A SHORT PROOF FOR THE CHARACTERIZATION BY ORDER AND DEGREE PATTERN OF $PGL(2,q)$ AND $L_2(q)$

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The degree pattern of a finite group $G$ is denoted by $D(G)$. In [14] and [19] the characterization of $L_2(q)$ and $PGL(2,q)$ by their orders and their degree patterns are proved. In this paper we give a very short proof for the main results of these papers.

Key words: projective special linear group, projective general linear group, degree pattern, prime graph.

1. INTRODUCTION

Let $N$ and $P$ denote the set of natural numbers and the set of prime numbers, respectively. If $n \in N$, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of orders of the elements of $G$ is denoted by $\pi_e(G)$. Obviously, $\pi_e(G)$ is closed and partially ordered by divisibility, hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. The prime graph of $G$ is a graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (we write $p \sim q$) if and only if $G$ contains an element of order $pq$. The prime graph of $G$ is denoted by $\Gamma(G)$. Denote by $t(G)$ the numbers of connected components of $\Gamma(G)$ and by $\pi_i(G)$, where $i = 1, 2, \ldots, t(G)$, the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then always we assume that $2 \in \pi_1$ and $\pi_2, \ldots, \pi_{t(G)}$ are called the odd component(s) of $\Gamma(G)$.

Let $\pi(G) = \{p_1, p_2, \ldots, p_m\}$ and $p_1 < p_2 < \cdots < p_m$. The degree pattern of $G$ is denoted by $D(G)$ and defined as follows: $D(G) = (\deg(p_1), \deg(p_2), \ldots, \deg(p_m))$, where $\deg(p_i)$ is the degree of vertex $p_i$ in $\Gamma(G)$. A group $G$ is called OD-characterizable if $G$ is uniquely determined by $|G|$ and $D(G)$.

It is proved that sporadic simple groups and their automorphism groups except $Aut(J_4)$ and $Aut(McL)$, the alternating groups $A_p, A_{p+1}, A_{p+2}$ and the symmetric groups $S_p$ and $S_{p+1}$, where $p \in P$ are OD-characterizable [7]. In [14], it is proved that all finite simple groups with exactly four prime divisors are OD-characterizable, except $A_{10}$. Also in [16, 17] finite groups with the same order and degree pattern as an almost simple group related to $L_2(49)$ or $U_3(5)$ are determined. Recently in [18] and [13] it is proved that every special linear group $L_2(q)$ and every projective general linear groups $PGL(2,q)$ are OD-characterizable. In fact, in this paper we give a very simple proof for these results. More results can be found in [1, 5, 6, 8, 9, 10, 15, 19, 20]. All further unexplained notations are standard and can be found in [2].

If $p \in P$ and $k, n \in N$, then $p^k \| n$ means that $p^k \mid n$ and $p^{k+1} \nmid n$.
2. PRELIMINARY RESULTS

The next lemma summarizes the structural properties of a Frobenius group and a 2-Frobenius group [2, 3]:

**Lemma 2.1.** (a) Let $G$ be a Frobenius group with Frobenius kernel $K$ and Frobenius complement $H$. Then $t(G) = 2$, $\pi(K)$ and $\pi(H)$ are the components of $\Gamma(G)$.

(b) Let $G$ be a 2-Frobenius group, i.e., $G$ has a normal series $1 \leq H \leq K \leq G$, such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. If $G$ has even order, then

(i) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(ii) $G/K$ and $K/H$ are cyclic, $|G/K|$ divides $|\text{Aut}(K/H)|$ and $(|G/K|, |K/H|) = 1$;

(iii) $H$ is a nilpotent group and $G$ is a solvable group.

By using [12, Theorem A] we have the following result:

**Lemma 2.2.** Let $G$ be a finite group with $t(G) \geq 2$. Then one of the following holds:

(a) $G$ is a Frobenius or 2-Frobenius group;

(b) there exists a nonabelian simple group $S$ such that $S \leq G/N \leq \text{Aut}(S)$ for some nilpotent normal $\pi_1$-subgroup $N$ of $G$ and $G/S$ is a $\pi_1$-group.

3. MAIN RESULTS

Throughout this section let $p \in P$, $n \in N$ and $q = p^n$.

**Theorem 3.1.** Let $p$ be an odd prime and $\varepsilon = 1$ or 2. If $G$ is a finite group such that $|G| = q(q^2 - 1)/\varepsilon$ and $\text{deg}(p) = 0$ in $\Gamma(G)$, then $L_2(q) \leq G \leq \text{Aut}(L_2(q))$.

*Proof.* We can easily see that if $|G| = 12$ or $24$ and $\text{deg}(3) = 0$ in $\Gamma(G)$, then $G \cong A_4$ or $S_4$, respectively. Also by using GAP we obtain that if $q = 5$, then $G \cong A_5$ or $S_5$, as required. Therefore let $q > 5$. First we show that $G$ is not a Frobenius or 2-Frobenius group.

**Step 1.** If $G$ is a Frobenius group with kernel $K$ and complement $C$, then by Lemma 2.1, $\pi(K)$ and $\pi(C)$ are the connected components of $\Gamma(G)$. Since $\text{deg}(p) = 0$ in $\Gamma(G)$, $\pi(K) = \{p\}$ or $\pi(C) = \{p\}$. If $\pi(K) = \{p\}$, then $|K| = q$ and $|C| = (q^2 - 1)/\varepsilon$. We know that $|K| \equiv 1(\text{mod}|C|)$, which is impossible. If $\pi(C) = \{p\}$, then $|C| = q$ and $|K| = (q^2 - 1)/\varepsilon$. Hence $(q^2 - 1)/\varepsilon \equiv 1(\text{mod} q)$, which is a contradiction, since $q > 5$. Therefore, $G$ is not a Frobenius group. Now let $G$ be a 2-Frobenius group with normal series $1 \leq H \leq K \leq G$, such that $G/H$ and $K$ are Frobenius groups with kernels $K/H$ and $H$, respectively. By using Lemma 2.1, $|K/H| = q$ and $|H||G/K| = (q^2 - 1)/\varepsilon$, since $\text{deg}(p) = 0$ in $\Gamma(G)$ and $p$ is an odd prime number. Also by Lemma 2.1, we have $|G/K|(p - 1)$, which is a divisor of $q - 1$. Therefore, $q - 1 = m|G/K|$, for some $m \geq 1$ and so $|H| = (q + 1)m/\varepsilon$. We know that $|H| \equiv 1(\text{mod}|K/H|)$. So $(q + 1)m/\varepsilon \equiv 1(\text{mod} q)$. Then $m \equiv \varepsilon(\text{mod} q)$ and so $m = \varepsilon$, since $1 \leq m \leq q$. Hence $|H| = q + 1$ and so $|G/K| = (q - 1)/\varepsilon$. Also $|G/K|(p - 1)$ and $(p - 1)|(q - 1)$, which implies that $q = p$. Therefore, $|H| = p + 1$ and $|K| = p(p + 1)$. Since $K$ is a solvable group, if $t$ is an odd prime divisor of $p + 1$, then $K$ has a $\{p, t\}$-Hall subgroup, say $T$. Let $s \in N$ and $t^s|(p + 1)$. Then $|T| = pt^s$ and if $n_t$ is the numbers of Sylow $t$-subgroups of $T$, then $n_t = 1$ or $p$. If $n_t = p$, then
1 + tr = p, for some r > 0, which is a contradiction, since t \| (p + 1) and t is odd. We note that 
\( t^i \equiv 1 \pmod{p} \), where \( 1 \leq i \leq s \), since \( p + 1 \) is even and so \( p > t^s \). Therefore \( n_p = 1 \), where \( n_p \) is the numbers of Sylow \( p \)-subgroups of \( T \). Hence by using Sylow Theorem it follows that \( T \) is a nilpotent subgroup of \( K \) and so \( p \sim t \) in \( \Gamma(G) \), which is a contradiction, since \( \deg(p) = 0 \) in \( \Gamma(G) \). Therefore, 
\( H \) is a \( \{2\} \)-group, i.e., there exists a natural number \( \alpha \) such that \( |H| = p + 1 = 2^\alpha \) (\( \alpha \geq 3 \), since by assumption \( p = q > 5 \)). Let \( P \) be a Sylow \( p \)-subgroup of \( G \). Since \( \Phi(H) \lneq G \), if \( \Phi(H) = \{1\} \), then \( |\Phi(H)| = 1 \pmod{p} \).

Since \( |\Phi(H)| \neq p + 1 \), \( \Phi(H) \cap C_G(P) = \{1\} \), which is a contradiction, since \( \deg(p) = 0 \) in \( \Gamma(G) \).

Hence \( \Phi(H) = \{1\} \) and so \( H \) is an elementary abelian \( 2 \)-group. Let \( F = GF(2^\alpha) \) and so \( H \) is the additive group of \( F \). Also \( |P| = p = 2^\alpha - 1 \) and so \( P \) is the multiplicative group of \( F \). Now \( G / K \) acts by conjugation on \( H \) and similarly \( G / K \) acts by conjugation on \( P \) and this action is faithful. Then \( G / K \) keeps the structure of the field \( F \) and so \( G / K \) is isomorphic to a subgroup of the automorphism group of \( F \). Hence \( |G / K| = 2^{2\alpha} - 2 \leq |Aut(F)| = \alpha \), which is impossible, since \( \alpha \geq 3 \). Therefore, \( G \) is not a 2-Frobenius group.

**Step 2.** By Lemma 2.2, there exists a nonabelian simple group \( S \) such that \( S \leq G / N \leq Aut(S) \) where \( N \) is a nilpotent subgroup of \( G \). Also by Lemma 2.2, since \( G / S \) is a \( \pi_1 \)-group and \( \deg(p) = 0 \) in \( \Gamma(G) \), we conclude that \( \{p\} \) is an odd component of \( \Gamma(S) \) and \( |S| = qm \), where \( m \mid (q^2 - 1) \). All of the nonabelian simple groups with at least two connected components are given in [11, Tables 1a, 2b and 2c]. Now we must consider each possibility separately. For convenience we omit the details of the proof and only state a few of them. We remark that in these tables, \( p' \in P \setminus \{2\} \), \( q' \) is a prime power and \( n' \in N \).

**Case 1.** Let \( S = A_{n'} \), where \( 6 < n' = p' \), \( p'+1 \) or \( p'+2 \); \( n' \) or \( n' - 2 \) is not prime. By using [11, Table 1a], we have \( p' = q \), since the odd component of \( \Gamma(A_{n'}) \) is \( \{p'\} \). As we mentioned above we have \( (p'-1)|p^{n-1} - 1 \), which is a contradiction, since in this case \( p' \geq 7 \).

**Case 2.** Let \( S = A_{n'} \), where \( 6 < n' = p' \), \( p' - 2 \) are primes. By using [11, Table 2b], we have \( p' = q \) or \( p' - 2 = q \), since the odd component of \( \Gamma(A_{n'}) \) are \( \{p'\} \) and \( \{p' - 2\} \). Then we must have \( \alpha(p'-1)(p'-3)! \) divides \( p'^2 - 1 \) or \( (p' - 2)^2 - 1 \), where \( \alpha = p' \) if \( p' - 2 = q \) and \( \alpha = p' - 2 \) if \( p' = q \), which is a contradiction.

**Case 3.** Let \( S = A_{p-1}(q') \), where \( (p',q') \neq (3,2),(3,4) \). We know that \( m \mid q^2 - 1 \), where \( |S| = qm \). By using [11, Table 1a], we have

\[
q = (q^{-p'} - 1) / (q' - 1)(p'q' - 1).
\]

We can easily see that \( q^2 - 1 < q^{2p'} \), which is a contradiction, since \( q^{(p'-1)/2} \mid (q^2 - 1) \) and \( q^{(p'-1)/2} \geq q^{2p'} \).

**Case 4.** Let \( S = G_2(q') \) be a Chevalley group, where \( q' \equiv 0 \pmod{3} \). By using [11, Table 2b], we have \( q^2 - q' + 1 = q \) or \( q^2 + q' + 1 = q \), since the odd components of \( \Gamma(S) \) are \( \pi(q^2 - q' + 1) \) or \( \pi(q^2 + q' + 1) \). Let \( q^2 + \beta q' + 1 = q \), where \( \beta = 1 \) or \( -1 \). Then \( q^{\beta} \mid (q^2 - 1) = ((q^2 + \beta q' + 1)^2 - 1) \). We can easily see that \( (q^2 \pm q' + 1)^2 - 1 < q^d \), which is a contradiction.
Similarly, we can prove that $S$ is not isomorphic to all other simple groups in Tables in [11], except $A_4(q')$.

**Case 5.** Let $S = A_4(q')$. If $q'$ is even and $q' > 2$, then by [11, Table 2b], the odd components of $\Gamma(S)$ are $\pi(q' - 1)$ or $\pi(q' + 1)$. If $\pi(q' - 1) = \{p\}$, then $q' - 1 = q$ and so $(q' + 1) \mid ((q' - 1)^2 - 1)$, which is impossible. If $\pi(q' + 1) = \{p\}$, then $q' + 1 = q$ and so $(q' - 1) \mid ((q' + 1)^2 - 1)$. So we have $q' = 4$ and $q = 5$, which is impossible, since $q > 5$. Hence $q'$ is not even.

Therefore $3 < q' \equiv \varepsilon \pmod{4}$, where $\varepsilon = 1$ or $-1$. By [11, Table 2b], the odd components of $\Gamma(S)$ are $\pi(q')$ and $\pi((q' + \varepsilon)/2)$. If $\pi((q' + \varepsilon)/2) = \{p\}$, then $(q' + \varepsilon)/2 = q$ and so $q' \mid (((q' + \varepsilon)/2)^2 - 1)$, which is a contradiction, since $q' = 3$. So we conclude that $\pi(q') = \{p\}$ and $q' = q$. Therefore we have $S = A_4(q) = L_2(q)$.

This argument shows that $L_2(q) \leq G/N \leq Aut(L_2(q))$ and so $|N| = 1$ or $2$. If $|N| = 2$, then we have $N \leq Z(G)$, which is a contradiction since $deg(p) = 0$ in $\Gamma(G)$. Therefore $|N| = 1$ and $L_2(q) \leq G \leq Aut(L_2(q))$. □

**COROLLARY 3.2.** The finite group $L_2(q)$ is OD-characterizable.

**Proof.** Let $G$ be a finite group such that $|G| = |L_2(q)|$ and $D(G) = D(L_2(q))$. We know that $|L_2(q)| = q(q^2 - 1)/d$, where $d = (2, q - 1)$. By using [11, Table 2b], we have $deg(p) = 0$ in $\Gamma(L_2(q))$.

In [5, Theorem 1.4] it is proved that if $q$ is even, then $G = L_2(q)$. Therefore, let $q$ be odd. So $|G| = q(q - 1)/2$ and $deg(p) = 0$ in $\Gamma(G)$. By using Theorem 3.1, we have $L_2(q) \leq G \leq Aut(L_2(q))$. On the other hand $|G| = |L_2(q)|$ and so $G = L_2(q)$. □

**THEOREM 3.3.** The finite group $PGL(2,q)$ is OD-characterizable.

**Proof.** Let $G$ be a finite group such that $|G| = |PGL(2,q)|$ and $D(G) = D(PGL(2,q))$. If $q$ is even, then $PGL(2,q) = L_2(q)$ and by Corollary 3.2, we have $G = PGL(2,q)$. Therefore, let $q$ be odd. We know that $|PGL(2,q)| = q(q^2 - 1)$ and $\mu(PGL(2,q)) = \{q - 1, p, q + 1\}$. Hence $deg(p) = 0$ and $deg(2) = |\pi(G)| - 2$ in $\Gamma(G)$.

By using Theorem 3.1, we have $L_2(q) \leq G \leq Aut(L_2(q))$. Thus $G$ is an extension of $L_2(q)$ by an involution, since $|G| = 2|L_2(q)|$. We know that $|Out(L_2(p^n))| = 2n$. In fact every element of $Out(L_2(p^n))$ is a product of a field automorphism and a diagonal automorphism. Let $\varphi \in G/L_2(q)$. If $\varphi$ is a field automorphism of order $2$, then $\varphi$ centralizes $L_2(p)$ and so $2 \sim p$ in $\Gamma(G)$, which is a contradiction. If $\varphi$ is a field-diagonal automorphism of order $2$, then $\Gamma(L_2(q)) = \Gamma(G)$ (see [3]), which is impossible, since in $\Gamma(L_2(q))$ we have $deg(2) < |\pi(L_2(q))| - 2$. Therefore $\varphi$ is a diagonal automorphism of $L_2(q)$ and so $G = PGL(2,q)$.

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