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MULTI-RESOLUTION ANALYSIS OF WAVELET LIKE SOLITON SOLUTION OF KdV EQUATION

Bharat BHOSALE[1], Luminita MORARU[2], Bouthina S. AHMED[3], David RISER[4], Anjan BISWAS[5,6]

[1] University of Mumbai, S H Kelkar College of Arts, Commerce and Science, Devgad, (M.S.), India
[2] "Dunărea de Jos" University of Galati, Physics Department, Galati, 800008, Romania
[3] Ain Shams University, Girls' College, Department of Mathematics, Cairo 11757, Egypt
[4] Delaware State University, Department of Physics and Engineering, Dover, DE 19901-2277, USA
[5] Delaware State University, Department of Mathematical Sciences, Dover, DE 19901-2277, USA
[6] King Abdulaziz University, Faculty of Science, Department of Mathematics, Jeddah, Saudi Arabia
Corresponding author: Anjan BISWAS, E-mail: biswas.anjan@gmail.com

Many physical phenomena described by non-linear partial differential equations have soliton solutions that have, in turn, Gaussian representations. Using this fact, we carried out a wavelet analysis and obtained a better approximation for solution of the Korteweg-de Vries equation.

Key words: wavelet like solitons, Gaussian representation, wavelet decomposition, data processing.

1. INTRODUCTION

Multi-resolution analysis uses wavelet functions as basis with an objective to specify the signal as a collection of its successive approximations [1–13]. The term wavelet was first coined by Grossman and Morlet [9]. The major breakthrough occurred due to D. Gabor [8], who introduced the Windowed Fourier Transform for the local spectral analysis of radar signals that actually laid pathway for use of wavelets from electrical engineering to mathematical physics. Wavelet transform (WT) with a wide window for low frequency signals and a narrow window for high frequency signals was introduced and formalized later by Grossman and Morlet, Daubechies [7] and many others. Recently, WT has been emerged as the most effective tool for signal processing and image analysis especially when the signals are random and comprised of fluctuations of different scales [11], or in problems involving singular potentials in quantum mechanics, in discussions concerning q-algebras, and even in nuclear structure studies [10].

Wavelets are used to analyze soliton solutions arising from nonlinear partial differential equations (NPDE) which display very strong interaction between the initial conditions and the dynamics and involve multiple scales [10], being able to produce self-similar or fractal-like patterns. Since the soliton-like solutions have infinite extent, it requires rather appropriate compactly supported basis functions to investigate such structures than the traditional nonlinear tools (inverse scattering, group symmetry, functional transforms). Such structures/patterns generally have a finite space-time extension and a multi-scale structure. The wavelet functions have a space-dependent scale that makes it as a powerful tool for analyzing multi-scale phenomena. This motivated us to employ the wavelet methods to analyze wavelet like soliton solutions of NPDEs such as Non-linear Schrodinger Equation (NLS), Sine-Gordon equation (SG), Korteweg-de Vries equation (KdV) [2–4]. However, in this work, we have confined our discussion to the most celebrated KdV equation that appears in the study of waves in shallow water in the fluid dynamics.

The paper is organized as follows. After the preliminary remarks and the genesis of the problem in the first introductory section, mathematical pre-requisites relevantly required in the present work are given in Sec. 2, the features of the KdV equation and its solution in Sec. 2.2, followed by mathematical analysis demonstrating the application of wavelet-Galerkin and wavelet-Petrokov-Galerkin methods in approximating the soliton solution in Sec. 2.3, experimental data processing in Sec. 3, and the conclusions in the last section.

2. MATHEMATICAL PRE-REQUISITES

2.1. Wavelet Transform

The classical wavelet transform, also called as Continuous Wavelet transform (CWT), is a decomposition of a function, $f(x)$, with respect to a basic wavelet $\Psi(x)$, given by the convolution of a function with a scaled and translated version of $\Psi(x)$

$$W_{\Psi}(a,b)[\mathrm{f}]=|a|^{-1/2}\int f(x)\Psi^{*}\left(\frac{x-b}{a}\right)\mathrm{d}x = <f,\frac{1}{\sqrt{|a|}}\Psi\left(\frac{x-b}{a}\right)> = \\ =<f,\Psi_{a,b}> = <f,U(a,\ b)\Psi> = W_{\Psi}f(a,b) \tag{1}$$

where $<.,.>$ is the inner product.

The functions, f and Ψ are square integrable functions and Ψ satisfies the admissibility condition: $C_{\Psi}=\int\frac{\left|\hat{\Psi}(\omega)\right|^{2}}{|\omega|}\mathrm{d}\omega<\infty$. C_{Ψ} is called admissibility constant. The subscript '*' denotes complex conjugation, a is the scale parameter, $a > 0$, b is the translation parameter. The term $1/\sqrt{|a|}$ is the energy conservative term that keeps energy of the scaled mother wavelet equal to the energy of the original wavelet. $\Psi_{a,b}$ are the wavelet elements defined by a unitary affine mapping: $U(a,b)\colon\ \Psi(x)\rightarrow\frac{1}{\sqrt{|a|}}\Psi\left(\frac{x-b}{a}\right)$. The function $f(x)$ can be recovered by the reconstruction formula called Inverse Transform:

$$f(x)=\frac{1}{C_{\Psi}}\iint W_{\Psi}f(a,b)\frac{1}{\sqrt{|a|}}\Psi\left(\frac{x-b}{a}\right)\frac{\mathrm{d}a\mathrm{d}b}{a^{2}}, \tag{2}$$

where the admissibility constant, $C_{\Psi}>0$ [12]. Although, this admissibility condition puts on us the constraints in the choice of analyzing basic wavelet, it still leaves a considerable degree of freedom because, in one dimensional case, the admissibility condition means the vanishing behavior of the Fourier image of the analyzing wavelet $\hat{\Psi}(\omega)$ in the neighborhood of $\omega=0$ and this requirement can be redundantly satisfied if $\hat{\Psi}(\omega)=0$; implying thereby vanishing zero moment, $\int\Psi(t)\mathrm{d}t=0$.

Therefore, any function with compact support which satisfies above requirement can be successfully used as a basic wavelet. We may note that, as an example, the derivatives of Gaussian $\exp(-x^{2})$ can be chosen as basic wavelet. Furthermore, with the substitution, $f(x)$ as the inverse Fourier Transform, $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(\mathrm{i}\omega x)\hat{f}(\omega)\mathrm{d}\omega$ in the definition of wavelet transform (1), we immediately get

$$W_{\Psi}\left[f(x)\right](a,b)=\ |a|^{1/2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(\mathrm{i}\omega b)\overline{\hat{\Psi}(a\omega)}\hat{f}(\omega)\mathrm{d}\omega. \tag{3}$$

This is a spectral representation of WT, which is useful for evaluating the transform integral in a more convenient way.

The Dual of the Wavelet. For the wavelets Ψ_{1} and Ψ_{2} satisfying the mutual admissibility condition, $0<\left|\Psi_{1}\Psi_{2}\right|<\infty$, the wavelet decomposition of $f\in L^{2}(R)$ is given by

$$f(x)=C_{\Psi_1\Psi_2}^{-1}\int\frac{1}{\sqrt{|a|}}\Psi_2\left(\frac{x-b}{a}\right)W_{\Psi_1}f(a,b)[f]\frac{\mathrm{d}a\mathrm{d}b}{a^2},$$

where $W_{\Psi_1}f(a,b)[f]$ is the wavelet transform of f with respect to Ψ_1, then Ψ_2 is called the wavelet dual of Ψ_1 and $C_{\Psi_1,\Psi_2}=\int\frac{\mathrm{d}\omega}{|\omega|}\overline{\hat{\Psi}_1(\omega)}\hat{\Psi}_2(\omega)$.

Discretized form of wavelet transforms. Wavelet analysis is particularly based on the property of self-similarity and used to process fractal like patterns. The direct evaluation of the convolution in the integral transform on the lattice size of $m \times n$, where m is the number of scales and n is the number of sampling points, is somewhat computationally involved. The convenient way for numerical implementation of WT is to use its discretized version, called Discrete Wavelet Transform (DWT). Depending on the discretization either of the transform domain parameters-scale and translation variables or of the independent variable of the function to be transformed, we have corresponding versions of DWT and could be used either depending on the requirement. In each case, DWT yields a countable set of coefficients in the transform domain that corresponds to points on a two dimensional grid or lattice $m \times n$ of discrete points in the scale-translation domain. With a and b as scale and translation parameters, taking scale a: $a=a_0^m$ and the translation b: $b=nb_0a_0^m$, where a_0 and b_0 are the discrete scale and translation step sizes, respectively, the DWT is given by [12],

$$\begin{aligned}\mathrm{W}_\Psi(m,n)[\mathrm{f}]&=\frac{1}{\sqrt{a_0^m}}\int_{-\infty}^{\infty}f(x)\Psi\left(\frac{x-nb_0a_0^m}{a_0^m}\right)\mathrm{d}x=\frac{1}{\sqrt{a_0^m}}\int_{-\infty}^{\infty}f(x)\Psi\left(a_0^{-m}x-nb_0\right)\mathrm{d}x=\\&=<f,\Psi_{m,n}>=<f,U(a_0^m,\ nb_0a_0^m)\Psi>.\end{aligned}\tag{4}$$

Unlike the CWT, the DWT is defined for positive scale values, $a_0>0$. For fast numerical algorithms, we take $a_0=2$. Here, the factor $1/\sqrt{a_0^m}$ preserves the unit energy property as referred earlier. Restricting the calculations on a discrete sub-lattice in the evaluation of DWT, the bi-parametric family of the wavelets $U(a_0^m,nb_0a_0^m)\Psi$ becomes the discrete set, $\Psi_{m,n}(x)=a_0^{-m/2}\Psi\left(a_0^{-m}x-nb_0\right)$, labeled by two integers $m,n\in Z$.

Under the condition $a_0=1$, the reconstruction of $f(x)$ is given by

$$f(x)\approx k\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathrm{W}_\Psi f(m,n)\right]a_0^{-\frac{m}{2}}\Psi\left(a_0^{-m}-nb_0\right)=k\sum_{m,n}\tilde{\Psi}_{m,n}<\Psi_{m,n},f>,\tag{5}$$

where k is the constant that depends upon the redundancy of the basic wavelet and the lattice combination which is ignored in many applications, and $\tilde{\Psi}_{m,n}$ is wavelet dual of $\Psi_{m,n}$. The objective of numerical implementation is to find a function $\Psi(x)$ such that the set of wavelet coefficients $<\Psi_{m,n},f>$. determines the function $f\in L^2(R)$ in unique way and provides a numerically stable reconstruction.

2.2. Korteweg-de Vries Equation

The generalized Korteweg–de Vries equation (gKdV) with time-dependent damping and dispersion [5], is given as

$$q_t+q^nq_x+a(t)q+b(t)q_{xxx}=0.\tag{6}$$

This equation arises in various physical situations including the study of coastal waves in ocean, liquid drops and bubbles and also in the context of atmospheric blocking phenomenon and in particular in the issue

of dipole blocking. The first term of the equation is the evolution term, the second term represents the nonlinear term, while the third term is the linear damping with a time-dependent coefficient $a(t)$ and the fourth term is the dispersion term with time-dependent coefficient $b(t)$. In equation (6), a, $b \in R$ while $n \in Z^+$.

The solitary wave solution to (6) is given as [5],

$$q(x,t) = \frac{A(t)}{\cosh^{p}\left[B(t)(x - v(t)t)\right]}. \tag{7}$$

where A represents the amplitude of the soliton, while B is the inverse width of the soliton and v represents the velocity of the soliton. Since damping and dispersion terms have time-dependent coefficients, in general, without loss of generality, we can write, $A = A(T)$, $B = B(t)$ and $v = v(t)$. Thus, for $p = 2/n$, the solitary wave solution (7) takes the form

$$q(x,t) = \frac{A}{\cosh^{\frac{2}{n}}\left[B(x - vt)\right]} = A\sec h^{\frac{2}{n}}\left[B(x - vt)\right]. \tag{8}$$

when $n = 1$,

$$q(x,t) = A\sec h^{2}\left[B(x - vt)\right]. \tag{9}$$

The same can also be written as

$$q(x,t) = q(s) \quad \text{with} \quad s = x - vt. \tag{10}$$

2.3. Mathematical analysis

Most often the signals have a Gaussian form and display self-similar fractal like patterns. It is reported that the soliton-like solution, $u(x,\ t) = u(s)$ with $s = x - v\ t$ has expansion in a Gaussian family of wavelets $\Psi(s) = Ne^{Q(s)}$, where $Q(s)$ is a polynomial and N, the normalization constant [10]. In particular, if we choose $Q(s) = -\mathrm{i}s - \frac{s^2}{2}$, we obtain a very particular wavelet with the support mainly confined in the (−1, 1) interval, namely $\Psi(s) = \exp\left[-\mathrm{i}s - \frac{s^2}{2}\right]\ \pi^{1/4}$. We shall use this fact in the development of procedure for approximating the soliton solution through wavelet decomposition and in the further application of experimental data processing. We consider the most celebrated gKdV equation obtained from (6) with $n = 1$

$$q_t + qq_x + \mu q + \nu q_{xxx} = 0. \tag{11}$$

The equation (11) can be written in differential operator form as

$$\hat{L}(q(x,t)) = 0 \ \text{ where } \ \hat{L} \equiv \frac{\partial}{\partial t} + q\frac{\partial}{\partial x} + \mu q + \nu\frac{\partial^3}{\partial x^3}. \tag{12}$$

In general, the stable solutions of the nonlinear-dispersive equations are dependent of the initial conditions, through their conservation laws. Consequently, they can generate a large class of patterns, shaped by the balance between nonlinear interaction and dispersion, among which the most interesting examples are solitons, breathers and kinks.

To apply the wavelet method or more appropriately, the *wavelet-Galerkin method*, the solution is decomposed with respect to the wavelet basis as

$$q(s) = q(x,t) + \sum\nolimits_{j,k} C_{j,k}(t)\Psi_{j,k}(x), \tag{13}$$

where $C_{j,k}(t)$ are the time dependent wavelet coefficients and $\Psi_{j,k}(x)$ is an admissible function/basic wavelet to be taken as $\Psi_{j,k}(x) = h^{-j/2}\Psi(h^{-j}x - k)$.

This is a discrete expansion or wavelet decomposition of soliton solution $q(s)$ in terms of integer translations (k) of Ψ which provides the analysis of localization, and in terms of dyadic dilations (h^j) of Ψ, which provides the description of different scales.

Substituting the decomposition (13) into (12), it yields the system of equations

$$\sum_{j,k} C_{j,k}(t)\hat{L}\Psi_{j,k}(x) = 0. \tag{14}$$

By scalar multiplication $\int \mathrm{d}x\bar{\Psi}_{l,m}$, where $\bar{\Psi}_{l,m}$ is dual wavelet, we obtain the orthogonal system of compactly supported wavelets $\Omega_{mk}^{lj} \equiv \int \mathrm{d}x\bar{\Psi}_{l,m}\hat{L}\Psi_{j,k}(x)$.

The system of equations (14) becomes

$$\sum_{j,k} \Omega_{mk}^{lj} C_{j,k} = 0. \tag{15}$$

This is a system of ordinary differential equations in the wavelet coefficients $C_{j,k}$. For the orthogonal Daubechies wavelets with compact support, only the matrix elements Ω_{mk}^{lj} with the basic functions of the same scale $l = j$ are different from zero. Thus, (15) provides a sparse structure of non-linear system suitable for numerical implementation. The main component of the wavelet-Galerkin solution is the evaluation of the matrix elements Ω_{mk}^{lj} of the differential operators in wavelet basis, $\Psi_{j,k}$. For this purpose, the analytically determined wavelets such as Mexican hat or Morlet wavelets are employed. For practical purposes, we can use the normalized Mexican hat wavelet, which is self dual, as the basic wavelet.

The wavelet–Galerkin scheme for equation (11) consists of substitution of discrete wavelet decomposition of the solution $q(x,t) = \sum_{j,k} C_{j,k}(t)\Psi_{j,k}(x)$ into (11), followed by the projection of the result onto orthogonal basis of $\Psi_{l,m}$,

$$\int \mathrm{d}x\bar{\Psi}_{l,m}(x)[\dot{C}_{j,k} - C_{s,r}\Psi_{s,r}(x)C_{j,k}\frac{\mathrm{d}}{\mathrm{d}x} + \mu C_{j,k} + \nu C_{j,k}\frac{\mathrm{d}^3}{\mathrm{d}x^3}]\Psi_{j,k}(x) = 0. \tag{16}$$

For the orthogonal Daubechies wavelets with compact support, this gives a system of nonlinear ordinary differential equations with unknown wavelet coefficients $C_{j,k}$ depending on time *t* only

$$\dot{C}_{l.m} - \Omega_{mrk}^{lsj} C_{j,k} + \mu\Omega_{1,mk}^{lj} C_{j,k} + \nu\Omega_{mk}^{lj} C_{j,k} = 0, \tag{17}$$

where the matrix elements are

$$\begin{aligned} &\Omega_{mrk}^{lsj} = \int \mathrm{d}x\bar{\Psi}_{l,m}(x)\Psi_{s,r}(x)\frac{\mathrm{d}}{\mathrm{d}x}\Psi_{j,k}(x); \quad \Omega_{1,mk}^{lj} = \int \mathrm{d}x\bar{\Psi}_{l,m}(x)\Psi_{j,k}(x); \\ &\Omega_{mk}^{lj} = \int \mathrm{d}x\bar{\Psi}_{l,m}(x)\Psi_{s,r}(x)\frac{\mathrm{d}^3}{\mathrm{d}x^3}\Psi_{j,k}(x). \end{aligned} \tag{18}$$

The direct integration in the matrix elements is numerically unstable for the irregularity of the basic function $\Psi(x)$. However, they can be evaluated analytically. When all coefficients of (18) are known, the system of ordinary differential equations (17) can be solved numerically by an implicit or explicit method.

In the simplest case, we have

$$C_{l,m}(t+\tau) = C_{l,m}(t) + \tau[\Omega_{mrk}^{lsj} C_{j,k}(t)C_{r,s}(t) + \mu\Omega_{1,mk}^{lj} C_{j,k} + \nu\Omega_{mk}^{lj} C_{j,k}(t)],$$

where τ is a time step of integration.

The evaluation of the matrix elements of all differential operators is provided by the knowledge of connection coefficients-the matrix elements of those operators in the basis of wavelet scaling function $\varphi(x)$, $\Lambda_{k_1 \dots k_n}^{(d_1 \dots d_n)} = \int \mathrm{d}x \varphi_{k_1}^{(d_1)} \dots \varphi_{k_n}^{(d_n)}$, where the superscripts of the parentheses stand for the order of differentiation. All the terms with the wavelet basic function Ψ are evaluated by the substitution $\Psi(x) = \sqrt{2} \sum g_n \varphi(2x-1)$. The general method of evaluation of connection coefficients is presented by Beylkin [1].

Alternative scheme of evaluating the system of differential equations is provided by modified method, known as *Wavelet-Petrokov-Galerkin* (WPG) method where the basic difference from the wavelet-Galerkin method is that of using the different functions as test and admissible wavelets [11]. To apply WPG method, we make the substitution $\Psi_{j,k}(x) = h^{-j/2} \Psi(h^{-j}x - k)$ in the expression (16) to write

$$\sum_{j,k} \int \mathrm{d}x h^{-j/2} \Psi(h^{-j}x - m) \left[\frac{\partial}{\partial t} C_{j,k} - C_{s,r} h^{-j/2} \Psi(h^{-j}x - r) C_{j,k} \frac{\mathrm{d}}{\mathrm{d}x} + \mu C_{j,k} \frac{\mathrm{d}}{\mathrm{d}x} + \right.$$
$$\left. + \nu C_{j,k} \frac{\mathrm{d}^3}{\mathrm{d}x^3} \right] h^{-j/2} \Psi(h^{-j}x - k) = 0. \tag{19}$$

Introducing the change of variable, $y = h^{-j}x - k$, the expression (19) becomes

$$\sum_{j,k} a(k) \frac{\mathrm{d}C_{j,k}}{\mathrm{d}t} + h^{-3j/2} \sum_{s,r} \sum_{j,k} b(l,k) C_{s,r} C_{j,k} + \mu h^{-j/2} \sum_{j,k} a(k) C_{j,k} + \nu h^{-3j} \sum_{j,k} \mathrm{d}(k) C_{j,k} = 0, \tag{20}$$

where $a(k) = \int \mathrm{d}y \Psi(y) \bar{\Psi}(y-k)$; $b(l,\ k) = \int \mathrm{d}y \frac{\mathrm{d}\Psi(y)}{\mathrm{d}y} \bar{\Psi}(y-k)$; and $c(k) = (-1)^{\beta} \int \mathrm{d}y \frac{\mathrm{d}^{\alpha}\Psi(y)}{\mathrm{d}y} \frac{\mathrm{d}^{\beta}\bar{\Psi}(y-k)}{\mathrm{d}y}$.

The unknown coefficients $C_{j,k}$ are determined from the system of ordinary differential equations written in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t} LC + C^T MC + NC + TC = 0, \tag{21}$$

where $C = C_{j,k}$, $L(l,k) = a(l-k)$, $M(l,k,s) = h^{-3j/2} b(l-k, l-s)$, $N(l,k) = \mu h^{-j/2} a(l-k)$, and $T(l,k) = \nu h^{-3j} c(l-k)$.

Note that the unknown coefficients are only the time dependent. By trapezoidal rule, $\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{C^{n+1} - C^n}{\Delta t}$, where $\Delta t = t_{n+1} - t_n$ is the time interval.

The equation (21) becomes

$$L\left(\frac{C^{n+1} - C^n}{\Delta t} \right) + C^T MC + NC + TC = 0. \tag{22}$$

Now setting $G(C) = C^T MC + NC + TC$, we have from (22)

$$L\left(C^{n+1} - C^n \right) + \frac{G(C^{n+1}) + G(C^n)}{2} \Delta t = 0. \tag{23}$$

This algebraic equation can be finally solved by Newton's iterative method using the recursive construct $U^{n+1} = U^n - \frac{f(U^n)}{f'(U^n)}$, $n = 0, 1, 2, \dots$ The solution thus obtained by approximation process can be eventually compared with some exact solution obtained from (9), $q(x,t) = A \sec h^2 \left[B(x - vt) \right]$, for specific values of the amplitude *A*, the inverse width *B* and the velocity *v* of the soliton computed at different positions depending on time *t*. This provides the alternative method for determining the soliton solution of KdV and like.

3. EXPERIMENTAL DATA PROCESSING

In experimental data processing, the central problem in either one dimensional or multi-dimensional set up is the separation of two or more signals from a noisy background. Most often these signals have a Gaussian form which itself is a wavelet. Therefore, the wavelet analysis provides robust method in the presence of noise especially if taken as wavelet image of the Gaussian $\exp(-x^2/2)$ with vanishing momenta wavelets $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2/2)$ known analytically, and the values of wavelet coefficients can be used to determine the amplitude and dispersion of the original signal (the soliton in this case), which has Gaussian representation. The central idea here is to assume the Gaussian distribution representing the soliton solution as the best fit for the experimental data and take the wavelet image of this Gaussian wavelet with appropriate analytically tested function as analyzing wavelet.

We assume that Gaussian function is the 'best fit' to describe the experimental data set to be processed as testing wavelet function. Then, the problem of fitting the distribution of Gaussian sources is to find the parameter set $(N^k, \sigma_k, x_k^m)_{k=1}^M$ that minimizes the difference

$$F(N,\sigma,x^m) = f_{\exp}(x) - \sum_{k=1}^{M} \frac{N_k}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(x-x_k^m)^2}{2\sigma_k^2}\right). \tag{24}$$

Applying wavelet transform to (24) with some analytically tested basic wavelet, say, Mexican hat or Morlet wavelet, one can precisely locate the position of the sources x_k^m.

Let us start with the wavelet image of a single Gaussian representing the soliton, located, without loss of generality, at $x^m = 0$.

$$q_{gauss}(x,\ t) = \frac{N}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{x^2}{2\sigma_k^2}\right) \text{ for all } t \text{ in space time coordinates } (x,\ t). \tag{25}$$

We need the wavelet images of the Gaussian with different vanishing momenta wavelets as analyzing wavelet, where the first m family of vanishing momenta wavelets [6] of basic wavelets Ψ, satisfies the condition $\int \mathrm{d}x x^m \Psi(x) = 0,\ \forall m,\ 0 \le m < n,\ n \in \mathbb{Z}$.

The wavelet images of the Gaussian with vanishing momenta wavelet are therefore given by

$$W_{g_n}(a,b)[q_{gauss}] = \int \overline{\frac{1}{\sqrt{a}} g_n\left(\frac{x-b}{a}\right)} q_{gauss}(x)\mathrm{d}x. \tag{26}$$

The integrals in (26) can be evaluated using the Fourier representation (3),

$$\mathrm{W}_{\Psi}(a,b)\left[\mathrm{q}_{\mathrm{gauss}}\right] = \frac{1}{2\pi}\ |a|^{1/2} \int_{-\infty}^{\infty} \exp(\mathrm{i}kb)\overline{\tilde{g}(ak)}\tilde{q}_{gauss}(k)\mathrm{d}k, \tag{27}$$

where $\tilde{g}_n(k) = \sqrt{2\pi}(\mathrm{i}k)^n \exp(-k^2/2)$.

Instead of evaluating integrals for each n separately, we can evaluate it once for the Morlet wavelet

$$\tilde{g}_n(\tau,k) = \sqrt{2\pi}\exp(\mathrm{i}k\tau - k^2/2) \tag{28}$$

and then take the nth derivative of $\tilde{g}$ with respect to the formal parameter τ at $\tau = 0$ to obtain the wavelet image of g_n family,

$$\tilde{g}_n(k) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Big|_{\tau=0} \tilde{g}_n(\tau,k) \tag{29}$$

$$\mathrm{W}_{g_n}(a,b)=\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n\Bigg|_{\tau=0}\mathrm{W}_{g(\tau)}(a,b). \tag{30}$$

Substituting (28) instead of $\tilde{g}_n(k)$ into (29) and taking into account the Fourier image of Gaussian (25) $\tilde{q}_{gauss}(k)=N\exp\left(-\frac{k^2\sigma^2}{2}\right)$, we arrive at

$$W_{g_n}(a,b)[q_{gauss}]=N\sqrt{\frac{a}{2\pi}}\int \mathrm{d}k\exp\left(\mathrm{i}k(b-a\tau)-\frac{k^2}{2}(a^2+\sigma^2)\right)=N\sqrt{a}\exp\left(-\frac{(b-a\tau)^2}{2(a^2+\sigma^2)}\right)\Big/\sqrt{a^2+\sigma^2}, \tag{31}$$

for the wavelet image of a single Gaussian with respect to the analyzing wavelet, vanishing momenta wavelet g_n in instant case. To find the distribution parameters for the case of single Gaussian source, we use the coefficients of its g_2 decomposition. Equation (30) for $n=2$, for example, leads to

$$W_{g_2}(a,b)[q_{gauss}]=Na\left(\frac{a}{a^2+\sigma^2}\right)^{3/2}\left(1-\frac{b^2}{a^2+\sigma^2}\right)\exp\left(-\frac{b^2}{2(a^2+\sigma^2)}\right). \tag{32}$$

Taking the derivative $\partial/\partial a$ of equation (32) at the central point $b=0$, we find the extremum of the g_2 coefficient at a scale $a_m=\sqrt{5\sigma}$.

The value of the wavelet coefficient at the extremal point is therefore

$$W_{g_2}(a_m,0)[q_{gauss}]=\frac{N}{\sqrt{\sigma}}5^{5/4}6^{-3/2}=\frac{N}{\sqrt{a_m}}\left(\frac{5}{6}\right)^{3/2}. \tag{33}$$

Thus, performing the convolution (1) or for numerical implementation (4) with $g_n(x)\equiv\Psi_{j,k}(x)=h^{-j/2}\Psi(h^{-j}x-k)$, and finding the maximum of the g_2 wavelet coefficient, we obtain the dispersion and amplitude of the original distribution q_{gauss} such that $q_{gauss}=\sum_{jk}\tilde{\Psi}_{j,k}<\Psi_{j,k},q_{gauss}>$.

4. CONCLUSIONS

Multi-resolution analysis employs wavelets as basis. Wavelet analysis is based on the property of self-similarity that makes wavelet as an effective tool to process fractal like patterns represented by signals that have Gaussian forms. Several non-linear partial differential equations describing various physical phenomena have wavelet like soliton solutions, which in turn, have Gaussian representations exhibiting self-similar fractal like patterns. As a result, wavelet methods are quite effective for approximating and analyzing such soliton solutions. Moreover, Gaussian representation of wavelet like solitons can be fruitfully used in experimental data processing.

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