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PROJECTIVITY OF $L^p(G,\omega)$ AS A $L^1(G)$-MODULE

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Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$. We show that projectivity of $L^p(G,\omega)$ as a Banach left $L^1(G)$-module is stable with respect to ω. In fact we prove that $L^p(G,\omega)$ is a projective $L^1(G)$-module if and only if G is compact.

Key words: Projective $L^1(G)$-module, weight function, weighted L^p-space.

1. INTRODUCTION

Let A be a Banach algebra, and denote by $A^{\#}$ the algebra formed by adjoining an identity, called e,to A as in [4, Definition 1.3.3]. We denote by $A-\text{mod}$, the category of all Banach left $A-$modules. We recall the definitions that we use in the present paper; for full details and proofs, see [5] and [12].

Suppose that $E \in A-\text{mod}$. Set

$$A.E = \{a.x,\ \ a \in A, x \in E\}$$

and let AE be the linear span of the set $A.E.$ Then E is called essential if $\overline{AE} = E.$ In the case where A has a bounded approximate identity, then $\overline{AE} = A.E$ [4, Corollary 2.9.26]. For Banach spaces E and $F,$ we denote by $B(E,F),$ the Banach space of all bounded linear operators from E to F. Let $T \in B(E,F)$. Then T is called admissible if there exists $S \in B(F,E)$ with $T \circ S \circ T = T.$ If $E,F \in A-\text{mod}$, then ${}_AB(E,F)$ is the closed linear subspace of $B(E,F),$ consisting of all left $A-$module morphisms. Let $T \in {}_AB(E,F).$ Then T is called a retraction if there exists $S \in {}_AB(F,E)$ with $T \circ S = I_F.$ Let $P \in A-\text{mod}$. Then P is called a projective $A-$module if, for all $E,F \in A-\text{mod}$ and each admissible epimorphism $T \in {}_AB(E,F)$ and for each $S \in {}_AB(P,F),$ there exists $R \in {}_AB(P,E)$ with $T \circ R = S.$ In our main results, we shall use the following proposition from [12, IV.1.1, IV.1.2, and IV.4.5] and also [16]. First, we recall the concept of approximation property (AP). In fact a Banach space is said to have the approximation property (AP), if every compact operator is a limit of finite-rank operators.

PROPOSITION 1.1. *Let A be a Banach algebra. Suppose that $E \in A-mod$ is projective and that at least one of the Banach spaces A and E has the approximation property. Then, for each $x \in E,$ such that $x \neq 0,$ there exists $T \in {}_AB(E,A^{\#})$ such that $T(x) \neq 0.$ In the case where E is essential, we may suppose that $T \in {}_AB(E,A).$*

2. PRELIMINARIES

Throughout the paper G is a locally compact group, all integrals are taken with respect to a fixed left Haar measure λ and $1 < p < \infty$. We call any strictly positive Borel measurable function ω on G a weight function and it is called locally integrable if $\omega \in L^1(A)$, for each compact subset A of G with $\lambda(A) > 0$. Two weight functions ω_1 and ω_2 on G are equivalent if there exist the scalars $\alpha, \beta > 0$ such that, locally almost everywhere on G,

$$\alpha \leq \frac{\omega_1(x)}{\omega_2(x)} \leq \beta.$$

The space $L^p(G,\omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [13]. Then $L^p(G,\omega)$ is a Banach space with the norm $\|.\|_{p,\omega}$, defined by $\|f\|_{p,\omega} = \|f\omega\|_p$, where $f \in L^p(G,\omega)$. The dual space of $L^p(G,\omega)$ is the Banach space $L^q(G,\frac{1}{\omega})$, consisting of all complex valued measurable functions g on G with $\frac{g}{\omega} \in L^q(G)$, under the duality

$$\langle f, g\rangle = \int_G f(x)g(x)\mathrm{d}\lambda(x) \quad f \in L^p(G,\omega), g \in (L^q(G,\frac{1}{\omega})),$$

where q is the exponential conjugate of p, defined by $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in G$, define the function θ_x on G by $\theta_x(y) = \frac{\omega(xy)}{\omega(y)}$, for all $y \in G$. We say that ω is of left moderate growth if for each $x \in G$, $\theta_x \in L^\infty(G)$. In fact

$$L(x) = \|\theta_x\|_\infty = ess\sup_{y\in G} \theta_x(y) = ess\sup_{y\in G} \frac{\omega(xy)}{\omega(y)} < \infty. \tag{1}$$

We refer to [13] for the definition of $L^\infty(G)$ and $\|.\|_\infty$. Immediate calculations show that if ω is of left moderate growth then $L(x) > 0$ and also $L(xy) \leq L(x)L(y)$, for all $x, y \in G$. In other words, L is a submultiplicative weight function on G. Moreover

$$L(x^{-1}) = \sup_{f\neq 0} \frac{\|{}_x f\|_{p,\omega}}{\|f\|_{p,\omega}} \quad (x \in G), \tag{2}$$

where ${}_x f(y) = f(xy)$, for all $y \in G$ [15]. Also by [3], if ω is of left moderate growth then ${}_x f \in L^p(G,\omega)$, for all $f \in L^p(G,\omega)$ and $x \in G$.

For measurable functions f and g on G, the convolution multiplication

$$f * g(x) = \int_G f(y)g(y^{-1}x)\mathrm{d}\lambda(y)$$

is defined at each point $x \in G$, for which this makes sense; *i.e.* the function $y \mapsto f(y)g(y^{-1}x)$ is λ-integrable.

Many authors have studied a lot of subjects related to the L^p-spaces and also the weighted case. It should be noted that we found many valuable results in the earlier publications. We only cite to some of

them such as [7, 8, 9, 10, 11]. We also refer to [1] and [15] as two recent works, which we will use some of their results in the present work.

In [1], it is shown that $L^p(G,\omega)$ is a Segal algebra (with an equivalent norm to its first norm) if and only if G is compact and ω is equivalent to a continuous function. Precisely, it has been pointed out in [1]. Corollary 1.4] that $L^p(G,\omega)$ is a left ideal in $L^1(G)$ if and only if $L^p(G,\omega)$ is a Segal algebra (with an equivalent norm). In fact [2, Theorem 2.3] has been used.

In [5, Theorem 5.1], it is proved that $L^p(G)$ is a projective $L^1(G)$-module if and only if G is compact. The aim of the present paper is to study projectivity of $L^p(G,\omega)$, for an arbitrary weight function ω, whenever $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. We first provide some valuable results on the weight functions, under which $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. Then as the main result of the present work, we prove that $L^p(G,\omega)$ is a projective $L^1(G)$-module if and only if G is compact.

3. MAIN RESULTS

Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$, such that $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. Then by [1, Proposition 1.1], ω is of left moderate growth. We use some arguments similar to [15, Lemma 2.1] to show that ω is also locally integrable. Suppose that A is a compact subset of G with $\lambda(A) > 0$. Consider the functions $\varphi = \frac{\chi_A}{\max\{1,\omega\}}$ and $\psi = \frac{\chi_{A^{-1}.A}}{\max\{1,\omega\}}$, where χ_A and $\chi_{A^{-1}.A}$ denote the characteristic functions of the sets A and $A^{-1}.A$, respectively. As $\varphi \in L^1(G)$ and $\psi \in L^p(G,\omega)$, then by the hypothesis, $f = \varphi * \psi \in L^p(G,\omega)$. At the same time $\varphi \in L^1(G)$ and $\psi \in L^\infty(G)$, so that f is continuous. Also since for each $x \in A$, $f(x) > 0$, we have $\min_A f = f_0 > 0$. Thus $\chi_A \le \frac{f}{f_0}$, which implies $\chi_A \in L^p(G,\omega)$ or, equivalently, $\omega \in L^p(A)$.

Consequently ω is locally integrable. Thus if $L^p(G,\omega)$ is a Banach left $L^1(G)-$module, then ω is of left moderate growth and locally integrable. Now [9, Theorem 2.7] implies that ω is equivalent to a continuous weight function ω_1. It follows that $L^p(G,\omega) = L^p(G,\omega_1)$, as two Banach spaces and with equivalent norms. It follows that $C_{00}(G)$ is a dense subspace of $L^p(G,\omega)$, where $C_{00}(G)$ is the space of all complex valued continuous functions on G with compact support.

Before stating the main result of this paper, let us give some main properties of the weight functions under which, $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. We commence with the following lemma which is very applicable in our arguments.

LEMMA 3.1. *Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$, such that $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. Then $L^p(G,\omega)$ is essential.*

Proof. By the explanations preceding the lemma, we may assume that ω is continuous. Consider the left approximate identity of $L^1(G)$, *i.e.* the net $e_V = \frac{\chi_V}{\lambda(V)}$, where V runs over the net of all relatively compact symmetric neighborhoods of the identity element of G. Now we use some arguments exactly

similar to the proofs of Lemma 4.1 and Theorem 4.1 in [15], to show that for each $f \in L^p(G,\omega)$, $\|e_V * f - f\|_{p,\omega} \to 0$. Let $f \in L^p(G,\omega)$ be arbitrary. Then for any $\varepsilon > 0$ there exists a compactly supported function $\varphi \in L^p(G,\omega)$, with $\|f - \varphi\|_{p,\omega} < \varepsilon$. Let $F = \sup p(\varphi) = \overline{\{x \in G : \varphi(x) \neq 0\}}$. Pick a symmetric compact neighborhood of identity U. For each $t \in U$, we have $\sup p_t\varphi \subseteq UF$. Since ω is continuous, it is bounded on every compact set and so $C = \sup_{x \in UF} \omega(x) < \infty$. Now for each $t \in$ U

$$\|\varphi - {}_t\varphi\|_{p,\omega} = \left(\int_{UF} |\varphi(x) - \varphi(tx)|^p \omega(x)^p \, d\lambda(x) \right)^{\frac{1}{p}} \leq C \|\varphi - {}_t\varphi\|_p.$$

By [13, Theorem 20.4] the latter norm is less than $\varepsilon > 0$, for t in a sufficiently small neighborhood of identity $V \subseteq U$. Since the function L defined in (1) is submultiplicative, then [3, Proposition 1.16] implies that L is bounded on every compact subset of G and so $D = \sup_{x \in U} L(x) < \infty$. Thus for each $t \in U$ we have

$$\|{}_t f - {}_t\varphi\|_{p,\omega} = \|{}_t(f - \varphi)\|_{p,\omega} \leq L(t^{-1}) \|f - \varphi\|_{p,\omega} \leq D \|f - \varphi\|_{p,\omega} < D\varepsilon,$$

and so for t in a sufficiently small neighborhood of identity $V \subseteq U$ we have

$$\|f - {}_t f\|_{p,\omega} \leq \|f - \varphi\|_{p,\omega} + \|\varphi - {}_t\varphi\|_{p,\omega} + \|{}_t(f - \varphi)\|_{p,\omega} < (1 + D + C)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, thus for each $f \in L^p(G,\omega)$, there exists a symmetric relatively compact neighborhood of identity V_0 such that $\|f - {}_t f\|_{p,\omega} < \varepsilon$, for each $t \in V_0$. Now for every $g \in L^q(G, \frac{1}{\omega})$ and $V \subseteq V_0$ we have

$$\begin{aligned} |\langle e_V * f - f, g \rangle| &\leq \int_G \int_V \frac{|f(t^{-1}x) - f(x)|}{\lambda(V)} \mathrm{d}\lambda(t) |g(x)| \mathrm{d}\lambda(x) \leq \\ &\leq \frac{1}{\lambda(V)} \int_V \langle |{}_{t^{-1}} f - f|, |g| \rangle \mathrm{d}\lambda(t) \leq \\ &\leq \sup_{t \in V_0} \|{}_{t^{-1}} f - f\|_{p,\omega} \|g\|_{q,\frac{1}{\omega}} \leq \varepsilon \|g\|_{q,\frac{1}{\omega}}. \end{aligned}$$

It follows that $\|e_V * f - f\|_{p,\omega} < \varepsilon$. Thus $\|e_V * f - f\|_{p,\omega} \to 0$. Therefore $\overline{L^1(G)L^p(G,\omega)} = L^p(G,\omega)$, which implies that $L^p(G,\omega)$ is an essential Banach left $L^1(G)$ -module.

LEMMA 3.2. *Let G be a locally compact group and ω be a weight function on G such that $L^p(G,\omega)$ is a Banach left $L^1(G)-$module. Then there exists $M > 0$ such that for each $x \in G$ and nonzero function $f \in L^p(G,\omega)$,*

$$\frac{1}{M} \|f\|_{p,\omega} \leq \|{}_x f\|_{p,\omega} \leq M \|f\|_{p,\omega}.$$

Moreover the weight function L, defined in (1) is bounded and bounded away from zero on G.

Proof. By the hypothesis, there exists $M > 0$ such that for all $f_1 \in L^1(G)$ and $f_2 \in L^p(G,\omega)$ $\|f_1 * f_2\|_{p,\omega} \leq M \|f_1\|_1 \|f_2\|_{p,\omega}$. Now take $f \in L^p(G,\omega)$ and $x \in G$. By Lemma 3.1 and Cohen factorization theorem [14, Theorem 32.22], Notes 32.25], we have $L^1(G) * L^p(G,\omega) = L^p(G,\omega)$. Precisely, for each

$\varepsilon > 0$, there exist $g \in L^1(G)$ and $h \in L^p(G,\omega)$ such that $f = g * h$ and $\|g\|_1 \le 1$ and also $\|f - h\|_{p,\omega} \le \varepsilon \|f\|_{p,\omega}$. It follows that $\|h\|_{p,\omega} \le (1+\varepsilon)\|f\|_{p,\omega}$. Also, since ω is of left moderate growth then $L^p(G,\omega)$ is left translation invariant and so ${}_x f \in L^p(G,\omega)$; see [3]. Moreover

$$\|{}_x f\|_{p,\omega} = \|{}_x g * h\|_{p,\omega} \le M\|g\|_1\|h\|_{p,\omega} \le M(1+\varepsilon)\|f\|_{p,\omega}. \tag{3}$$

Similarly for ${}_x f \in L^p(G,\omega)$, there exist $g_1 \in L^1(G)$ and $h_1 \in L^p(G,\omega)$ such that ${}_x f = g_1 * h_1$ and $\|g_1\|_1 \le 1$ and also $\|{}_x f - h_1\|_{p,\omega} \le \varepsilon \|{}_x f\|_{p,\omega}$. It follows that $\|h_1\|_{p,\omega} \le (1+\varepsilon)\|{}_x f\|_{p,\omega}$. Consequently

$$\|f\|_{p,\omega} = \|{}_{x^{-1}}({}_x f)\|_{p,\omega} = \|{}_{x^{-1}} g_1 * h_1\|_{p,\omega} \le M\|h_1\|_{p,\omega} \le M(1+\varepsilon)\|{}_x f\|_{p,\omega}. \tag{4}$$

The inequalities (3) and (4) imply that

$$\frac{1}{M(1+\varepsilon)}\|f\|_{p,\omega} \le \|{}_x f\|_{p,\omega} \le M(1+\varepsilon)\|f\|_{p,\omega}.$$

Since ε > 0 is arbitrary, it follows that

$$\frac{1}{M}\|f\|_{p,\omega} \le \|{}_x f\|_{p,\omega} \le M\|f\|_{p,\omega}.$$

Furthermore, (2) implies that the function L is bounded and bounded away from zero on G. We draw the attention of the readers to some valuable points, which are provided in the following remarks.

Remarks 3.3. Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$ such that $L^p(G,\omega)$ is a Banach left $L^1(G)-$module. By the before explanations, ω is of left moderate growth and also locally integrable and so it is equivalent to a continuous weight function ω_1 [9, Theorem 2.7]. We show that in some cases $L^p(G,\omega) = L^p(G)$, as two Banach spaces with equivalent norms. Take $f \in L^p(G,\omega)$ to be a nonzero compactly supported function.

(i) Suppose that ω is submultiplicative. We follow some arguments exactly similar to [11],Lemma 2.2] to show that ω is bounded and bounded away from zero on G. For each $x \in G$,

$$\|{}_{x^{-1}} f\|_{p,\omega} = \left(\int_G |f(y)|^p\, \omega(xy)^p \,\mathrm{d}\lambda(y)\right)^{\frac{1}{p}} \le \omega(x)\|f\|_{p,\omega}.$$

Now Lemma (3.2) implies that $\omega(x) \ge \frac{1}{M}$, for all $x \in G$. Also

$$\|{}_{x^{-1}} f\|_{p,\omega} = \left(\int_G |f(y)|^p\, \omega(xy)^p \,\mathrm{d}\lambda(y)\right)^{\frac{1}{p}} \ge \omega(x)\left(\int_{\sup p(f)} \frac{|f(y)|^p}{\omega(y^{-1})^p}\mathrm{d}\lambda(y)\right)^{\frac{1}{p}},$$

and so

$$\|{}_{x^{-1}} f\|_{p,\omega} \ge \frac{1}{K}\omega(x)\|f\|_{p,\omega},$$

where $K = \sup_{y \in \sup p(f)} \omega(y)\omega(y^{-1})$. Again note that since ω is submultiplicative so it is bounded and bounded away from zero on compacta, by [3, Proposition 1.16]. Now Lemma 3.2 and the above inequalities yield that $\frac{1}{M} \le \omega(x) \le KM$, for all $x \in G$. It follows that $L^p(G,\omega) = L^p(G)$.

(ii) Suppose that G is abelian. Since L is submultiplicative, thus it is bounded and bounded away from zero on compacta, by [3, Proposition 1.16]. Moreover ω is equivalent to continuous weight function ω_1 and so ω_1 is bounded and bounded away from zero on compacta. Thus $D = \sup_{y\in \sup p(f)} \frac{L_1(y)}{\omega_1(y)} < \infty$, where $L_1(x) = ess\sup_{y\in G} \frac{\omega_1(xy)}{\omega_1(y)}$. Since ω_1 is continuous, then for each $x \in G$

$$\left\| {}_{x^{-1}} f \right\|_{p,\omega_1} = \left(\int_G |f(y)|^p \, \omega_1(xy)^p \, \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} = \left(\int_G |f(y)|^p \, \omega_1(yx)^p \, \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} \leq$$

$$\leq \omega_1(x) \left(\int_{\sup p(f)} |f(y)|^p \, L_1(y)^p \, \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} \leq D\omega_1(x) \left\| f \right\|_{p,\omega_1}.$$

Now Lemma 3.2 implies that $\omega_1(x) \geq \frac{1}{DM}$, for each $x \in G$. Furthermore

$$\left\| {}_{x^{-1}} f \right\|_{p,\omega_1} = \left(\int_G |f(y)|^p \, \omega_1(xy)^p \, \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} = \left(\int_G |f(y)|^p \, \omega_1(yx)^p \, \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} \geq$$

$$\geq \omega_1(x) \left(\int_{\sup p(f)} \frac{|f(y)|^p}{L_1(y^{-1})^p} \mathrm{d}\lambda(y) \right)^{\frac{1}{p}} \geq \frac{1}{C} \omega_1(x) \left\| f \right\|_{p,\omega_1},$$

where $C = \sup_{y\in \sup p(f)} L_1(y^{-1})\omega_1(y)$. Again Lemma 3.2 implies that $\omega_1(x) \leq CM$, for all $x \in G$. So $\frac{1}{DM} \leq \omega_1(x) \leq CM$, $(x \in G)$. Since ω_1 and ω are equivalent, it follows that $L^p(G,\omega) = L^p(G)$.

By Remarks (3.3), in the case where G is abelian or ω is submultiplicative, $L^p(G,\omega)$ is just $L^p(G)$ and so by [5, Theorem 5.1], $L^p(G,\omega)$ is a projective $L^1(G)-$module if and only if G is compact. At the present time we are not able to determine, under which circumstances $L^p(G,\omega) = L^p(G)$. Anyway, we again present the proof of [5, Theorem 3.1], with some minor differences, for the weighted case to obtain in general, $L^p(G,\omega)$ is a projective $L^1(G)-$module if and only if G is compact.

THEOREM 3.4. *Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$ such that $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. Then $L^p(G,\omega)$ is a projective $L^1(G)$-module if and only if G is compac*t.

Proof. First suppose that G is compact. We may assume that ω is continuous. Since G is compact, then ω is bounded and also bounded away from zero on G. Consequently ω is in fact equivalent to the constant weight function 1. It follows that $L^p(G,\omega) = L^p(G)$, as two Banach spaces, with the equivalent norms. Therefore $L^p(G,\omega)$ is a projective $L^1(G)-$module, by [5, Theorem 5.1]. For the converse, suppose on the contrary that G is not compact. We follow the proof of [5, Theorem 3.1] to get a contradiction. Let V and W be compact, symmetric neighborhoods of identity element e such that $V^2 \subseteq W$. Take $f \in C_{00}(G)$ with $f(G) \in [0,1]$, with $f(e) = 1$ and also with $\mathrm{supp}(f) \subseteq V$ and $\|f\|_\infty = 1$ and $\|f\|_1 = 1$. It is clear that $f * f \neq 0$ and also $\mathrm{supp}(f * f) \subseteq W$. Since $L^1(G)$ has the approximation property (AP) [6,

Example 11, p. 245] and also by Lemma 3.1 $L^p(G,\omega)$ is essential, thus Proposition 1.1 implies that there exists $T\in_{L^1(G)} B(L^p(G,\omega),L^1(G))$ such that $T(f*f)\neq 0$.

Set $\eta=\frac{\| f*T(f)\|_1}{2}>0$ and fix $k\in\mathbb{N}$. Choose the function $g\in C_{00}(G)$ with support K, such that $\|T(f)-g\|_1<\frac{1}{k}$ and $\|f*(T(f)-g)\|_1<\eta$. It follows that $\|f*g\|_1>\eta$. Since G is not compact, thus there exist $s_1,...s_k\in G$ such that the sets $s_j(V\cup K)^2$ are pointwise disjoint for $j=1,...,k$. Indeed, set $s_1=e$ and consider the set $s_1(V\cup K)^2(V\cup K)^{-2}$ which is a compact subset of G. Since G is not compact, there exists $s_2\in G\setminus s_1(V\cup K)^2(V\cup K)^{-2}$. Thus one can choose inductively each $s_j\in G\setminus\cup_{i=1}^{j-1}s_i(V\cup K)^2(V\cup K)^{-2}$. Set $f_j=_{s_j^{-1}}f$ and $\lambda=\frac{\sum_{j=1}^{k}f_j*f}{k^{\frac{1}{p}}}$. Since $s_1V,...,s_kV$ are pairwise disjoint, we have $\left\|\sum_{j=1}^{k}f_j\right\|_\infty=1$ and $\left\|\sum_{j=1}^{k}f_j*g\right\|_1=k\|f*g\|_1$. Then

$$\|T(\lambda)\|_1=\frac{\left\|\sum_{j=1}^{k}f_j*T(f)\right\|_1}{k^{\frac{1}{p}}}\geq\frac{k\|f*g\|_1-1}{k^{\frac{1}{p}}}\geq\frac{k\eta-1}{k^{\frac{1}{p}}}.$$

On the other hand by Lemma 3.2 L is bounded from above by a constant $M>0$. Thus

$$\begin{aligned}\left\|\sum_{j=1}^{k}f_j*f\right\|_{p,\omega}^p&=\int_G\left(\sum_{j=1}^{k}f_j*f(x)\right)^p\omega(x)^p\,\mathrm{d}\lambda(x)=\\&=\int_G\sum\nolimits_{j=1}^{k}\left(f_j*f(x)\right)^p\omega(x)^p\,\mathrm{d}\lambda(x)=\\&=\sum_{j=1}^{k}\int_G\left(f*f(x)\right)^p\omega(s_jx)^p\,\mathrm{d}\lambda(x)\leq\\&\leq\sum_{j=1}^{k}L(s_j)^p\int_G(f*f(x))^p\omega(x)^p\,\mathrm{d}\lambda(x)\leq\\&\leq kM^p\int_G\left(f*f(x)\right)^p\omega(x)^p\,\mathrm{d}\lambda(x).\end{aligned}$$

Since $f\in C_{00}(G)$, it follows that $f*f\in L^p(G,\omega)$ and consequently

$$\left\|\sum_{j=1}^{k}f_j*f\right\|_{p,\omega}\leq k^{\frac{1}{p}}M\|f*f\|_{p,\omega}$$

and so $\|\lambda\|_{p,\omega}\leq M\|f*f\|_{p,\omega}$. Thus

$$\frac{k\eta-1}{k^{\frac{1}{p}}}\leq\|T(\lambda)\|_1\leq\|T\|\|\lambda\|_{p,\omega}\leq M\|T\|\|f*f\|_{p,\omega}.$$

Consequently $k^{\frac{1}{q}}\eta \le M\|T\|\|f * f\|_{p,\omega} + \frac{1}{k^{\frac{1}{p}}}$. This holds for each $k \in N$, the required contradiction.

Therefore G should be compact, as claimed.

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