THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 13, Number 3/2012, pp. 215–224

SOLITONS AND CONSERVED QUANTITIES OF THE ITO EQUATION

Ghodrat EBADI [1], A. H. KARA [2], Marko D. PETKOVIC [3], Ahmet YILDIRIM [4], Anjan BISWAS [5]

[1] University of Tabriz, Tabriz, Faculty of Mathematical Sciences, 51666-14766; Iran
[2] University of the Witwatersrand, School of Mathematics, Wits 2050, Johannesburg, South Africa
[3] University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia
[4] Ege University, Department of Mathematics, 35100 Bornova, Izmir, Turkey
University of South Florida, Department of Mathematics and Statistics, Tampa, FL 33620-5700, USA
[5] Delaware State University, Department of Mathematical Sciences, Dover, DE 19901-2277, USA
E-mail: biswas.anjan@gmail.com

This paper obtains the soliton solutions of the Ito integro-differential equation. The G'/G method will be used to carry out the solutions of this equation and then the solitary wave ansatz method will be used to obtain a 1-soliton solution of this equation. Finally, the invariance and multiplier approach will be applied to recover a few of the conserved quantities of this equation.

Key words: conserved quantities; solitary wave ansatz method; solitons; Ito integro-differential equation; G'/G method.

1. INTRODUCTION

The study of nonlinear evolution equation (NLEE) has been going on for the past few decades [1-6]. During this time, there has been a measurable progress that has been made. There are lots of equations that have been integrated. There are various methods of integrability that has been developed so far. In addition to NLEEs, there has been a growing interest in the nonlinear integro-differential evolution equations. Some of these commonly studied integro-differential evolution equations are the Ito equation, the generalized shallow water wave equation and many others. There are various analytical methods of solving these NLEEs that has also been developed in the past couple of decades. Some of these methods are exp-function method, Fan's F-method, Riccati's equation method, Adomian decomposition method and many others. In this paper, the G'/G method will be developed to solve the Ito equation. Also, the solitary wave ansatz method will integrate the Ito equation. Finally, the conserved quantities for the Ito equation will be computed using the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates the total divergence.

2. THE G'/G METHOD

In this section, the G'/G method will be described and applied to obtain traveling wave solution of the Ito equation. This equation is studied in (1+1) dimensions as well as (1+2) dimensions. therefore, this work will be done in two following subsections.

2.1. (1+1) Dimensions

The (1+1)-dimensional form of the generalized Ito integro-differential equation that is going to be studied in this subsection is given by

$$q_{tt} + q_{xxxt} + a\left(2q_x q_t + q q_{xt}\right) + a q_{xx}\int_{-\infty}^{x} q_t \mathrm{d}x' = 0 \tag{1}$$

Here, in (1), q is the dependent variable while x and t are the independent variables. The coefficient a is constant. The eq. (1) can reduced to

$$v_{ttx} + v_{xxxxt} + a\left(2v_{xx}v_{xt} + v_x v_{xxt} + v_{xxx}v_t\right) = 0, \tag{2}$$

with using the potential $q = v_x$ .The eq. (2) is converted to the ODE

$$c^2 eu''' - ce^4 u^{(v)} + a\left(-2ce^3 u''u'' - ce^3 u'u''' - ce^3 u'''u'\right) = 0 \tag{3}$$

or equivalently

$$cu''' - e^3 u^{(v)} - ae^2\left((u')^2\right) = 0, \tag{4}$$

by the wave variables $v = u(\xi),\ \xi = ex - ct$, where primes denote the derivatives with respect to ξ, and e, c are real constants to be determined later.

The eq. (4) is then integrated twice. This converts it to

$$cu' - e^3 u''' - ae^2\left((u')^2\right) = 0, \tag{5}$$

In G'/G method, the solution $u(\xi)$ of equation (5) is considered in the finite series form

$$u(\xi) = \sum_{i=0}^{N} A_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{6}$$

where A_i are positive integers with $A_N \neq 0$ that will be determined. N is a positive integer that can be accomplished by balancing the linear term of highest order derivative with the highest order nonlinear term in equation (5). Here balancing u''' with $(u')^2$ gives $N = 1$. Therefore, we can write the solution of equation (5) in the form of

$$u(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right),\ \ A_1 \neq 0. \tag{7}$$

The function $G(\xi)$ in (6) is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{8}$$

where λ and μ are real constants to be determined. By using (8) and the general solution of (6), we can find u', u''' and $(u')^2$ as polynomials of G'/G. Substituting the general solution of (8) together with u', u''' and $(u')^2$ as polynomials of G'/G into equation (5) yield algebraic equations involving powers of G'/G. Equating the coefficients of each power of G'/G to zero gives a system of algebraic equations for $A_i, \lambda,\ \mu,\ e$ and c. Then, solving this system by Maple gives

$$A_1 = \frac{6e}{a},\ c = e^3(\lambda^2 - 4\mu), \tag{9}$$

where e and μ are the arbitrary constants. Next, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the three types of the following traveling wave solutions of the equation (4) are obtained.
When $\Delta = \lambda^2 - 4\mu > 0$, the hyperbolic function traveling wave solution is

$$u(\xi) := A_0 + \frac{6e}{a}\left(\frac{-\lambda}{2} + \gamma h(\xi)\right) \tag{10}$$

or equivalently

$$q(\xi):=\frac{6e^2\gamma^2}{a}h(\xi)(1-h(\xi)), \tag{11}$$

where $\gamma=\frac{\sqrt{\lambda^2-4\mu}}{2}$, $\xi=ex-e^3(\lambda^2-4\mu)t$, $h(\xi)=\frac{c_1\sinh(\gamma\xi)+c_2\cosh(\gamma\xi)}{c_2\sinh(\gamma\xi)+c_1\cosh(\gamma\xi)}$, and A_0 is an arbitrary constant. In (10) c_1, c_2 are arbitrary constants. If them are taken as special values, the various known results in the literature can be rediscovered, for instance, setting $c_2=0,\ c_1=1,\ e=1,\ a=2$ and $\alpha=A_0+\frac{-6\lambda}{2}$, then the solution of (2) can be written as $v_1(\xi):=\alpha+\sqrt{c}\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right]$ that it is the solution (41) in [6]. Setting $c_1=0,\ c_2=1,\ e=1,\ a=2$ and $\alpha=A_0+\frac{-6\lambda}{2}$, then the solution of (2) can be written as $v_2(\xi):=\alpha+\sqrt{c}\coth\left[\frac{\sqrt{c}}{2}(x-ct)\right]$.

From (5) we can extract other exact solutions of Eq. (5) reported earlier. When $\Delta=\lambda^2-4\mu=0$, the rational function traveling wave solution is

$$u(\xi):=A_0+\frac{6e}{a}(\frac{-\lambda}{2}+h(\xi)) \tag{12}$$

or equivalently

$$q(\xi):=\frac{-6e^2}{a}h^2(\xi), \tag{13}$$

where $\xi=ex$, $h(\xi)=\frac{c_1}{c_2+c_1\xi}$, and A_0 is an arbitrary constant.

Finally, when $\Delta=\lambda^2-4\mu<0$, the trigonometric function traveling wave solution is

$$u(\xi):=A_0+\frac{6e}{a}(\frac{-\lambda}{2}+\gamma h(\xi)) \tag{14}$$

or equivalently

$$q(\xi):=\frac{-6e^2\gamma^2}{a}(1+h^2(\xi)), \tag{15}$$

where $\gamma=\frac{\sqrt{4\mu-\lambda^2}}{2}$, $\xi=ex+e^3(4\mu-\lambda^2)t$, $h(\xi)=\frac{-c_1\sin(\gamma\xi)+c_2\cos(\gamma\xi)}{c_2\sin(\gamma\xi)+c_1\cos(\gamma\xi)}$, and A_0 is an arbitrary constant.

2.2. (1+2) Dimensions

The (1+2)-dimensional form of the generalized Ito integro-differential equation that is going to be studied in this subsection is given by [5]

$$q_{tt}+q_{xxxt}+a\left(2q_xq_t+qq_{xt}\right)+aq_{xx}\int_{-\infty}^{x}q_t\,dx'+bq_{yt}+dq_{xt}=0. \tag{16}$$

Here, in (16), q is the dependent variable while x, y and t are the independent variables. The coefficients a, b and d are constants. The eq. (16) can reduced to

$$v_{ttx} + v_{xxxxt} + a\left(2v_{xx}v_{xt} + v_x v_{xxt} + v_{xxx}v_t\right) + bv_{xyt} + dv_{xxt} = 0, \tag{17}$$

with using the potential.

The eq. (17) is converted to the ODE

$$c^2 eu''' - ce^4 u^{(v)} + a\left(-2ce^3 u''u'' - ce^3 u'u''' - ce^3 u'''u'\right) - cbefu''' - cde^2 u''' = 0 \tag{18}$$

or equivalently

$$(c - bf - de)u''' - e^3 u^{(v)} - ae^2\left((u')^2\right) = 0, \tag{19}$$

by the wave variables $v = u(\xi)$, $\xi = ex + fy - ct$, where primes denote the derivatives with respect to ξ, and e, f and c are real constants to be determined later. The eq. (19) is then integrated twice. This converts it into

$$(c - bf - de)u' - e^3 u''' - ae^2\left((u')^2\right) = 0. \tag{20}$$

Here balancing u''' with $(u')^2$ gives $N = 1$. Therefore, we can write the solution of equation (20) in the form of

$$u(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right), \quad A_1 \neq 0. \tag{21}$$

The function $G(\xi)$ in (21) is the solution of the auxiliary linear ordinary differential equation (20). Substituting the general solution of (20) together with u', u''' and $(u')^2$ as polynomials of G'/G into equation (21) yield algebraic equations involving powers of G'/G. Equating the coefficients of each power of G'/G to zero gives a system of algebraic equations for A_i, λ, μ, e, f and c. Then, solving this system by Maple gives

$$A_1 = \frac{6e}{a}, \quad c = bf + de + e^3(\lambda^2 - 4\mu), \tag{22}$$

where e, f and μ are the arbitrary constants. Next, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the three types of the following traveling wave solutions of the equation (20) are obtained.

When $\Delta = \lambda^2 - 4\mu > 0$, the hyperbolic function traveling wave solution is

$$u(\xi) := A_0 + \frac{6e}{a}\left(\frac{-\lambda}{2} + \gamma h(\xi)\right) \tag{23}$$

or equivalently

$$q(\xi) := \frac{6e^2\gamma^2}{a} h(\xi)(1 - h(\xi)), \tag{24}$$

where $\xi = ex + fy - \left(bf + de + e^3(\lambda^2 - 4\mu)\right)t$, $h(\xi) = \dfrac{c_1\sinh(\gamma\xi) + c_2\cosh(\gamma\xi)}{c_2\sinh(\gamma\xi) + c_1\cosh(\gamma\xi)}$, $\gamma = \dfrac{\sqrt{\lambda^2 - 4\mu}}{2}$, and A_0 is an arbitrary constant. In (23) c_1, c_2 are arbitrary constants. If them are taken as special values, the various known results in the literature can be rediscovered, for instance, setting $c_2 = 0$, $c_1 = 1$, $e = 1$, $a = 2$, $f = 1$, $c = b + d + (\lambda^2 - 4\mu)$ and $\alpha = A_0 + \dfrac{-6\lambda}{2}$, then the solution (17) can be written as $v_1(\xi) := \alpha + \sqrt{c-(b+d)}\tanh\left[\frac{\sqrt{c-(b+d)}}{2}(x+y-ct)\right]$, that it is the solution (73) in [6]. Setting $c_1 = 0,\ c_2 = 1,\ e = 1,\ a = 2,\ f = 1,\ c = b + d + (\lambda^2 - 4\mu)$ and $\alpha = A_0 + \frac{-6\lambda}{2}$, then the solution of (17) can be written as $v_2(\xi) := \alpha + \sqrt{c-(b+d)}\coth\left[\frac{\sqrt{c-(b+d)}}{2}(x+y-ct)\right]$.

From (20) we can rediscover some other exact solutions of eq. (16). When $\Delta = \lambda^2 - 4\mu = 0$, the rational function traveling wave solution is

$$u(\xi) := A_0 + \frac{6e}{a}\left(\frac{-\lambda}{2} + h(\xi)\right) \tag{25}$$

or equivalently

$$q(\xi) := \frac{-6e^2}{a}h^2(\xi), \tag{26}$$

where $\xi = ex + fy,\ h(\xi) = \frac{c_1}{c_2 + c_1\xi}$, and A_0 is an arbitrary constant. Finally, when $\Delta = \lambda^2 - 4\mu < 0$, the trigonometric function traveling wave solution is

$$u(\xi) := A_0 + \frac{6e}{a}\left(\frac{-\lambda}{2} + \gamma h(\xi)\right) \tag{27}$$

or equivalently

$$q(\xi) := \frac{-6e^2\gamma^2}{a}(1 + h^2(\xi)), \tag{28}$$

where $\xi = ex + fy - \left(bf + de - e^3(4\mu - \lambda^2)\right)t,\quad h(\xi) = \frac{-c_1\sin(\gamma\xi) + c_2\cos(\gamma\xi)}{c_2\sin(\gamma\xi) + c_1\cos(\gamma\xi)},\quad \gamma = \frac{\sqrt{4\mu - \lambda^2}}{2}$,

and A_0 is an arbitrary constant.

3. THE ANSATZ METHOD

Ito equation is a generalization of the bilinear Korteweg-de Vries (KdV) equation. It is sometimes referred to as the extension of the NLEE of KdV or modified KdV (mKdV) type to higher order. This equation is studied in (1+1) dimensions as well as (1+2) dimensions. This study will therefore be split in the following two subsections.

3.1 (1+1) Dimensions

The dimensionless form of the Ito equation in 1+1 dimensions is given by [5]

$$q_{tt} + q_{xxxt} + a\left(q_x q_t + q q_{xt}\right) + b q_{xx}\int_{-\infty}^{x} q_t \,\mathrm{d}x' = 0. \tag{29}$$

Here in (29) a and b are all constants. In this section, the search will be for 1-soliton solution to (29). The technique that will be used to carry out the calculations is the solitary wave ansatz method. Therefore, the starting ansatz for the 1-soliton solution to (29) is taken as

$$q(x,t) = A\,\mathrm{sech}^p \tau, \tag{30}$$

where

$$\tau = B(x - vt). \tag{31}$$

Here in (30), A is the amplitude of the soliton while B is the inverse width of the soliton. Also, v is the velocity of the soliton while the value of the unknown exponent p will be determined while deriving the solution to the Ito equation. Substituting (30) into (29) yields

$$\begin{gathered}
p^2 v^2 AB^2 \mathrm{sech}^p \tau - p(p+1) v^2 AB^2 \mathrm{sech}^{p+2} \tau - \\
-p^4 vAB^4 \mathrm{sech}^p \tau + 2p(p+1)\left(p^2 + 2p + 2\right) vAB^4 \mathrm{sech}^{p+2} \tau - \\
-p(p+1)(p+2)(p+3) vAB^4 \mathrm{sech}^{p+2} \tau + ap(p+1) vA^2 B^2 \mathrm{sech}^{2p+2} \tau - \\
-2ap^2 vA^2 B^2 \mathrm{sech}^{2p} \tau + ap^2 vA^2 B^2 \mathrm{sech}^{2p+2} \tau - \\
-bvp^2 A^2 B^2 \mathrm{sech}^{2p} \tau + bvp(p+1) A^2 B^2 \mathrm{sech}^{2p+2} \tau = 0.
\end{gathered} \tag{32}$$

From (38) equating the exponents $2p$ and $p+2$ gives

$$p = 2. \tag{33}$$

It needs to be noted that the same value of p is obtained when the exponents $2p+2$ and $p+4$ are equated with each other. Now, from (32) setting the coefficients of the linearly independent functions $\mathrm{sech}^{p+j}\tau$ for $j = 0, 2, 4$ to zero yields

$$v = 4B^2, \tag{34}$$

$$A = \frac{24B^2}{2a+b}, \tag{35}$$

$$A = \frac{60B^2}{5a+3b}. \tag{36}$$

Now, equating the two values of A from (35) and (36) yields the condition $b = 0$. In such case, both expressions (35) and (36) reduce to

$$A = \frac{12B^2}{a}. \tag{37}$$

Therefore the 1-soliton solution to Ito equation is given by

$$q(x,t) = A\,\mathrm{sech}^2[B(x - vt)], \tag{38}$$

where the amplitude-width relation is given by (37) and the velocity of the soliton is given by (44).

3.2. (1+2) Dimensions

The dimensionless form of the Ito equation in (1+2) dimensions is given by [5]

$$q_{tt} + q_{xxxt} + a\left(q_x q_t + q q_{xt}\right) + b q_{xx} \int_{-\infty}^{x} q_t \, dx' + c q_{yt} + d q_{xt} = 0. \tag{39}$$

Here, a, b, c and d are all constants. In order to seek 1-soliton solution to (39) the starting ansatz is still given by (30), where in this case

$$\tau = B_1 x + B_2 y - vt. \tag{40}$$

Here in (40), A represents the amplitude of the soliton, while B_1 and B_2 represents the inverse width in the x- and y-directions respectively. Finally, v represents the velocity of the soliton and the exponent p, which is unknown at this point, will be determined in terms of n during the course of derivation of the soliton solution. Substituting (30) into (39) again yields

$$\begin{aligned}&p^2v^2A\operatorname{sech}^p\tau - p(p+1)v^2A\operatorname{sech}^{p+2}\tau - p^4vAB_1^3\operatorname{sech}^p\tau + 2p(p+1)\left(p^2+2p+2\right)vAB_1^3\operatorname{sech}^{p+2}\tau -\\ &-p(p+1)(p+2)(p+3)vAB_1\operatorname{sech}^{p+4}\tau + ap(p+1)vA^2B_1\operatorname{sech}^{2p+2}\tau -\\ &-2ap^2vA^2B_1\operatorname{sech}^{2p}\tau + ap^2vA^2B_1\operatorname{sech}^{2p+2}\tau - bvp^2A^2B_1\operatorname{sech}^{2p}\tau + bvp(p+1)A^2B_1\operatorname{sech}^{2p+2}\tau -\\ &-cvp^2AB_2\operatorname{sech}^p\tau + cvp(p+1)AB_2\operatorname{sech}^{p+2}\tau - dvp^2AB_1\operatorname{sech}^p\tau + dvp(p+1)AB_1\operatorname{sech}^{p+2}\tau = 0\end{aligned} \tag{41}$$

Similiarly as in the previous sub-section, the same value of p as in (33) is yielded. Proceeding as in the (1+1) dimensional case, equation (39) gives

$$v = cB_2 + dB_1 + 4B_1^3, \tag{42}$$

$$A = \frac{24B_1^2}{2a+b}, \tag{42}$$

$$A = \frac{60B_1^2}{5a+3b}. \tag{44}$$

Now, equating the two values of A from (43) and (44) yields to the condition $b = 0$ as in the previous subsection. In such case, both expressions (43) and (44) reduce to

$$A = \frac{12B_1^2}{a}. \tag{45}$$

Finally, the 1-soliton solution to (39) is given by

$$q(x,y,t) = A\operatorname{sech}^2\left(B_1x + B_2y - vt\right), \tag{46}$$

where the amplitude A is related to the width B_1 of the soliton as given by (45) and the velocity of the soliton is given by (42).

4. CONSERVATION LAWS

In the following subsection, more conserved quantities are derived using the *multiplier approach*. For this, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence. That is, if (T^t, T^x). is a conserved vector corresponding to conservation law [1]

$$D_tT^t + D_xT^x = 0, \tag{47}$$

along the solutions of the differential equation in question, then

$$E_q[D_tT^t + D_xT^x] = 0, \tag{48}$$

where E_q is the appropriate Euler-Lagrange operators. Moreover, if there exists a nontrivial differential function Q, called a ‘multiplier’, such that

$$Q.(\text{equation}) = D_tT^t + D_xT^x \tag{49}$$

for some conserved vector, then

$$E_q[Q.(\text{equation})] = 0, \tag{50}$$

from which a knowledge of each multiplier Q leads to a conserved vector via a Homotopy operator. The elaborate and tedious calculations reveal that a number of multipliers and, hence, nontrivial conservation laws (and densities/conserved quantities) are available.

4.1. (1+1) Dimensions

In this case, following (2), the Euler operator is E_v and multipliers obtained are

$$Q_1 = v,\ Q_2 = x,\ Q_3 = xt,\ Q_4 = g(t),$$

where $g(t)$ is an arbitrary function including the case $Q = 1$. We note that there are no derivative dependent multipliers upto third order. The corresponding conserved densities are, respectively, via the homotopy operator

$$T_t^1 = \frac{1}{90}(-30v_t v_x - 10av_x^3 + 60vv_{xt} + 9v_{xx}^2 + 6v_x(5avv_{xx} - 3v_{xxx}) + 10av^2v_{xxx} + 18vv_{xxxx}), \tag{51}$$

$$T_t^2 = \frac{1}{30}(-10v_t - 5av_x^2 + 20xv_{xt} - 5avv_{xx} + 15axv_xv_{xx} - 6v_{xxx} + 5axvv_{xxx} + 6xv_{xxxx}), \tag{52}$$

$$T_t^3 = \frac{1}{30}(-10tv_t - 10xv_x - 5atv_x^2 + 20txv_{xt} + 15atxv_xv_{xx} - 6tv_{xxx} + 5v(4 - atv_{xx} + atxv_{xxx}) + 6txv_{xxxx}), \tag{53}$$

$$T_t^4 = \frac{1}{30}(-10g'v_x + g(20v_{xt} + 15av_xv_{xx} + 5avv_{xxx} + 6v_{xxxx})). \tag{54}$$

The Lie point symmetry algebra has basis

$$X_1 = \partial_x,\ X_2 = \partial_t,\ X_3 = \partial_v,\ X_4 = x\partial_x + 3t\partial_t - v\partial_v. \tag{55}$$

Hence, the respective conserved quantities are

$$\int_{-\infty}^{\infty} T_t^1 \mathrm{d}x = \frac{608A^2B^3}{105}, \tag{56}$$

$$\int_{-\infty}^{\infty} T_t^2 \mathrm{d}x = 0, \tag{57}$$

$$\int_{-\infty}^{\infty} T_t^3 \mathrm{d}x = \frac{2A}{B}, \tag{58}$$

$$\int_{-\infty}^{\infty} T_t^4 \mathrm{d}x = 0. \tag{59}$$

These conserved quatities are all computed using the 1-soliton solution that is given by (46).

4.2. (1+2) Dimensions

The condition on Q for the multidimensional case is based on (17) leading the zero$^{\text{th}}$ order multipliers

$$Q_1 = \alpha(y)v - b4ax^2\alpha'(y),\ Q_2 = \beta(y)x,\ Q_3 = \gamma(y - bt),\ Q_4 = \delta(t, y), \tag{60}$$

with corresponding densities

$$T_t^1 = -\frac{1}{360a}(-15b^2x^2\alpha'' v_x + 120a\alpha v_t v_x + 60ab\alpha v_y v_x + 60ad\alpha v_x{}^2 +$$
$$+40a^2\alpha v_x{}^3 - 36a\alpha v_{xx}{}^2 - 40a^2v^2\alpha v_{xxx} + 72a\alpha v_x v_{xxx} +$$
$$+3b\alpha'(-20xv_t - 10bxv_y - 20dxv_x - 10axv_x{}^2 + 20x^2v_{xt} + 10bx^2v_{xy} +$$
$$+12v_{xx} + 10dx^2v_{xx} + 15ax^2v_xv_{xx} - 12xv_{xxx} + 6x^2v_{xxxx}) + 3v(5b\alpha'(4d - 2axv_{xx} + ax^2v_{xxx}) -$$
$$-4(-5b^2x\alpha'' + 2a\alpha(10v_{xt} + 5bv_{xy} + 5dv_{xx} + 5av_xv_{xx} + 3v_{xxxx})))),$$

$$T_t^2 = \frac{1}{30}(-5bx\beta' v_x + 5v(2b\beta' + a\beta(-v_{xx} + xv_{xxx})) +$$
$$+\beta(-10v_t - 5bv_y - 10dv_x - 5av_x{}^2 + 20xv_{xt} + 10bxv_{xy} + 10dxv_{xx} + 15axv_xv_{xx} - 6v_{xxx} + 6xv_{xxxx})),$$

$$T_t^3 = \frac{1}{30}(5bx\gamma' v_x - 5v(2b\gamma' + a\gamma(v_{xx} - xv_{xxx})) + \gamma(-10v_t - 5bv_y - 10dv_x - 5av_x{}^2 + 20xv_{xt} + 10bxv_{xy} +$$
$$+10dxv_{xx} + 15axv_xv_{xx} - 6v_{xxx} + 6xv_{xxxx})),$$

$$T_t^4 = \frac{1}{30}(-5b\delta_y v_x - 10\delta_t v_x + \delta(20v_{xt} + 10bv_{xy} + 10dv_{xx} + 15av_xv_{xx} + 5avv_{xxx} + 6v_{xxxx})).$$

Particular cases of these, for e.g., would be from choosing

$$Q_1^p = yv,\ Q_2^p = yx,\ Q_3^p = (y - bt)x,$$

with densities

$$P_t^1 = \frac{1}{90}(-30v_tv_x - 15bv_yv_x - 15dv_x{}^2 - 10av_x{}^3 + 60vv_{xt} +$$
$$+30bvv_{xy} + 30dvv_{xx} + 30avv_xv_{xx} + 9v_{xx}{}^2 + 10av^2v_{xxx} - 18v_xv_{xxx} + 18vv_{xxxx}),$$
$$P_t^2 = \frac{1}{30}(-10yv_t - 5byv_y - 5bxv_x - 10dyv_x - 5ayv_x{}^2 + 20xyv_{xt} + 10bxyv_{xy} +$$
$$+10dxyv_{xx} + 15axyv_xv_{xx} - 6yv_{xxx} + 5v(2b - ayv_{xx} + axyv_{xxx}) + 6xyv_{xxxx}),$$
$$P_t^3 = \frac{1}{30}(10(bt - y)v_t + 5b^2tv_y - 5byv_y + 10bdtv_x +$$
$$+5bxv_x - 10dyv_x + 5abtv_x{}^2 - 5ayv_x{}^2 - 20btxv_{xt} + 20xyv_{xt} - 10b^2txv_{xy} +$$
$$+10bxyv_{xy} - 10bdtxv_{xx} + 10dxyv_{xx} - 15abtxv_xv_{xx} + 15axyv_xv_{xx} +$$
$$+6btv_{xxx} - 6yv_{xxx} + 5v(-2b + a(bt - y)v_{xx} + ax(-bt + y)v_{xxx}) - 6btxv_{xxxx} + 6xyv_{xxxx}).$$

5. CONCLUSIONS

In this paper, the Ito equation, which is a nonlinear integro-differential evolution equation is studied.

First the G'/G method is used to carry out the integration of this equation and its solutions are obtained. Subsequently the 1-soliton solution of the Ito equation in 1+1 and 1+2 dimensions are obtained. In this context it was proved thatthe soliton solution will exist provided the nonlocal term collapses to zero. This 1-soliton solution is then used to compute the non-trivial conserved quantities where the corresponding conserved densities are computed using the multiplier approach.

ACKNOWLEDGMENTS

The third author (Marko D. Petkovic) gratefully acknowledge the support from the research project 144011 of the Serbian Ministry of Science.

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Received March 15, 2012