ON ABELIAN HOPFION OF THE CP² MODEL ON R⁵

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We propose a generalization of the three dimensional Skyrme-Fadde’ev model consisting in Abelian Hopfions defined on R⁵. An explicit ansatz is presented and the reduced action of the model is computed. We also evaluate the topological charge of the configurations.

Key words: Skyrme-Fadde’ev model, Abelian Hopfions, topological charge.

1. INTRODUCTION

The usual solitons, including instantons, and monopoles and vortices as well as their gauge decoupled versions (when these exist in the model at hand) are stabilised by topological charges which with the appropriate normalisations take on integer values [1]. The field theoretic models which support instantons are gauge field theories typified by asymptotically pure gauge connection, while the monopoles are (non-Abelian) gauged Higgs theories typified by asymptotically half pure gauge connection resulting in Dirac-Yang fields in that domain. Vortices are also supported by Higgs theories, but with Abelian connection. Both vortices and monopoles can have gauge decoupling limits [2, 3]. Apart from vortices, which are by necessity defined on R² only, monopoles and instantons can be defined on all R⁵ (see [4]). These topological charges are Chern-Pontryagin (CP) charges or their descendents, or simply winding numbers.

What distinguishes Hopfions from the above mentioned usual solitons is, that they are not stabilised by CP charges, but rather by Chern--Simons (CS) charges, namely the volume integral of the CS density in the given dimensions. These are solutions to the O(3) (or CP¹) sigma model on R⁵, which were thoroughly investigated in the literature starting with the pioneering work [5]. These are topological solitons in systems involving a field Φ:R² → S². Such a field configuration is classified topologically by its Hopf number \( N \in \pi_3(S^2) \). It is important to distinguish the CS densities in play here, from what one might call dynamical CS densities as those in [6], which appear as part of a Lagrangian density on a Minkowski space. The latter are defined in terms of Yang-Mills (YM) fields, while the CS densities used to stabilise Hopfions are defined instead in terms of composite connections (and their curvatures), constructed from a nonlinear sigma model of scalar fields on a Euclidean space. Of course, both dynamical CS densities [6] and those pertaining to Hopfions, can be defined only on odd dimensional spaces.

The familiar Hopfions mentioned above are defined on R³, but since CS densities can be defined just as well on any R²n+1, it may be worth considering such solutions. To date, models supporting such solutions have not been considered. This is essentially a problem of academic interest, although knots in 5 dimensions have been considered recently [7]. Nevertheless, relatively little is known about higher dimensional generalisations of Hopfions. It is our intention here to investigate this question in the simplest next case, namely that of an Abelian Hopfion on R⁵. To this end, one can employ either the O(5) sigma model, or the CP² sigma model, on R⁵. We have chosen to work with the second of these, but the equivalent analysis can be readily carried out in the first case also.
2. THE CP^n MODELS ON R^{2n+1}: THE CASE n=2

We start with the generic structure of models that can support Abelian Hopfion on R^{2n+1}. These are described by complex n-tuplets

\[ Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{bmatrix} \equiv z_a ; \quad \bar{Z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{n+1} \end{bmatrix} \equiv \bar{z}^a \quad a = 1, 2, \ldots, n+1, \]

subject to the constraint

\[ Z^\dagger Z \equiv \bar{z}^a z_a = 1, \quad (2) \]

taking their values in \( \frac{U(n+1)}{U(n) \times U(1)} \), such they are described by 2n + 1 real parameters that parametrise R^{2n+1}. In (1), \( \bar{z}^a \) is the complex conjugate of \( z_a \), transforming with an index that is contravariant to the covariant index of \( z_a \), and \( Z^\dagger \) in (2) is the transpose of \( \bar{Z} \). This leads to the definition of the projection operator

\[ P = (\mathbb{1} - ZZ^\dagger) = (\delta^a_{a} - z_a \bar{z}^a) \quad (3) \]

The most interesting feature of these models is that when the field \( Z \) is subjected to a local U(1) gauge transformation \( g(\Lambda) \), then the quantity defined as

\[ B_i = Z^\dagger \partial_i Z , \quad i = 1, 2, \ldots, 2n + 1 \quad (4) \]

transforms like an Abelian composite connection under \( g(\Lambda) \), which leads to the definition of the covariant derivative of \( Z \) and the Abelian curvature of this connection,

\[ D_i Z = \partial_i Z - B_i Z, \quad (5) \]
\[ G_{ij} = \partial_i B_j - \partial_j B_i. \quad (6) \]

The Abelian CS density on R^{2n+1} is then readily defined in terms of the quantities (6) and (5). This is what makes these models well suited to describing Abelian Hopfions in all odd dimensions.

\[ \Omega_{CS} \cong \varepsilon_{i_1\ldots i_{2n+1}} B_{i_2} G_{i_1} G_{i_3} G_{i_4} \ldots G_{i_{2n-1} i_{2n}}. \quad (7) \]

Let us first dispose of the well studied case \( n = 1 \), namely the CP^1 models on R^3. The most general model supporting finite energy solutions, consistent with the Derrick scaling requirement is

\[ H_3 = \kappa_0^2 V + \frac{1}{2} \kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2, \quad (8) \]

with \( D_i Z \) and \( G_{ij} \) given by (5) and (6). The constants \( \kappa_0, \kappa_1, \) and \( \kappa_2 \) each have the dimension of length, and \( V \) is some pion mass type potential, which can most naturally be chosen to be

\[ V = 1 + Z^\dagger \sigma_3 Z. \quad (9) \]

In the special case with \( \kappa_0 = 0 \), (8) reduces to the Skyrme-Fadde'ev model [5].

We proceed to the problem at hand, namely the study of the Hopfions of CP^2 models on R^5. The most general model supporting finite energy solutions, consistent with the Derrick scaling requirement is
\[ H_4 = \kappa_0^2 V + \frac{1}{2} \kappa_2^2 D_i Z^i D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{8} \kappa_2^4 (G_{ij} D_k Z)^j (G_{ij} D_k Z) + \frac{1}{16} \kappa_4^8 G_{ijkl}^2, \quad (10) \]

with \( D_i Z \) and \( G_{ij} \) given by (5) and (6), and the 4-from \( G_{ijkl} \) being the totally antisymmetrised product of this curvature. The constants \( \kappa_0, \kappa_1, \kappa_2, \kappa_3 \) and \( \kappa_4 \) each have the dimension of length, and \( V \) is some pion mass type potential. According to the scaling requirement for finite energy, it is necessary to retain at least one of the constants \( (\kappa_0, \kappa_1, \kappa_2) \) and at least one of the constants \( (\kappa_3, \kappa_4) \), with the option of setting the rest equal to zero.

The virial identity resulting from the usual Derrick-type scaling requirement [8] that must be satisfied is

\[ 5 \| V \| + 3 \| D_i Z \|^2 + \| G_{ij} \|^2 - 3 \| G_{ijkl} \|^2 = 0, \quad (11) \]

where the dimensional constants and the detailed normalisations have been suppressed, and where each of the quantities \( \| . \|^2 \) is the positive definite integral of the corresponding density in (10).

The simplest truncation of the model (10), consistent with finite energy and the definition of a knot-charge, is

\[ H_{(2,4)} = \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{16} \kappa_4^8 G_{ijkl}^2, \quad (12) \]

featuring the quartic and octic terms multiplying the constants \( \kappa_2^4 \) and \( \kappa_4^8 \), respectively. It is however reasonable to include the usual quadratic term multiplying the constant \( \kappa_2^2 \), such that the system to be studied is

\[ H_{(1,2,4)} = \frac{1}{2} \kappa_2^2 D_i Z^i D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{16} \kappa_4^8 G_{ijkl}^2. \quad (13) \]

### 3. AN HOPFION ANSATZ: IMPOSITION OF BI-AZIMUTHAL SYMMETRY

The most general Hopfion Ansatz would depend on five independent functions, which will solve a complicated set of second order partial differential equations (PDE’s). To facilitate the construction of the solutions and the evaluation of the topological charge it is desirable to reduce this number. Just as the Hopfion solutions of the \( \mathbb{CP}^1 \) model on \( \mathbb{R}^3 \) can be constructed by subjecting the field \( Z \) to azimuthal symmetry in the \( (x_1, x_2) \) plane, here the corresponding imposition of symmetry on the \( \mathbb{CP}^2 \) field \( Z \) is bi-azimuthal symmetry, namely to two azimuthal symmetries in the \( (x_1, x_2) \) and \( (x_3, x_4) \) planes separately. This will reduce the 5th order field equations to 3rd order PDE’s, just as in the Skyrme-Faddeev case 3rd order field equations are reduced to 2nd order PDE’s. Moreover, this corresponds to the simplest ansatz leading to a nonvanishing topological charge. The choice of the particular symmetry imposition in both cases is predicated on the resulting demonstration that the Hopf charge can be defined as a topological charge integral in the residual dimensions. This also yields a verification of the validity of the Ansatz used.

The resulting Ansatz used for the field (1) on \( \mathbb{R}^5 \) is

\[ Z = \begin{bmatrix} a(p, \sigma, z) + ib(p, \sigma, z) \\ c(p, \sigma, z)e^{i\alpha} \\ d(p, \sigma, z)e^{i\beta} \end{bmatrix} = \begin{bmatrix} \sin \frac{1}{2} f(p, \sigma, z)e^{i(\alpha(p, \sigma, z))} \\ \cos \frac{1}{2} f(p, \sigma, z) \sin g(p, \sigma, z)e^{i\alpha} \\ \cos \frac{1}{2} f(p, \sigma, z) \cos g(p, \sigma, z)e^{i\beta} \end{bmatrix}, \quad (14) \]
in terms of the variables \( \rho = \sqrt{x_\alpha^2} \), \( \sigma = \sqrt{x_\alpha^2} \) with \( \alpha = 1, 2, A = 3, 4 \) and \( z = x_5 \). \( \varphi \) and \( \chi \) are the azimuthal angles in the \((x_1, x_2)\) and \((x_3, x_4)\) planes, respectively, \((n, m)\) being the winding (vortex) numbers of planes, respectively (therefore the line element on \( \mathbb{R}^5 \) reads \( ds^2 = dz^2 + d\sigma^2 + \rho^2 d\varphi^2 + d\alpha^2 + \sigma^2 d\chi^2 \)).

The resulting composite Abelian connections descending from (4) are (with \( \hat{x}_\alpha = x_\alpha / \rho \), \( \hat{x}_A = x_A / \sigma \))

\[
B_\alpha = i \left( (a_\rho^{\alpha} - b_\rho^{\alpha}) \hat{x}_\alpha + \frac{n}{\rho} c^2 (\hat{x}\varepsilon)^\alpha \right),
\]
\[
B_A = i \left( (a_\sigma^{\alpha} - b_\sigma^{\alpha}) \hat{x}_A + \frac{m}{\sigma} d^2 (\hat{x}\varepsilon)^A \right),
\]
\[
B_z = i (ab_z - ba_z),
\]

leading to the following components of the Abelian curvature

\[
G_{\alpha\beta} = 2i \frac{n}{\rho} cc_\rho e_{\alpha\beta},
\]
\[
G_{AB} = -2 \frac{m}{\sigma} dd_\sigma e_{AB},
\]
\[
G_{\alpha A} = 2i \left[ a_\rho^{\alpha} b_\rho^{\alpha} \hat{x}_A - \frac{n}{\rho} cc_\rho \hat{x}_\alpha (\hat{x}\varepsilon)^\alpha + \frac{m}{\sigma} dd_\sigma \hat{x}_\alpha (\hat{x}\varepsilon)^A \right],
\]
\[
G_{az} = 2i \left[ a_\rho^{\alpha} b_\rho^{\alpha} \hat{x}_A - \frac{n}{\rho} cc_\rho (\hat{x}\varepsilon)^\alpha \right],
\]
\[
G_{Az} = 2i \left[ a_\sigma^{\alpha} b_\sigma^{\alpha} \hat{x}_A - \frac{m}{\sigma} dd_\sigma (\hat{x}\varepsilon)^A \right].
\]

Note that these Abelian connections and curvatures are pure imaginary, as per the definition (4) of the composite connection \( B_i \). Once the connection and the curvature are known, we can evaluate the individual pieces which enter (10). First, we give the quadratic kinetic term multiplying \( \kappa^2_1 \)

\[
D_iZ^iD_jZ = \partial_iZ^i\partial_jZ - B^2_i = (\partial_\alpha Z^i \partial_\alpha Z - B^2_\alpha) + (\partial_A Z^i \partial_A Z - B^2_A) + (\partial_z Z^i \partial_z Z - B^2_z),
\]
yielding

\[
D_iZ^iD_jZ = (a_\rho^2 + b_\rho^2 + c_\rho^2 + d_\rho^2) + (a_\sigma^2 + b_\sigma^2 + c_\sigma^2 + d_\sigma^2) + (a_z^2 + b_z^2 + c_z^2 + d_z^2) - (ab_\rho - ba_\rho)^2 - (ab_\sigma - ba_\sigma)^2 - (ab_z - ba_z)^2
\]
\[
+ \frac{n^2}{\rho^2} c^2(1 - c^2) + \frac{m^2}{\sigma^2} d^2(1 - d^2).
\]

Next, the term with coupling strength \( \kappa^4_2 \) in (10), namely

\[
G^2_{ij} = G_{\alpha\beta}^2 + G_{AB}^2 + 2G_{\alpha A}^2 + 2G_{az}^2 + 2G_{Az}^2,
\]
can be calculated immediately to yield

\[
\frac{1}{8} G^2_{ij} = \left[ (a_\rho b_\rho)^2 + (a_\rho b_z)^2 + (a_\sigma b_\sigma)^2 \right] + \frac{n^2}{\rho^2} c^2 \left( c_\rho^2 + c_\sigma^2 + c_z^2 \right) + \frac{m^2}{\sigma^2} d^2 \left( d_\rho^2 + d_\sigma^2 + d_z^2 \right).
\]

To calculate the term with coupling strength \( \kappa^8_4 \) we first evaluate the three distinct components of \( G_{ijkl} \), namely
\[ G_{ijkl} = (G_{a|b|AB}, G_{a|b|C}, G_{ABaz}), \]

with

\[ G_{a|b|AB} = 4 \frac{nm}{\rho \sigma} cdc_{|p|d|\sigma|} e_{a|b|}, \]

\[ G_{a|b|C} = 4 - \frac{n}{\rho} c \left( (c_{|\lambda|b|\alpha|} + c_{\rho a|b|\alpha|} + c_{\alpha a|b|\rho|}) \hat{x}_a \right), \]

\[ G_{ABaz} = 4 - \frac{m}{\sigma} \left( -d_c a_{|\rho|b|\alpha|} + d_\rho a_{|\alpha|b|\rho|} + d_\rho a_{|\alpha|b|\rho|} \hat{x}_a + \frac{n}{\rho} c \rho \sigma \sigma \sigma \sigma \right). \]

Substituting these into

\[ \frac{1}{2} G_{ijkl}^2 = 3G_{a|b|AB}^2 + 5G_{a|b|C} + 5G_{ABaz}, \]

we find the simple compact expression

\[ \frac{1}{3} \cdot 2^7 G_{ijkl}^2 = \left[ \frac{n^2}{\rho^2} c^2 (c_{|\rho|b|\alpha|})^2 + \frac{m^2}{\sigma^2} d^2 (d_{|\rho|b|\alpha|})^2 \right] + \]

\[ + \frac{n^2}{\rho} c^2 d^2 \left[ (c_{|\rho|d|\sigma|})^2 + (c_{\rho|d|\sigma|})^2 + (c_{\sigma|d|\sigma|})^2 \right]. \]

With these relations, the derivation of the corresponding equations of motion for the scalars \( a, b, c \) and \( d \) within the generic model (10) is straightforward. These equations are very complicated and we shall not present them here.

### 4. Chern-Simons Density on \( \mathbb{R}^5 \) and the Topological Charge

As mentioned in the previous section, imposition of the appropriate symmetry is crucial in ensuring the existence of a knotted (nontrivial) Hopfion. In this context, it is clear that the imposition of spherical symmetry is inappropriate since this leaves no room for any useful winding. In our imposition of bi-azimuthal symmetry on the \( \mathbb{C}P^2 \) system on \( \mathbb{R}^5 \) here, we are guided by the application of axial (mono-azimuthal) symmetry on the corresponding \( \mathbb{C}P^1 \) system on \( \mathbb{R}^3 \) [9].

The Chern-Simons density on \( \mathbb{R}^5 \), denoting the coordinates \( x_i = (x_\mu, x_5) \), is given by the following expression

\[ \Omega^{(5)}_{CS} = e_{ijkl} B_{\alpha i} G_{|\mu|} G_{|\delta|} = e_{\mu \nu \rho \sigma} \left( B_\beta G_{\mu \nu}, G_{|\rho|} + 4B_\mu G_\nu \right) = \]

\[ = 2e_{\alpha i} e_{|\beta|} B_\beta \left( G_{|\alpha|} G_{|\delta|} - 2G_{|\alpha|} G_{|\delta|} \right) + \]

\[ + 2 \left[ B_\alpha \left( G_{|\beta|} G_{|\delta|} - 2G_{|\beta|} G_{|\delta|} \right) + B_\beta \left( G_{|\alpha|} G_{|\delta|} - 2G_{|\alpha|} G_{|\delta|} \right) \right]. \]

Substituting the bi-azimuthally symmetric Ansatz (14) yields the following simple expression of (21)

\[ \frac{1}{2} \Omega^{(5)}_{CS} = \det \begin{vmatrix} a & b & c & d \\ a_\rho & b_\rho & c_\rho & d_\rho \\ a_\sigma & b_\sigma & c_\sigma & d_\sigma \\ a_z & b_z & c_z & d_z \end{vmatrix}. \]

Then it is clear that if any one of the functions \( a, b, c \) and \( d \) vanishes, \( \Omega^{(5)}_{CS} \) vanishes.
As usual, the topological charge \( Q^{(S)}_{CS} \) of the solutions is the integral of the Chern-Simons density. In evaluating \( Q^{(S)}_{CS} \), it is convenient to work with the trigonometric parametrisation in (14), namely the parametrisation in which the sigma model constraint is already imposed. Then the integral of (22) reduces to the simple expression

\[
Q^{(S)}_{CS} = 4 \cdot (2\pi)^2 n_1 n_2 \int \left[ \partial_\rho (\cos f) \partial_\sigma g \partial_\alpha + \text{cyc}.(\rho, \sigma, z) \right] dp d\rho dz. \tag{23}
\]

Denoting the coordinates \((\rho, \sigma, z) = (\xi_i, i = 1, 2, 3, \text{i.e.}, \)

\[
\xi_i = \begin{pmatrix} r \sin \psi \sin \theta \\
 r \sin \psi \cos \theta \\
r \cos \psi
\end{pmatrix},
\]

with \(0 \leq r < \infty, 0 \leq \psi \leq \pi \) and \(0 \leq \theta \leq \frac{\pi}{2}\), the charge (23) can be re-expressed as

\[
Q^{(S)}_{CS} = 4 \cdot (2\pi)^2 n_1 n_2 \int \hat{e}_{ijk} \partial_\rho (\cos f) \partial_\sigma g \partial_\alpha d^3 \xi =
4 \cdot (2\pi)^2 n_1 n_2 \int \hat{e}_{ijk} \left((\cos f) \partial_\rho g \partial_\alpha \right) \mid_{r \rightarrow \infty} \hat{\xi}_j dS \tag{25}
\]

in an obvious notation, where \( dS = r^2 \sin \psi d\psi d\theta \), and where we have applied Gauss' Theorem.

The result of the integration is

\[
Q^{(S)}_{CS} = 4 \cdot (2\pi)^2 n_1 n_2 \int_{\psi=0}^{\pi} \int_{\theta=0}^{\frac{\pi}{2}} \cos f \left( \partial_\rho g \partial_\alpha \alpha - \partial_\rho \alpha \partial_\alpha g \right) \mid_{r \rightarrow \infty} d\psi d\theta. \tag{26}
\]

Finally, requiring the boundary values (with \( m \) an arbitrary integer)

\[
\lim_{r \rightarrow \infty} g = 0, \lim_{r \rightarrow \infty} \alpha = m \pi,
\]

the relation (25) yields the following simple expression for the charge of the Abelian Hopfion on \( \mathbb{R}^5 \)

\[
Q^{(S)}_{CS} = -32\pi^2 n_1 n_2 m. \tag{28}
\]

5. CONCLUSIONS

The purpose of this work was to propose a generalization of the well-known Skyrme-Fadde'ev Abelian Hopf model in \( \mathbb{R}^3 \) to the case of five dimensional space. These configurations are stabilized by the Abelian CS term. An explicit ansatz subject to bi-azimuthal symmetry in a four dimensional subspace has also been proposed. This ansatz contains three independent functions with dependence of three variables. Finally, we have evaluated the topological charge of the Hopfions and show that it can be written as a product of three winding numbers which enters the Hopfions' Ansatz and the boundary conditions.

For the case of the \( D = 3 \) Skyrme-Fadde'ev Abelian Hopf model, there are explicit constructions of solutions of the equations of motion corresponding to global energy minima [10, 11]. (Note that all these solutions were found numerically, no exact results existing so far, see the review work [12]). The model discussed in this work should also possess finite energy solutions, those within the Ansatz (1) subject to the bi-azimuthal symmetry being the simplest class. Although our preliminary attempts to construct them have met with severe difficulties, we think this is a numerical problem only, and such solutions should exist.

Finally, let us remark that the model in this work together with Skyrme-Fadde'ev one can be viewed as the first two members of an hierarchy of Abelian Hopfions of the \( \mathbb{C}P^n \) sigma model, on \( \mathbb{R}^{2n+1} \). Indeed, a natural generalisation of the Ansätze in the \( n = 1, 2 \) cases here exists. This is \( n \)-fold-azimuthal symmetry on \( \mathbb{R}^{2n+1} \). The only crucial check is that the CP density reduces to a total divergence subject to that symmetry, about which one may be quite confident.
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