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MOTION OF A PARTICLE IN A RESISTING MEDIUM USING FRACTIONAL CALCULUS APPROACH

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In this manuscript we propose a fractional differential equation to describe the vertical motion of a body through the air. The order of the derivative was considered to be $0 < \gamma \leq 1$. To keep the dimensionality of the physical parameter in the system, an auxiliary parameter σ is introduced. This parameter characterizes the existence of fractional components in the given system. We prove that there is a relation between γ and σ through the physical parameter of the system and that, due to this relation the analytical solutions are given in terms of the Mittag-Leffler function depending on the order of the fractional differential equation.

Key words: resisting medium; vertical motion; fractional calculus.

1. INTRODUCTION

The discipline of the mathematical analysis dedicated to the study of derivatives and integrals of arbitrary order, real or complex, is called Fractional Calculus (FC) [1–6]. In the past few decades the FC and fractional differential equations have found applications in various disciplines [1–29]. Fractional differential equations have assumed an important role in modeling the anomalous dynamics of many processes related to complex systems in the most diverse areas of science and engineering.

Fundamental physical considerations in favor of the use of models based on derivatives of non-integer order are given in [7–10].

Usually, some authors replace the integer derivative by a fractional one on a purely mathematical or heuristic basis. However, from the physical and engineering point of view this is not completely correct because the physical parameters contained in the differential equation should not have the dimensionality measured in the laboratory [11, 12].

One possible way to clarify these things is to replace the ordinary derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$ by a fractional operator $\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}$ leaving its dimensionality s^{-1} invariant.

This can be done as given below [13]:

$$\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{1}{\sigma^{1-\gamma}} \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} \tag{1}$$

where σ has dimension of seconds (in the case of fractional time derivative and has dimension of length in the case of spatial fractional derivative) in order to keep the dimensionality of the fractional derivative operator.

Having all the above things in mind we organize the manuscript as follows: In section two some basic definitions of fractional derivatives are given. Section three deals with falling body problem within fractional derivative. Finally, section four presents our conclusions.

2. BASIC TOOLS

The left Riemann-Liouville fractional integral is defined as below

$$ {}_aI_t^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\mathrm{d}\tau. \tag{2}$$

The right Riemann-Liouville fractional integral can be written as

$$ {}_tI_b^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)\mathrm{d}\tau. \tag{3}$$

The left Riemann-Liouville fractional derivative has the form

$$ {}_aD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau. \tag{4}$$

Similarly, the right Riemann-Liouville fractional derivative is given by

$$ {}_xD_b^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\int_x^b \frac{f(\tau)}{(\tau-x)^{\alpha-n+1}}\mathrm{d}\tau. \tag{5}$$

α denotes the order of the derivative in such way that $n-1\le\alpha\le n$ and is not equal to zero. If α is an integer, the fractional derivatives become the classical ones, namely

$$ {}_aD_x^{\alpha}f(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha}f(x), \tag{6}$$

$$ {}_xD_b^{\alpha}f(x)=\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha}f(x)\quad;\quad\alpha=1,2........ \tag{7}$$

One of the important formula is so called the fractional Leibniz rule, namely

$$ {}_aD_t^{\alpha}[f(t)g(t)]=\sum_{k=0}^{\infty}\binom{\alpha}{k}\left({}_aD_t^{\alpha-k}f(t)\right)\left(\frac{\partial^k g(t)}{\partial t^k}\right). \tag{8}$$

As a result, the fractional of an exponential has the form

$$ {}_aD_t^{\alpha}[e^{\lambda t}]=\lambda^{\alpha}e^{\lambda t}. \tag{9}$$

To analyze the dynamical behavior of a fractional system it is necessary to use an appropriate definition of fractional derivative.

In many applied problems, it is required to use the definitions of fractional derivatives that allow the utilization of physically interpretable initial conditions, which contain $x(0),\dot{x}(0),\ldots$.

The Caputo representation for fractional order derivative satisfies these requirements. In the Caputo case, the derivative of a constant is zero, therefore we can define, properly, the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer case. As a result, in this manuscript we use the Caputo fractional derivative for a function of time, $f(t)$ defined as [2]

$$ {}_0^C D_t^\gamma f(t) = \frac{1}{\sigma^{1-\gamma}} \frac{\mathrm{d}^\gamma f(t)}{\mathrm{d}t^\gamma} = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} \mathrm{d}\eta, \tag{10} $$

where $\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} = {}_0^C D_t^\gamma$, $\Gamma(\cdot)$ is the Euler Gamma function, $n = 1, 2, \ldots \in N$ and $n-1 < \gamma \le n$.

We consider the case when $n = 1$, i.e., in the integrand there is only first derivative. In this case γ such that, $0 < \gamma \le 1$, is the order of the fractional derivative.

The Caputo derivative operator satisfies the following relations

$$ {}_0^C D_t^\gamma \left[f(t) + g(t) \right] = {}_0^C D_t^\gamma f(t) + {}_0^C D_t^\gamma g(t), \qquad {}_0^C D_t^\gamma c = 0, \tag{11} $$

where c is a constant.

For example, in the case $f(t) = t^k$, where k is arbitrary number and $0 < \gamma \le 1$, we have the following expression for the fractional derivative operation

$$ {}_0^C D_t^\gamma t^k = \frac{k\Gamma(k)}{\Gamma(k+1-\gamma)} t^{k-\gamma} \quad (0 < \gamma \le 1), \tag{12} $$

where $\Gamma(k)$ and $\Gamma(k+1-\gamma)$ are the Gamma functions. If $\gamma = 1$ the expression (12) yields the ordinary derivative

$$ {}_0^C D_t^\gamma t^k = \frac{\mathrm{d}t^k}{\mathrm{d}t} = kt^{k-\gamma}. \tag{13} $$

The Laplace transform of the derivative of Caputo sense is defined as [10] :

$$ L\left[{}_0^C D_t^\gamma f(t) \right] = s^\gamma F(s) - \sum_{k=0}^{n-1} s^{\gamma-k-1} f^{(k)}(0). \tag{14} $$

In the following we apply the fractional operator given by (1) to investigate the falling body problem.

3. THE FRACTIONAL FALLING BODY PROBLEM

Let us consider a particle of mass m falling through the air with initial velocity v_0 from a height h in a constant gravitational field. Then, it experiences a resisting force that opposes to the relative motion in which the particle moves relatively to the air. It is know that this resisting force is related to the relative velocity. For slow speeds the resisting force is in magnitude proportional to the speed. However, in other cases it may be proportional to the square (or some other power) of the speed. Applying the Newton second law we obtain [14]:

$$ m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg - mkv, \tag{15} $$

where k represents a positive constant that species the strength of the retarding force, its dimensionality is the inverse of seconds and, $-mkv$, is a positive upward force because we take z and $v = \dot{z}$ to be positive upward, and the motion is downward, that is, $v < 0$, so that, $-kmv > 0$.

The solution of the equation (15) is given by:

$$v(t)=-\frac{g}{k}+\left(v_0+\frac{g}{k}\right)e^{-kt}. \tag{16}$$

Integrating again with the condition $z(0)=h$, we have:

$$z(t)=h-\frac{gt}{k}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)\left[1-e^{-kt}\right]. \tag{17}$$

The expression (8) shows that as the time becomes very long, the velocity approaches the limiting value $-\frac{g}{k}$; this is called the terminal velocity, v_f. Moreover, if in expression (15), $v=-\frac{g}{k}$, then we obtain the same result, since the force is zero and the acceleration disappears. If the initial velocity exceeds the terminal velocity in magnitude, then the body immediately begins to slow down and v approaches the terminal speed from the opposite direction.

After that we can write the fractional differential equation corresponding to the falling body problem. By using (1), the fractional differential equation corresponding to (15) is given as:

$$\frac{m}{\sigma^{1-\gamma}}\,{}_0^C D_t^\gamma v+mkv+-mg, \tag{18}$$

where m m is the physical mass of the body measured in kg, unlike the work [11]. We can write the equation (18) as follows :

$${}_0^C D_t^\gamma v+av=-b, \tag{19}$$

where $a=k\sigma^{1-\gamma}$ and $b=g\sigma^{1-\gamma}$.

Applying the direct (14) and inverse Laplace transform [15] with the condition $v(0)=v_0$, we obtain:

$$v(t)=-\frac{g}{k}+\left(v_0+\frac{g}{k}\right)E_\gamma\left(-k\sigma^{1-\gamma}t^\gamma\right), \tag{20}$$

where $E_\gamma(\cdot)$ is the Mittag-Leffler function, defined as:

$$E_\beta(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(\beta m+1)}. \tag{21}$$

When $\beta=1$, from (21) we have

$$E_1(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+1)}=\sum_{m=0}^{\infty}\frac{t^m}{m!}=e^t. \tag{22}$$

The Mittag-Leffler function plays the analogy role in the linear fractional differential equations, like the exponential functions plays in ordinary linear differential equations.

Applying once more the fractional derivative to (20) we have:

$${}_0^C D_t^\gamma z=-A+BE_\gamma\left(-Ct^\gamma\right). \tag{23}$$

Here, we define $A=\frac{\sigma^{1-\gamma}}{k}$, $B=\sigma^{1-\gamma}\left(\frac{g}{k}+v_0\right)$ and $C=k\sigma^{1-\gamma}$.

Using the direct and inverse Laplace formula [15], we obtain:

$$z(t)=h-\frac{g\sigma^{1-\gamma}t^\gamma}{k\Gamma(\gamma+1)}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)\left[1-E_\gamma\left(-k\sigma^{1-\gamma}t^\gamma\right)\right]. \tag{24}$$

It is interesting to note, that the order of the fractional derivative γ is related with the physical parameter k of the system through the parameter σ. This relation is given by:

$$\gamma = \sigma k\,; \quad 0 < \sigma \le \frac{1}{k} \tag{25}$$

Then, the magnitude

$$\delta = 1 - \gamma \tag{26}$$

characterizes the existence of the fractal structure in the system.

From $\gamma = 1$, i.e., $\sigma = \frac{1}{k}$ it follows $\delta = 0$, that is, in the system there is no fractal structure. However, in the interval $0 < \sigma < \frac{1}{k}$ the δ grows tending to unity, because in the system there are increasingly fractal excitations.

Taking into account the expression (25), the solutions (20) and (24) can be rewritten through γ

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) E_\gamma\left(-\gamma^{1-\gamma}\tau^\gamma\right) \tag{27}$$

and

$$z(t) = h - \frac{g\gamma^{1-\gamma}}{k^2\Gamma(\gamma+1)}\tau^\gamma + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left[1 - E_\gamma\left(-\gamma^{1-\gamma}\tau^\gamma\right)\right], \tag{28}$$

where $\tau = kt$ is a dimensionless parameter.

In the particular case $\gamma = 1$ the expressions (27) and (28), becomes (16) and (17).

4. CONCLUSIONS

Fractional calculus started to be a useful tool within the modeling of complex systems. One of the main difficulties when we apply the fractional differential operators on a function is related to the change in the dimensionality of the obtained results. Having these issues in mind, in this manuscript we analyze the fractional differential equation counterpart of the vertical motion of a body through the air. The obtained results depend on the order of the fractional differential equation (19) and (20).

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