$

where $Sp(\cdot)$ means a trace of the matrix, $S(\alpha, L)$ is a solution of the matrix algebraic Lyapunov equation (MALE)

$$(F + GL)'S(\alpha, L) + S(\alpha, L)(F + GL) = -(Q + L'RL + \alpha[0 \quad L_2]'[0 \quad L_2]). \tag{14}$$

It is difficult to express the solutions of the equation (14) through the elements of the matrix L. For this purpose the adjoint gradient method is used. Initial approximation of the matrix L is calculated with the use of the stabilizing solution of the equation (10) that provides stability of the closed-loop system (1–4). The solution of (14) by $\alpha \to \infty$ we find in the form

$$L = [L_1 \quad 0], \quad K = L_1.$$

Thus the following algorithm is proposed to the solution of the problem (1–4).

Algorithm 2. 1. Given matrices F, G, Q, C, R find the stabilizing solution S from MARE (10).

2. Calculate initial L^1 from (9). Suppose that the $(i-1)$ iterations have been done.

3. Take enough large number α and solve the equation

$$(F + GL^i)'S + S(F + GL^i) = -\left(Q + (L^i)'RL^i + \alpha[0 \quad L_2^i]' \ [0 \quad L_2^i]\right),$$

$$U(F + GL^i)' + (F + GL^i)U = -X_0.$$

4. Calculate

$$\frac{\partial Sp(S(\alpha, L))}{\partial L_{kj}} = Sp\left[2\frac{\partial(F + GL)}{\partial L_{kj}}SU + U\frac{\partial}{\partial L_{kj}}(Q + L'RL + \alpha[0 \quad L_2]'[0 \quad L_2])\right]_{L=L^i}.$$

5. Construct the vector

$$L_q = \frac{dSp(S)}{dL}\bigg|_{L=L^i}, \quad q = 2, 3, ..., \hat{p}, \quad 1 \le \hat{p} \le p, \ p = mn.$$

6. Construct the matrix

$$\hat{L}_i = \begin{bmatrix} L_{11}^q & L_{12}^q & ... & L_{1n}^q \\ L_{21}^q & L_{22}^q & ... & L_{2n}^q \\ ... & ... & ... & ... \\ L_{m1}^q & L_{m2}^q & ... & L_{mn}^q \end{bmatrix}.$$

7. The next L^{i+1} is calculated by the relation

$$L^{i+1} = L^i - \gamma^i \hat{L}_i, \quad i = 1, 2, ..., \hat{p},$$

$$L^{\hat{p}+i+1} = L^{\hat{p}+i} + \beta^i (L^{\hat{p}+i} - L^{\hat{p}+1-i}),$$

where γ_i, β_i are calculated by the "Golden section" method.

8. Taking small real number ε check up the condition

$$\left\| L^{2\hat{p}+1} - L^{2\hat{p}} \right\| < \varepsilon,$$

where $\|\cdot\|$ is a matrix norm. If the condition is satisfied the procedure stops, otherwise choose another $\hat{p}$, take $L^{2\hat{p}+2}$ as L^i and go to step 3.

4. HIGH ACCURACY ALGORITHM FOR THE SOLUTION OF THE ALGEBRAIC LYAPUNOV EQUATIONS

The equations (6), (7) are the algebraic Lyapunov equations on the set K [14]. As one can see both proposed above algorithms require solving of these equations. For their solution there exist various computing algorithms, as well as, method of infinite series [12], Schur's method, matrix sign-function

method. From these the matrix sign function method is the most comprehensible for use of Symbolic Toolbox procedures of package Matlab. Therefore, we shall use further this method for the solution of the Lyapunov equation. Here we describe this method.

Let's the matrix algebraic Lyapunov equation be given

$$AX + XA' = C, \tag{15}$$

and it is required to find a symmetric matrix X on the set of Hurwitz matrices A (any square matrix) and C (any negatively defined symmetric matrix).

It is known that, if A is Hurwitz matrix, then the solution of the equation (15) exists. Here we give the algorithm from [12] for the solution the equation (15) with the help of matrix sign function. Matrix sign function sign A of the matrix A is determined as follows [14]

$$\text{sign}A = \lim_{i \to \infty} A_i, \quad A_{i+1} = \frac{1}{2}\left(A_i + A_i^{-1}\right), \quad A_0 = A.$$

Here is the algorithm from [12] to the solution of the equation (15).

Algorithm 3. 1. Take $A_0 = A'$; $C_0 = C$. Suppose n iterations have been done.

2. Calculate

$A_{i+1} = \alpha_i A_i + \beta_i A_i$, where $\alpha_i = \dfrac{1}{1 + \left|\det A_i\right|^{1/n}}$,

$\beta_i = 1 - \alpha_i$, n-dimension of the matrix A,

$C_{i+1} = \alpha_i C_i + \beta_i (A_i')^{-1} \quad C_i A_i^{-1}$;

3. If $\left\|A_{i+1} + E\right\| < \varepsilon$,   then $X = -\dfrac{1}{2}C_{i+1}$ and the procedure is stopped. Otherwise take $A_i = A_{i+1}$, $C_i = C_{i+1}$

and go to step 2. Here ε is a given constant.

This algorithm is easily realized in package Matlab by use of procedures Symbolic Toolbox.

Note that, there exist various algorithms to calculate the inverse matrix [15]. For the purpose and to calculate the determinant of the matrix we use the procedures inv.m and det.m of the Matlab package Symbolic Toolbox. The procedure alyap.m is developed on the base of offered algorithm that is also realizable by Symbolic Toolbox Matlab.

Thus the following algorithm for the solution of the problem (1–4) may be offered.

Algorithm 4. 1. The matrices F, G, Q, C, R are given.

2. Use algorithms 2 and 3 for choice the initial approach K_0.

3. By chosen K_0 using the algorithms 1 and 3 calculate the sought solution K_N, where $K_N \to K$, by $N \to \infty$.

All calculations are done by the Symbolic Toolbox Matlab.

The example below illustrates this result.

Example 1. The initial data of this example are taken from [8, 16]. In this case the matrices F, G, N, D indeed are

$$F = \begin{bmatrix} 0 & 1 & 0 \\ -79.2851 & -0.113 & 0 \\ 28.564 & 0.041 & 0 \end{bmatrix}; G = \begin{bmatrix} 0 & 0 \\ 0.0410 & -0.0047 \\ -0.0300 & -0.0016 \end{bmatrix}; N = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Appearing in (3) matrices Q and R are taken as

$$Q = N'N, \ R = D'D.$$

Then the observable vector is

$$y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; x = Cx.$$

Solving this problem by Matlab using the procedure Symbolic Toolbox by the help of Algorithm 2 the coefficient of the optimal regulator and minimal value of the corresponding functional are obtained as

$$K = \begin{bmatrix} -7.1828 & 7.8565 \\ 0.528700 & 2.228100 \end{bmatrix}, \; J = 47.86183.$$

Using the Algorithm1 the following values are obtained

$$K_A = \begin{bmatrix} 6.421582 & 0.29845 \\ -0.052946 & 0.113614 \end{bmatrix}, \; J_A = 45.989114.$$

Gradient of the functional for the given case is calculated as

$$\left\| \frac{\partial J}{\partial K_A} \right\| = 1.135114\text{e-}029 .$$

The equations (6), (7) which contain K, are solved with accuracy of order 5.3311e-029 in [16]. In [8] the coefficient of the optimal regulator is calculated as

$$K_G = \begin{bmatrix} -2.4448 & -1.3049 \\ -0.5232 & -0.2921 \end{bmatrix}.$$

Corresponding minimum value of the functional is $J_G = 56.1632$ and the gradient is $\left\| \dfrac{\partial J}{\partial K_0} \right\| = 0.3172$.

Such essential deviations of the minimum values of the functional and gradients demonstrates efficiency [16] of the offered here algorithms.

***Example* 2.** Let's take

$$F = \begin{bmatrix} 0 & a \\ -\frac{1}{a} & 0 \end{bmatrix}; \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \; R = 1; \; G = \begin{bmatrix} 0 \\ \frac{1}{a} \end{bmatrix}; \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Results of the solution of this problem for various values of a solved by usual Matlab and Symbolic Toolbox are given in Table 1.

Table 1

Comparison table

a	λ - eigenvalue of the closed-loop system		K – coefficient of the optimal regulator	
	Usual calculations	Symbolic calculations	Usual calculations	Symbolic calculations
1	−0.408±0.92i	−0.408±0.92i	−0.8165	−0.8165
10^{-5}	−0.0007e+4; −5198e+4	−0.0000e+4; −1.5180e+4	−0.152	−0.1518012
10^{-9}	---	−1.1450231e-6; −895449.0643	---	−0.689549e-2
10^{-15}	---	−0.1319; −7.5838	---	−0.39022e-24
10^{-24}	---	−1.490186e-8; −1.11346e+8	---	−0.11134e-15
10^{-25}	---	---	---	---

Here "---" means that the condition of the asymptotical stability is not satisfied.

Solution of the discrete analogue of the problem (1)-(4) is reduced to the solution of the discrete algebraic Lyapunov equation. From existing methods for the solution of the discrete algebraic Lyapunov equation the method of infinite series is more easily realizable in the package Matlab by use of procedures Symbolic Toolbox. Let's describe this method [12].

Consider discrete algebraic Lyapunov equation

$$S - \Psi' S \Psi = R. \tag{16}$$

It is required to find a symmetric matrix S on the set of matrices Ψ (any square matrix) and R (any symmetric matrix). Here the following statement is valid: if eigenvalues of the matrix Ψ are inside of unit circle, then

$$Y = R + (\Psi') R \Psi + (\Psi')^2 R \Psi^2 + \dots . \tag{17}$$

Series (17) converges to the solution of the equation (16). Partial sums of this series are calculated by the following iterative formulas

$$Y_0 = R, \quad Y_{k+1} = Y_k + (\Psi')^{2k} Y_k \Psi^{2k}, \quad Y_k = \sum_{i=1}^{2k} (\Psi')^{i-1} R \Psi^{i-1}.$$

All steps of this scheme may be easily solved with the help of symbolic arithmetic. Another method is to reduce the equation (16) with the help of Cayley transformation

$$A = (\Psi - E)(E + \Psi)^{-1}; \; C = 2(\Psi' + E)^{-1} R (\Psi + E)^{-1}$$

to the continuous algebraic Lyapunov equation (14), that then may be solved by the sign-function method. Let us illustrate this by the following example.

Example 3. In [9] the matrices Ψ, Γ, C, Q, R are taken as

$$\Psi = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -0.1 & 1 \\ 0 & 0 & 3 \end{bmatrix}; \; \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \; Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}; \; R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case solving the discrete problem it is obtained

$$F = \begin{bmatrix} -1.74277688047887 & -0.37934272471665 \\ 0.0006658209882 & -2.8350876761572 \end{bmatrix}.$$

The minimal value of the functional is $J = 78.28046546698863$.

Solving this problem in package Matlab by use of procedures Symbolic Toolbox we obtain the optimality condition as

$$\left\| \frac{\partial J}{\partial K} \right\| = 1.1.728742597191068e\text{-}024 .$$

Corresponding equations by the obtained K are solved with accuracy 1.7423e-023.

In [9] for the same problem is obtained

$$K = \begin{bmatrix} -1.9 & -0.137 \\ 0.00082 & -2.9 \end{bmatrix}, \; J = 79.344866$$

Comparison of these two results demonstrates that the offered here algorithms improve the results of the work [9].

5. CONCLUSION

In the work high accuracy algorithms are offered for the solution of the optimal regulator output problem both in the stationary and periodic cases that allows one to obtain solutions with necessary accuracy.

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