SOLITONS, KINKS AND SINGULAR SOLUTIONS OF COUPLED KORTEWEG-DE VRIES EQUATIONS

Bouthina S. AHMED¹, Anjan BISWAS²

¹ Ain Shams University, Department of Mathematics, College of Girls, Cairo, Egypt
² Delaware State University, Department of Mathematical Sciences, Dover, DE 19901-2277, USA
E-mail: biswas.anjan@gmail.com; Ahmed_Bouthina@live.com

This paper studies the coupled Korteweg-de Vries (KdV) equations that describe shallow two-layered water waves in ocean shores. The ansatz method is employed to retrieve the solitary wave solutions, topological and singular solitons of this coupled system of equations. The constraint conditions will fall out for the existence of these types of solitons. The study is also generalized to coupled KdV equations with power law nonlinearity. The numerical simulations supplement the analytical schemes.
Key words: solutions, integrability, ansatz method, numerics.

1. INTRODUCTION

Unique nonlinear phenomena exist in all research fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. In the past work, the research of travelling wave solutions of several nonlinear evolution equations played an important role in the analysis of such nonlinear phenomena [1–15]. For the travelling wave solutions, many methods were attempted, such as the inverse scattering transform method [2], Hirota’s bilinear transformation [9], the tanh method [6,11,12], sine-cosine method [15], homogeneous balance method [15], exp-function method [15] and so on. The above methods derived many types of solutions from most nonlinear evolution equations. It is now time to start focusing on such equations which appear in coupled vector format. They have practical applications too. Quite commonly, vector coupled evolutions are studied in two-layered or rather multiple-layered shallow water waves, or optical solitons in birefringent fibers and/or dense-wavelength division multiplexed systems [8]. The two-layered fluid waves near ocean shores can be modeled by coupled Korteweg-de Vries (KdV) equations or the Gear-Grimshaw models [3]. This paper will focus on coupled KdV equations; exact solitary waves, kinks (topological solitons) and singular solitons will be obtained.

2. GOVERNING EQUATIONS

The coupled KdV equations which describe the interaction process of two long waves are given by
Type I

\[ q_t + a_1 q q_x + b_1 q_{xxx} + c_1 r r_x = 0 , \]  
\[ r_t + a_2 q r_x + b_2 r_{xxx} = 0 , \]  

and the coupled KdV equations with power law nonlinearity are
Type II

\[ q_t + a_1 q^n q_x + b_1 q_{xxx} + c_1 r^n r_x = 0 , \]
\[ r_t + a_2 q^m r_x + b_2 r_{xxx} = 0. \] (4)

These are the two models that represent the coupled KdV equations. The dependent variables are \( q(x,t) \) and \( r(x,t) \) which respectively represent the wave profile for the two layers of fluid. The spatially independent variable is \( x \) while the temporal independent variable is \( t \). The constant coefficients are \( aj, bj, j=1,2 \) and \( c_1 \). In (1), the first term represents the evolution term, while the second term along with the fourth term are the nonlinear terms. The third term is the dispersion. Then for (2), the first term again represents evolution, the second term is the cross-nonlinear term while the third term is again the dispersion. For Type-II, the two additional parameters are \( m \) and \( n \) that represent the power law nonlinearity parameters, i.e., we get coupled KdV equations with power law nonlinearity. Thus, the power law nonlinearity is introduced on a generalized setting that could represent the regular KdV equation or the modified KdV (mKdV) depending on the values of \( m \) and \( n \).

3. ANSATZ APPROACH

One of the most popular approaches to integration, that has lately surfaced is the \textit{ansatz method} [3, 4, 7]. This method is similar to the trial solution method that is very commonly used in solving ordinary differential equations. Such a trial solution, for solitons, is assumed based on the known results of the single KdV equation. This trial solution is then substituted into the coupled system. The linearly independent functions are then identified, after the application of the balancing principle, and their respective coefficients are all set to zero. These lead to the computation of the soliton parameters.

3.1. Solitary waves

In this section the search is going to be for 1-soliton solution to the coupled KdV-type I equations given by (1), (2) and type II equations (3) and (4). To begin, let us assume the following solitary wave ansatz in two cases of the KdV equation.

3.1.1. Coupled KdV equations

To obtain 1-soliton solution of (1) and (2) we assume that

\[ q(x,t) = \frac{A_1}{\cosh^{p_1} \tau}, \] (5)

\[ r(x,t) = \frac{A_2}{\cosh^{p_2} \tau}, \] (6)

where \( A_1 \) and \( A_2 \) represent the amplitudes of \( q \) and \( r \) solitons, respectively, and \( \tau \) is given by

\[ \tau = B(x - vt), \] (7)

while \( B \) is another free parameter of the soliton and \( v \) is the velocity of the soliton.

Substituting (5) and (6) into (1) and (2) gives:

\[ \frac{p_1 A_1 B \tanh \tau}{\cosh^{p_1} \tau} - \frac{a_1 p_1 A_1^2 B \tanh \tau}{\cosh^{2p_1} \tau} - h_1 \left\{ \frac{3 p_1 A_1 B^3 \tanh \tau}{\cosh^{p_1} \tau} + \frac{p_1 (p_1 + 1) (p_1 + 2) A_1 B^3 \tanh \tau}{\cosh^{(p_1 + 2)} \tau} \right\} = 0, \] (8)

\[ \frac{p_2 A_2 B \tanh \tau}{\cosh^{p_2} \tau} - \frac{a_2 p_2 A_2 A_1 B \tanh \tau}{\cosh^{(p_1 + p_2)} \tau} - h_2 \left\{ \frac{3 p_2 A_2 B^3 \tanh \tau}{\cosh^{p_2} \tau} + \frac{p_2 (p_2 + 1) (p_2 + 2) A_2 B^3 \tanh \tau}{\cosh^{(p_2 + 2)} \tau} \right\} = 0. \] (9)
Now, by equating the exponent $p_1 + 2$ and $2p_1$ and the exponent $p_2 + 2$ and $p_1 + 2$ in (8), (9) we get

$$p_1 = p_2 = 2,$$  
(10)

By equating the coefficients of $\frac{\tanh \tau}{\cosh^{P+j} \tau}$, $j = 0, 2$ in (8) and (9) we have

$$v - 4b_2B^2 = 0,$$  
(11)

$$-a_1A_1^2 + 24b_1A_1B^2 - 2c_1A_2^2 = 0,$$  
(12)

$$v - 4b_2B^2 = 0,$$  
(13)

$$-a_2A_1 + 12b_2B^2 = 0.$$  
(14)

From the above equations we have

$$b_2 = b_1 = b,$$  
(15)

$$B = \frac{1}{2} \sqrt{\frac{v}{b}},$$  
(16)

$$A_1 = \frac{3v}{a_2},$$  
(17)

$$A_2 = \frac{3v}{a_2} \sqrt{\frac{a_2 - a_1}{c_1}}, \quad \frac{a_2 - a_1}{c_1} > 0.$$  
(18)

The following Fig. 1 displays 1-soliton solution $q(x,t)$, $r(x,t)$ with the parameters $a_1 = 1$, $a_2 = 2$, $b = 1$, $c = 2$, $v = 0.5$, $-20 \leq x \leq 20$, $0 \leq t \leq 20$.

3.1.2. Coupled KdV equation with power-law nonlinearity

Starting with the same ansatz method as (5) and (6), equations (3) and (4) give
\[ \frac{p_1 q A_1 B \tanh \tau}{\cosh p_1 \tau} - \frac{a_1 p_1 A_1^{n+1} B \tanh \tau}{\cosh (np_1 + p_1) \tau} - b_1 \left\{ \frac{p_1^3 A_1 B^3 \tanh \tau}{\cosh p_1 \tau} + \frac{p_1(p_1 + 1)(p_1 + 2)A_1 B^3 \tanh \tau}{\cosh (p_1 + 2) \tau} \right\} = 0 \]  
\[ \frac{c_1 p_2^2 A_2^{n+1} B \tanh \tau}{\cosh (np_2 + p_2) \tau} = 0 \]  
(19)

Equating the exponent of \( np_1 + p_1 \), \( p_1 + 2 \) and exponent of \( np_2 + p_1 \), \( p_1 + 2 \) in (19) and (20) we get

\[ p_1 = p_2 = \frac{2}{n} . \]  
(21)

From (19) and (20), by equating the coefficients of \( \cosh^{p_1 + j} \tau \), \( j = 0, 2 \) we get

\[ v - \frac{4}{n^2} b_1 B^2 = 0 , \]  
(22)

\[ a_1 A_1^{n+1} - \frac{(n + 2)(2n + 2)}{n^2} b_1 A_1 B^2 - c A_1^{n+1} = 0 , \]  
(23)

\[ v - \frac{4}{m^2} b_2 B^2 = 0 , \]  
(24)

\[ a_2 A_2^m - \frac{(m + 2)(2m + 2)}{m^2} b_2 B^2 = 0 . \]  
(25)

From the above equations we get

\[ n = m , \]  
(26)

\[ b_2 = b_1 = b , \]  
(27)

\[ B = \frac{m}{2} \sqrt{\frac{v}{b_1}} , \]  
(28)

\[ A_1 = \left( \frac{v(m + 2)(2m + 2)}{4a_2} \right) \frac{1}{m} , \]  
(29)

\[ A_2 = \left( \frac{v(m + 2)(2m + 2)b_1 A_1 B^2}{m^2} - a_1 A_1^{n+1} \right) \frac{1}{m+1} . \]  
(30)
In Fig. 2 we display 1-soliton solution \( q(x,t) \), \( r(x,t) \) with the parameters:

\[
m = 2, \quad a_1 = 1, \quad a_2 = 2, \quad b_1 = b_2 = 1, \quad c_1 = -2, \quad v = 0.5, \quad -20 \leq x \leq 20, \quad 0 \leq t \leq 20
\]

### 3.2. Topological solitons

Topological solitons are also known as *kinks* or *shock waves* [13, 14]. A topological soliton is a nonlinear wave where we have a transition from one stable state to another. Thus, these are typically modeled by the \( \tanh \) functions. Hence, the following ansatz will be utilized to get a solution of the coupled KdV equations.

\[
q(x,t) = A_1 \tanh^{p_1} \tau, \quad (31)
\]

\[
r(x,t) = A_2 \tanh^{p_2} \tau, \quad (32)
\]

where \( A_1 \) and \( A_2 \) are free parameters of \( q \) and \( r \) soliton respectively and \( \tau \) is given by (7).

Substituting (31) and (32) into (1) and (2) respectively yields

\[
\left\{ (p_1 - 1)(p_1 - 2) \tanh^{p_1 - 3} \tau - (p_1 + 1)(p_1 + 2) \tanh^{p_1 + 3} \tau - \{2p_1^2 + (p_1 - 1)(p_1 - 2)\} \tanh^{p_1 - 1} \tau + \{2p_1^2 + (p_1 + 1)(p_1 + 2)\} \tanh^{p_1 + 1} \tau + c_1p_2A_2^2\tanh^{2p_2 - 1} \tau - \tanh^{2p_2 + 1} \tau \right\} = 0,
\]

\[
p_1vA_2B(\tanh^{p_2 + 1} \tau - \tanh^{p_2 - 1} \tau) + a_2p_2A_1^2B(\tanh^{2p_2 - 1} \tau - \tanh^{2p_2 + 1} \tau) + p_1A_1B^3
\]

\[
+ b_2p_2A_2B^3\{(p_2 - 1)(p_2 - 2) \tanh^{p_2 - 3} \tau - (p_2 + 1)(p_2 + 2) \tanh^{p_2 + 3} \tau
\]

\[
- \{2p_2^2 + (p_2 - 1)(p_2 - 2)\} \tanh^{p_2 - 1} \tau + \{2p_2^2 + (p_2 + 1)(p_2 + 2)\} \tanh^{p_2 + 1} \tau \right\} = 0.
\]

Now, by equating the exponent \( 2p_1 + 1 \) and \( p_1 + 3 \) in (33) equating the exponent \( 2p_2 + 1 \) and \( p_1 + 3 \) in (34) give

\[
p_1 = p_2 = 2. \quad (35)
\]

Fig. 2 – Soliton solution for \( m = 2 \): – left: \( q(x,t) \), right: \( r(x,t) \).
By equating the coefficients of \( \tanh^{p_1+j} \tau \), \( j = \pm 1, \pm 3 \) in (33), by equating the coefficients of \( \tanh^{p_2+j} \tau \), \( j = \pm 1, \pm 3 \) in (34) and put \( p_1 = p_2 = 2 \) we have

\[
v + 8b_1B^2 = 0, \quad (36)
\]

\[
vA_1 + a_1A_1^2 + 20b_1A_1B^2 + c_1A_2^2 = 0, \quad (37)
\]

\[
a_1A_1^2 + 12b_1A_1B^2 + c_1A_2^2 = 0, \quad (38)
\]

\[
v + 8b_1B^2 = 0, \quad (39)
\]

\[
v + a_2A_1 + 20b_2B^2 = 0, \quad (40)
\]

\[
a_2A_1 + 12b_2B^2 = 0. \quad (41)
\]

Using Maple program to solve the above equations we obtain the solutions of the form

\[
b_1 = b_2 = b, \quad (42)
\]

\[
B = \sqrt{-\frac{v}{8b}}, \quad (43)
\]

\[
A_1 = \frac{3v}{2a_2}, \quad (44)
\]

\[
A_2 = \frac{3v}{2a_2} \sqrt{\frac{a_2 - a_1}{c_1}}, \quad \frac{a_2 - a_1}{c_1} > 0. \quad (45)
\]

Figure 3 displays topological 1-soliton solution \( q(x,t) \), \( r(x,t) \), respectively, with the parameters \( a_1 = 1, a_2 = 2, b = -2, c = 2, v = 0.5, -20 \leq x \leq 20, \ 0 \leq t \leq 20 \)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{q(x,t).png}
\includegraphics[width=0.4\textwidth]{r(x,t).png}
\caption{Topological 1-soliton solution: left: \( q(x,t) \), right: \( r(x,t) \).}
\end{figure}

3.2.2. Coupled KdV equation with power-law nonlinearity

For the coupled KdV equation with power law nonlinearity, the starting hypothesis stays the same as in (33), (34). Thus, (3) and (4) give
\[ p_1 v A_1 B (\tanh^{p_1+1} \tau - \tanh^{p_1-1} \tau) + a_1 p_1 A_1^{p_1+1} B (\tanh^{np_1+p_1-1} \tau - \tanh^{np_1+p_1+1} \tau) + p_1 A_1 B^3 ((p_1 - 1)(p_1 - 2) \tanh^{p_1-3} \tau - (p_1 + 1)(p_1 + 2) \tanh^{p_1+3} \tau - \{2 p_1 + (p_1 - 1)(p_1 - 2)\} \tanh^{p_1-1} \tau) + \{2 p_1 + (p_1 + 1)(p_1 + 2)\} \tanh^{p_1+1} \tau] + c_1 p_2 A_2^2 (\tanh^{np_2+p_2-1} \tau - \tanh^{np_2+p_2+1} \tau) = 0, \]

\[ p_2 v A_2 B (\tanh^{p_2+1} \tau - \tanh^{p_2-1} \tau) + a_2 p_2 A_2 A_1 B (\tanh^{np_1+p_2-1} \tau - \tanh^{np_1+p_2+1} \tau) + b_2 p_2 A_2 B^3 ((p_2 - 1)(p_2 - 2) \tanh^{p_2-3} \tau - (p_2 + 1)(p_2 + 2) \tanh^{p_2+3} \tau - \{2 p_2 + (p_2 - 1)(p_2 - 2)\} \tanh^{p_2-1} \tau + \{2 p_2 + (p_2 + 1)(p_2 + 2)\} \tanh^{p_2+1} \tau} = 0. \]

Now, by the aid of balancing principle, equating the exponent \( np_1 + p_1 + 1 \) and \( p_1 + 3 \) and equating the exponent \( np_2 + p_2 + 1 \) in (46) gives

\[ p_2 = \frac{2}{n}. \]

By setting the coefficients of \( \tanh^{p_1+j} \tau, \tanh^{p_2+j} \tau, \ j = \pm 1, \pm 3 \) in (46) and (47) to zero, since these are linearly independent functions, we have

\[ v + (2 p_1^2 + (p_1 - 1)(p_1 - 2)) b_1 B^2 = 0, \]

\[ v A_1 + a_1 A_1^{p_1+1} + (2 p_1^2 + (p_1 + 1)(p_1 + 2)) b_1 A_1 B^2 + c_1 A_2^{p_1+1} = 0, \]

\[ a_1 A_1^{p_1+1} + (p_1 + 1)(p_1 + 2) b_1 A_1 B^2 + c_1 A_2^{p_1+1} = 0, \]

\[ p_1 (p_1 - 1)(p_1 - 2) b_1 A_1 B^3 = 0, \]

\[ v + (2 p_2^2 + (p_2 - 1)(p_2 - 2)) b_2 B^2 = 0, \]

\[ v + a_2 A_2^{p_2+1} + (2 p_2^2 + (p_2 + 1)(p_2 + 2)) b_2 B^2 = 0, \]

\[ a_2 A_2^{p_2+1} + (p_2 + 1)(p_2 + 2) b_2 B^2 = 0. \]

It follows from (52) that \( p_1 = 1 \) or \( p_1 = 2 \), i.e., the case 3.2.1. Now, substituting \( p_1 = 1 \) in (49), (53) we find

\[ n = m = 2. \]

Now, we put \( p_1 = p_2 = 1 \) and \( n = m = 2 \) in (49–55) and we get

\[ B = \sqrt{-v/2b}, \]

\[ A_1 = \sqrt{v/a_2}, \]

\[ A_2 = (A_1(a_2 - a_1)/a_2c_1)^{1/3}. \]

Figure 4 displays the kink soliton solution with the following parameters:

\[ a_1 = 1, \ a_2 = 2, \ b = -2, \ c = 2, \ v = 0.5, \ 20 \leq x \leq 20, \ 0 \leq t \leq 20. \]
3.3. Singular solitons

There are several nonlinear evolution equations that support singular solutions and they are applicable in various areas of mathematics and physics. Let us assume the following singular solutions:

\[ q(x,t) = \frac{A_1}{\sinh^{p_1} \tau}, \]
\[ r(x,t) = \frac{A_2}{\sinh^{p_2} \tau}, \]

where \( A_1 \) and \( A_2 \) are constants of the \( q \) and \( r \) soliton, respectively, and \( \tau \) is given by (7).

From (60) and (61) we have

\[ \frac{p_1 v A_1 B \coth \tau}{\sinh^{p_1} \tau} - \frac{a p_1 A_1^2 B \coth \tau}{\sinh^{2p_1} \tau} - b_1 \left\{ \frac{p_1^3 A_1 B^3 \coth \tau}{\sinh^{p_1} \tau} + \frac{p_1 (p_1 + 1)(p_1 + 2) A_1 B^3 \coth \tau}{\cosh^{(p_1 + 2)} \tau} \right\} = 0 \]

\[ -\frac{c}{1} \frac{p_2 A_1^2 B \coth \tau}{\sinh^{2p_2} \tau} = 0 \]

\[ \frac{p_2 v A_2 B \coth \tau}{\sinh^{p_2} \tau} - \frac{a_2 p_2 A_1 A_2 B \coth \tau}{\cosh^{(p_2 + 2)} \tau} - b_2 \left\{ \frac{p_2^3 A_1 A_2^3 \coth \tau}{\sinh^{p_2} \tau} + \frac{p_2 (p_2 + 1)(p_2 + 2) A_2 B^3 \coth \tau}{\sinh^{(p_2 + 2)} \tau} \right\} = 0. \]

Finally we get

\[ B = \frac{1}{2} \sqrt{\frac{v}{b}}, \]
\[ A_1 = -\frac{3v}{a_2}, \]
\[ A_2 = \frac{3v}{a_2} \sqrt{\frac{(a_1 + a_2)}{c_1}}, \quad \frac{a_1 + a_2}{c_1} > 0. \]
3.3.2. Coupled KdV equations with power-law nonlinearity

In this case, (3) and (4) give

\[
\begin{align*}
  & \frac{p_1 v A_1 B \coth \tau}{\sinh^{p_1} \tau} - a_1 p_1 A_1^{p_1+1} B \coth \tau \frac{\sinh^{p_1} \tau}{\sinh^{p_1+1} \tau} - b_1 \left\{ \frac{p_1^3 A_1 B^3 \coth \tau}{\sinh^{p_1} \tau} + \frac{p_1 (p_1 + 1)(p_1 + 2) A_1 B^2 \coth \tau}{\sinh^{p_1+2} \tau} \right\} \\
  & - \frac{c_1 p_2 A_2^{p_1+1} B \coth \tau}{\sinh^{p_2} \tau} = 0 \\
  & \frac{p_2 v A_2 B \coth \tau}{\sinh^{p_2} \tau} - a_2 p_2 A_2^{p_2} B \coth \tau \frac{\sinh^{p_2} \tau}{\sinh^{p_2+1} \tau} - b_2 \left\{ \frac{p_2^3 A_2 B^3 \cot \tau}{\sinh^{p_2} \tau} + \frac{p_2 (p_2 + 1)(p_2 + 2) A_2 B^2 \coth \tau}{\sinh^{p_2+2} \tau} \right\} = 0.
\end{align*}
\]

(67)

(68)

Doing the same calculation of subsection 3.1.2 we get

\[
B = m \sqrt{\frac{v}{2b}},
\]

(69)

\[
A_1 = \left( \frac{-(m+1)(m+2)v}{2a_2} \right)^{1/m},
\]

(70)

\[
A_2 = \left( \frac{-a_1 (m+1)(m+2)v}{2a_2} \right)^{1/m} - \left( \frac{(m+2)^2 v}{4} \right)^{1/(1+m)}.
\]

(71)

4. CONCLUSIONS

This paper studied the coupled KdV equations that appear in shallow two-layered water waves in lakes, rivers and ocean shores. Solitary waves, kinks, and singular soliton solutions were obtained. The ansatz method was applied in order to obtain these special solutions. The soliton solutions also introduced several constraint conditions that must remain valid in order for these solutions to exist. The numerical simulations clearly illustrate the solutions that were analytically obtained. Other approaches that will be used to study these coupled KdV equations which will reveal further nonlinear wave solutions will be reported elsewhere.

REFERENCES


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