THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 16, Number 2/2015, pp. 160–167

THE FIRST INTEGRAL METHOD FOR WU-ZHANG NONLINEAR SYSTEM WITH TIME-DEPENDENT COEFFICIENTS

Dumitru BALEANU[1,2], Bulent KILIC[3], Mustafa INC[3]

[1]Çankaya University, Department of Mathematics, Öğretmenler Cad. 14 06530, Balgat, Ankara, Turkey
[2]Institute of Space Sciences, Magurele, Bucharest, Romania
[3]Fırat University, Science Faculty, Department of Mathematics, 23119 Elazığ, Turkey
E-mail: dumitru@cankaya.edu.tr

The first integral method is used to construct traveling wave solutions of Wu-Zhang nonlinear dynamical system with time-dependent coefficients. We obtained different types of exact solutions by using two types of variable transformations. The method is an effective tool to construct the different types of exact solutions of nonlinear partial differential equations having real world applications.

Key words: first integral method, Wu-Zhang system, analytical solutions, traveling wave solutions.

1. INTRODUCTION

Mathematical modeling of dynamical systems is expressed by *nonlinear evolution equations*, thus, is crucial to reach the general solutions of nonlinear partial differential equations (NPDEs). The solutions of these equations provide much information about the features and structure of the nonlinear evolution equations. Hence, these equations are modeling dynamical systems that have been occurred in mechanics, physics and some other branches of science and engineering. We mention that Wu and Zhang reported three kinds of equations in order to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters having a uniform depth [1].

We recall that many efficient methods have been improved to provide more information for physicists and engineers. Most of these methods, *e.g.*, Tanh [2], G'/G expansion [3], Jacobi elliptic function [4], inverse scattering [5], Hirota bilinear [6], exp-function [7], and first integral [8], use $\xi = x \mp ct$ wave variable transformation to reduce the NPDEs to ordinary differential equations (ODEs) to acquire the solution. We recall that an accurate Legendre collocation scheme for coupled hyperbolic equations with variable coefficients was presented in [9]. A one-dimensional nonlinear coupled system of equations in the theory of thermoelasticity was investigated in [10]. The dynamics of two-layered shallow water waves with coupled KdV equations was discussed in [11].

The first integral method (FIM) has been initially presented in the literature by Feng [8], by solving Burgers-KdV equation. The prominent aspect of the FIM from other ones is that it could be successfully implemented to NPDEs and some fractional differential equations for obtaining several types of exact solutions. In recent years, many studies on this method have been made. Raslan [12] has used this method for the Fisher equation. Tascan and Bekir [13] have used this method for Cahn-Allen equation. Abbasbandy and Shirzadi [14] have investigated Benjamin Bona-Mohany equation by this method. Jafari *et al*. [15] has researched for Biswas-Milovic equation.

The rest of this paper is organized as follows. First, we present the first integral method. Second, this method is applied to the Wu-Zhang nonlinear dynamical system [16–17] and after then in the last section we give the conclusions of this paper.

2. THE FIRST INTEGRAL METHOD

The core structure of the FIM is presented below [8].

Step 1. We consider a usual NPDE in the form:

$$Q\left(q,q_t,q_x,q_{xx},q_{tt},q_{xt},\ldots\right)=0. \tag{1}$$

Then the Eq. (1) transforms to the ODE with $\xi = x \mp ct$ as

$$H\left(Q,Q^{'},Q^{''},Q^{'''}\ldots\right)=0. \tag{2}$$

Step 2. It could be taken of ODE (2) as:

$$q\left(x,t\right)=q\left(\xi\right). \tag{3}$$

Step 3. A new independent variable is presented by

$$Q\left(\xi\right)=q\left(\xi\right),\ \ G\left(\xi\right)=\frac{\partial q\left(\xi\right)}{\partial \xi}, \tag{4}$$

which leads to a new system of ODEs

$$\begin{cases} \dfrac{\partial Q\left(\xi\right)}{\partial \xi}=G\left(\xi\right) \\ \dfrac{\partial F\left(\xi\right)}{\partial \xi}=P\left(Q\left(\xi\right),G\left(\xi\right)\right) \end{cases}. \tag{5}$$

Step 4. In accordance with the qualitative theory of ODE [18], if it is possible to get the first integrals for system (5), it could be obtained immediately the solutions of system (5). The *division theorem* (DT) [19] gives us an idea how to obtain the first integrals.

3. APPLICATIONS

Here we illustrate the FIM for Wu-Zhang nonlinear dynamical system with time-dependent coefficients**.** Let's consider the Wu-Zhang system with time-dependent coefficients:

$$q_t+a(t)qq_x+b(t)p_x=0 \tag{6}$$

$$p_t+c\left(t\right)(pq)_x+d(t)q_{xxx}=0. \tag{7}$$

Equations (6) and (7) turn out to the following ODEs by using $q\left(x,t\right)=Q\left(\xi\right)$, $p\left(x,t\right)=P\left(\xi\right)$ and $\xi=x-\lambda(t)$:

$$f(t)Q_\xi+a(t)QQ_\xi+b(t)P_\xi=0 \tag{8}$$

$$f(t)P_\xi+c\left(t\right)(QP_\xi+PQ_\xi)+d(t)Q_{\xi\xi\xi}=0, \tag{9}$$

where $f\left(t\right)=\dfrac{\partial\lambda(t)}{\partial t}$. Equations (6) and (7) are integrated once and by replacing Eq. (8) into Eq. (9) we get

$$\left(\frac{mc\left(t\right)}{b\left(t\right)}-\frac{f\left(t\right)^2}{b(t)}\right)Q+\left(\frac{a(t)f\left(t\right)}{2b\left(t\right)}-\frac{c(t)f\left(t\right)}{b(t)}\right)Q^2-\frac{a(t)c\left(t\right)}{2b\left(t\right)}Q^3+d\left(t\right)Q_{\xi\xi}=n+\frac{mf\left(t\right)}{b\left(t\right)}, \tag{10}$$

where n and m are arbitrary constants of integration.

Then, using (3) and (4) we have

$$\begin{cases} \dot{Q}(\xi)=G(\xi) \\ \dot{G}(\xi)=\dfrac{1}{d(t)}\left(n+\dfrac{mf(t)}{b(t)}-\left(\dfrac{mc(t)}{b(t)}-\dfrac{f(t)^2}{b(t)}\right)Q(\xi)-\left(\dfrac{a(t)f(t)}{2b(t)}-\dfrac{c(t)f(t)}{b(t)}\right)Q^2(\xi)+\dfrac{a(t)c(t)}{2b(t)}Q^3(\xi)\right). \end{cases} \tag{11}$$

In accordance with the FIM, it is supposed that $Q(\xi)$ and $G(\xi)$ are non-trivial solutions of system (11) and $F(Q,G)=\sum_{i=0}^{r}a_i(Q)G^i$ is an irreducible function in $C[Q,G]$ such that

$$F(Q(\xi),G(\xi))=\sum_{i=0}^{r}a_i(Q)G^i=0, \tag{12}$$

where $a_r(Q)\neq 0$. Equation (12) is the first integral for system (12); owing to the DT, there exists $g(Q)+h(Q)G$ in $C[Q,G]$ such that:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}\xi}&=\frac{\mathrm{d}F}{\mathrm{d}Q}\frac{dQ}{d\xi}+\frac{\mathrm{d}F}{\mathrm{d}G}\frac{\mathrm{d}G}{\mathrm{d}\xi}= \\ &=\left[g(Q)+h(Q)G\right]\sum_{i=0}^{r}a_i(Q)G^i. \end{aligned} \tag{13}$$

In this study, we consider $r=1$ and $r=2$ cases in Eq. (12).

Case 1.1. If we equate the coefficients of $G^i\,(i=0,1,\ldots,r)$ on both sides of Eq. (12) for $r=1$, we have

$$\dot{a}_1(X)=h(Q)a_1(Q) \tag{14}$$

$$\dot{a}_0(X)=a_1(Q)g(Q)+h(Q)a_0(Q) \tag{15}$$

$$a_0(Q)g(Q)=a_1(Q)\left(a\lambda Q(\xi)-(b+c)Q^3(\xi)+n\right). \tag{16}$$

From Eq. (14), if we choose $a_1(Q)$ as a constant, then $h(Q)=0$. For convenience we take $a_1(Q)=1$ and from equalization of the degrees of $g(Q)$ and $a_0(Q)$ we conclude that the degree of $g(Q)$ is equal to zero. Then, we assume that $g(Q)=A_0Q+A_1$ and we obtain from Eq. (15) as follows

$$a_0(Q)=\frac{A_0}{2}Q^2+A_1Q+A_2. \tag{17}$$

Introducing $a_0(Q)$, $a_1(Q)$, and $g(Q)$ in Eq. (16) and equating the coefficients of Q^i to zero, we have:

Family 1:

$$a(t)=-2c(t),\ \alpha=n=0,\ \mathrm{d}(t)=1, A_0=\pm\frac{\mathrm{i}\sqrt{2}c(t)}{\sqrt{b(t)}},\ A_1=0,\ \lambda(t)=\pm\mathrm{C}_0\pm\int_1^t\sqrt{A_0A_2b(s)}\mathrm{d}s. \tag{18}$$

Setting (18) in (12), we have the following ordinary differential equations

$$G(\xi)=\pm\frac{\mathrm{i}\sqrt{2}c(t)}{2\sqrt{b(t)}}Q^2(\xi)\mp\frac{\mathrm{i}f^2(t)}{\sqrt{2}\sqrt{b(t)}c(t)}. \tag{19}$$

If we solve Eq. (19) by using (3) and (4), respectively, we have the analytical solutions of the Wu-Zhang system:

$$q(x,t)=\frac{\left(e^{\frac{i\sqrt{2}f(t)(x+\lambda(t))}{\sqrt{b(t)}}}+e^{2c(t)Cf(t)}\right)f(t)}{\left(e^{\frac{i\sqrt{2}f(t)(x+\lambda(t))}{\sqrt{b(t)}}}-e^{2c(t)Cf(t)}\right)c(t)},\ p(x,t)=\frac{1}{b(t)}\left(\lambda q(x,t)-\frac{a(t)}{2}q^2(x,t)\right)+m \tag{20}$$

where C is an arbitrary constant.

Family 2:

$$a(t)=4c(t),\ m=n=0,\ A_0=\pm\sqrt{\frac{a(t)c(t)}{d(t)b(t)}},\ \lambda(t)=\pm C_0\pm\int_1^t\frac{A_1a(s)+2A_1c(s)}{3A_0}ds. \tag{21}$$

Setting (23) in (12), we have the following ordinary differential equations

$$G(\xi)=\pm\frac{3A_0f(t)}{a(t)+2c(t)}Q(\xi)\mp\sqrt{\frac{a(t)c(t)}{4d(t)b(t)}}Q^2(\xi). \tag{22}$$

If we solve Eq. (22) by using (3) and (4), respectively, we have the analytical solutions of the Wu-Zhang system:

$$q(x,t)=\frac{-6\sqrt{a(t)}f(t)}{e^{6\sqrt{a(t)}\left(C-\frac{\sqrt{c(t)}(x+\lambda(t))}{2\sqrt{b(t)d(t)}(a(t)+2c(t))}\right)f(t)}-\sqrt{c(t)}\left(a(t)-2c(t)\right)},\ p(x,t)=\frac{1}{b(t)}\left(\lambda q(x,t)-\frac{a(t)}{2}q^2(x,t)\right)+m, \tag{23}$$

where C is an arbitrary constant.

Family 3:

$$a(t)=-2c(t),b(t)=-2c^2(t),\alpha=n=A_1=0,\ d(t)=1,A_0=ic(t)\sqrt{\frac{2}{b(t)}},\ A_2=\frac{f^2(t)}{Ab(t)},\lambda(t)=C_0\int_1^t\sqrt{b(s)}ds. \tag{24}$$

Setting (24) in (3.5), we have Riccati differential equations as follows

$$G(\xi)=\frac{1}{2}Q^2(\xi)-\frac{1}{2}. \tag{25}$$

If we solve Eq. (25) by using (3) and (4), we have:

$$q(x,t)=\tanh\left[x+\lambda(t)\right]\pm i\,\mathrm{sech}\left[x+\lambda(t)\right],\ p(x,t)=\frac{1}{b(t)}\left(\lambda q(x,t)-\frac{a(t)}{2}q^2(x,t)\right)+m \tag{26}$$

or

$$q(x,t)=\coth\left[x+\lambda(t)\right]\pm i\,\mathrm{csch}\left[x+\lambda(t)\right],\ p(x,t)=\frac{1}{b(t)}\left(\lambda q(x,t)-\frac{a(t)}{2}q^2(x,t)\right)+m. \tag{27}$$

Case 1.2. If we equate the coefficients of $G^i\left(i=0,1,2,\ldots,r\right)$ on both sides of Eq. (3.6) for $r=2$, we have

$$\dot{a}_2(Q)=h(Q)a_2(Q), \tag{28}$$

$$\dot{a}_1(Q) = a_2(Q)g(Q) + h(Q)a_1(Q), \tag{29}$$

$$a_1(Q)g(Q) + h(Q)a_0(Q) = \dot{a}_0(Q) + 2a_2(Q)\left(a\lambda Q(\xi) + (b+c)Q^3(\xi) + n\right), \tag{30}$$

$$a_1(Q)\dot{G} = a_0(Q)g(Q). \tag{31}$$

If $a_2(Q)$ is constant then $h(Q)=0$ from (29). For convenience, we take $a_2(Q)=1$ and from equalization of the degrees of $g(Q)$, $a_1(Q)$, and $a_2(Q)$ we conclude that the degree of $g(Q)$ is equal to one. Then, we suppose that $g(Q) = A_0Q + A_1$ and we obtain from Eq. (30) as follows

$$a_1(Q) = \frac{A_0}{2}Q^2 + A_1Q + A_2, \tag{32}$$

$$a_0(Q) = \frac{\left(-\frac{1}{2}a(t)c(t) + \frac{1}{4}A_0{}^2 b(t)d(t)\right)}{2b(t)d(t)}Q^4 + \frac{\left(A_0A_1b(t)d(t) + \frac{2}{3}\left(a(t)+2c(t)\right)\lambda(t)\right)}{2b(t)d(t)}Q^3 +$$

$$\frac{\left[\left(A_1{}^2 + A_0A_2\right)b(t)d(t) - 2\left(\lambda^2(t) - \alpha c(t)\right)\right]}{2b(t)d(t)}Q^2 - \frac{\left[2b(t)\left(2n - A_1A_2d(t)\right) + 4\alpha\lambda(t)\right]}{2b(t)d(t)}Q - k, \tag{33}$$

where k is an integration constant from Eq. (30). Replacing $a_0(Q), a_1(Q)$, and $g(Q)$ in Eq. (30), then equating the coefficients of Q^i to zero, we have:

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0. \tag{34}$$

Setting (33) and (34) in (12), we have the differential equation as follows

$$G^2(\xi) = k + \frac{2\left(nb(t) + \alpha\lambda(t)\right)}{b(t)d(t)}Q(\xi) + \frac{\left(\lambda^2(\mathrm{t}) - \alpha c(t)\right)}{b(t)d(t)}Q^2(\xi) - \frac{\lambda(t)\left(a(t)+2c(t)\right)}{3b(t)d(t)}Q^3(\xi) + \frac{a(t)c(t)}{4b(t)d(t)}Q^4(\xi). \tag{35}$$

If we solve the Eq. (28) by using (3) and (4), respectively, we have the analytical solutions of Eq. (3.1). Eq. (28) admits the following rational solution,

$$q(x,t) = \frac{12\sqrt{m}\,b(t)d(t)(a(t)+2c(t))}{\sqrt{c(t)}\left((x+\lambda(t))^2 m\,\mathrm{a}^2(t) + 4(x+\lambda(t))^2 m\,\mathrm{c}^2(t) + a(t)(4(x+\lambda(t))^2 m\,c(t) - 9b(t)d(t))\right)}, \tag{36}$$

where $p(x,t) = \frac{1}{b(t)}\left(\lambda q(x,t) - \frac{a(t)}{2}q^2(x,t)\right) + m$, $n = \frac{\sqrt{m^3c(t)}}{b(t)}$, and $\lambda(t) = -\frac{nb(t)}{m}$.

Equation (35) admits the following trigonometric, hyperbolic, cnoidal, and snoidal solutions, respectively

$$q(x,t) = \sqrt{\frac{2\left(n^2b^2(t) - m^3c(t)\right)}{m^2c^2(t)}}\sec\left[\left(x+\lambda(t)\right)\sqrt{\frac{m^3c(t) - n^2b^2(t)}{\alpha^2b(t)d(t)}}\right] \tag{37}$$

$$q(x,t) = \sqrt{\frac{m^3c(t) - n^2b(t)^2}{\alpha^2c(t)^2}}\tan\left[\left(x+\lambda(t)\right)\sqrt{\frac{n^2b(t)^2 - m^3c(t)}{2m^2b(t)d(t)}}\right] \tag{38}$$

$$q(x,t) = \sqrt{\frac{2\left(n^2b^2(t) - m^3c(t)\right)}{m^2c^2(t)}}\mathrm{sech}\left[\left(x+\lambda(t)\right)\sqrt{\frac{n^2b^2(t) - m^3c(t)}{\alpha^2b(t)d(t)}}\right] \tag{39}$$

$$q(x,t)=\sqrt{\frac{n^2b^2(t)-m^3c(t)}{\alpha^2c^2(t)}}\tanh\left[(x+\lambda(t))\sqrt{\frac{m^3c(t)-n^2b^2(t)}{2m^2b(t)d(t)}}\right] \tag{40}$$

$$q(x,t)=\sqrt{\frac{2k^2\left(n^2b^2(t)-m^3c(t)\right)}{m^2c^2(t)\left(2k^2-1\right)}}\mathrm{cn}\left[(x+\lambda(t))\sqrt{\frac{(n^2b^2(t)-m^3c(t))}{m^2b(t)d(t)(2k^2-1)}},k\right] \tag{41}$$

$$q(x,t)=\sqrt{\frac{2k^2\left(n^2b^2(t)-m^3c(t)\right)}{m^2c^2(t)\left(1+k^2\right)}}\mathrm{sn}\left[(x+\lambda(t))\sqrt{\frac{(n^2b^2(t)-m^3c(t))}{m^2b(t)d(t)(1+k^2)}},k\right]. \tag{42}$$

where k is the modulus of cn and sn, $p(x,t)=\frac{1}{b(t)}\left(\lambda q(x,t)-\frac{a(t)}{2}q^2(x,t)\right)+m$, $a(t)=-2c(t)$, and $\lambda(t)=-\frac{nb(t)}{m}$. Now, we deduce that $\deg g(Q)=0$, only. Then, we suppose that $g(Q)=A_2$. For convenience, we take $a_2(Q)=1$, and from the equalization the degrees of $g(Q)$, $a_1(Q)$, and $a_2(Q)$ we obtain from Eq. (29) as follows

$$a_0(Q)=\frac{a(t)c(t)}{4b(t)d(t)}Q^4+\frac{(a(t)+2c(t))\lambda(t)}{3b(t)d(t)}Q^3+$$

$$+\frac{\left(\alpha c(t)+\frac{1}{2}A_2{}^2b(t)d(t)-\lambda(t)^2\right)}{b(t)d(t)}Q^2+\frac{\left(b(t)\left(-2n+A_2A_1d(t)\right)-2\alpha\lambda(t)\right)}{b(t)d(t)}Q-k,a_1(Q)=A_2Q+A_1 \tag{43}$$

Replacing $a_0(Q), a_1(Q), a_2(Q)$, and $g(Q)$ in Eq. (30) then equating all the coefficients of Q^i to zero, we have

$$A_2=0,\ \ A_1=0. \tag{44}$$

Setting (43) in (12), we have the ordinary differential equation as follows

$$G^2(\xi)=k+\frac{2\left(nb(t)+m\lambda(t)\right)}{b(t)d(t)}Q(\xi)+\frac{\lambda^2(t)-\alpha c(t)}{b(t)d(t)}Q^2(\xi)-\frac{\left(a(t)+2c(t)\right)\lambda(t)}{3b(t)d(t)}Q^3(\xi)+\frac{a(t)c(t)}{4b(t)d(t)}Q^4(\xi)\cdot \tag{45}$$

It is seen that (45) is identical to (35) so we have the same solutions as (36)–(42). In the literature, in general the wave variable transformation has been used as $\xi=x-\lambda(t)t$ with the time-dependent coefficient of wave velocity. If we choose this transformation for the Wu-Zhang equation, it becomes the following ODE

$$\left(\lambda'(t)t+\lambda(t)\right)Q_\xi-a(t)QQ_\xi-b(t)P_\xi=0 \tag{46}$$

$$\left(\lambda'(t)t+\lambda(t)\right)P_\xi-c(t)(QP_\xi+PQ_\xi)-d(t)Q_{\xi\xi\xi}=0. \tag{47}$$

If we take $z(t)=\lambda'(t)t+\lambda(t)$, each side of Eqs. (6) and (7) is integrated once and then by replacing Eq. (8) into Eq. (9) we get

$$\left(\frac{mc(t)}{b(t)}-\frac{z^2(t)}{b(t)}\right)U+\left(\frac{a(t)z(t)}{2b(t)}-\frac{c(t)z(t)}{b(t)}\right)Q^2-\frac{a(t)c(t)}{2b(t)}Q^3+d(t)Q_{\xi\xi}=n+\frac{mz(t)}{b(t)}, \tag{48}$$

where n and m are the constants of integration. Then, using (3) and (4) we have

$$\begin{cases} \dot{Q}(\xi) = G(\xi) \\ \dot{G}(\xi) = \dfrac{1}{d(t)}\left(n + \dfrac{mz(t)}{b(t)} - \left(\dfrac{mc(t)}{b(t)} - \dfrac{z^2(t)}{b(t)}\right)Q(\xi) - \left(\dfrac{a(t)(t)}{2b(t)} - \dfrac{c(t)z(t)}{b(t)}\right)Q^2(\xi) + \dfrac{a(t)c(t)}{2b(t)}Q^3(\xi)\right). \end{cases} \tag{49}$$

It is seen that (10) and (11) are the same with (48) and (49), excepting the terms $f(t)$ and $z(t)$. Namely, for *Family 1*, the only changed term in all obtained coefficients in (18) is as follows

$$\lambda(t) = \mathrm{C}_0 + \int_1^t \sqrt{A_0 A_2 b(s)}\mathrm{ds} \text{ instead of } \lambda(t) = \frac{1}{t}\left(\mathrm{C}_0 + \int_1^t \sqrt{A_0 A_2 b(s)}\mathrm{ds}\right),$$

and for *Family 2* the only changed term in all obtained coefficients in (21) is as follows

$$\lambda(t) = \mathrm{C}_0 + \int_1^t \frac{A_1\left(a(t) + 2\mathrm{c}(\mathrm{t})\right)}{3A_2}\mathrm{ds} \text{ instead of } \lambda(t) = \frac{1}{t}\left(\mathrm{C}_0 + \int_1^t \frac{A_1\left(a(t) + 2\mathrm{c}(\mathrm{t})\right)}{3A_2}\mathrm{ds}\right).$$

4. CONCLUSIONS

We used the FIM for finding some new exact solutions for the Wu-Zhang nonlinear system with time-dependent coefficients. We have obtained different types of traveling wave solutions of this dynamical system. The different types of obtained solutions are denoted in terms of trigonometric, cnoidal, snoidal, and exponential functions. Some of the exact solutions reported in this paper are new, to the best of our knowledge. Consequently, the FIM is a highly effective method to construct different types of exact solutions of NPDEs and systems of NPDEs [20–29].

REFERENCES

1. T.Y. WU, J.E. ZHANG, *On modeling nonlinear long wave*, in: L.P. Cook, V. Roytbhurd, M. Tulin (Eds.), *Mathematics is for Solving Problems*, SIAM, 1996, p. 233.
2. E.J. PARKES, B.R. DUFFY, *An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations*, Computer Physics Communications, **98**, pp. 288–300, 1996.
3. M. WANG, X. LI, J. ZHANG, *The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Physics Letters A, **372**, pp. 417–423, 2008.
4. Z. FU, S. LIU, Q. ZHAO, *New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations*, Physics Letters A, **290**, pp. 72–76, 2001.
5. P.G. DRAZIN, R.S. JHONSON, *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.
6. X.B. HU, W.X. MA, *Application of Hirota's bilinear formalism to the Toeplitz lattice-some special soliton-like solutions*, Physics Letters A, **293**, pp. 161–165, 2002.
7. X.H. WU, J.H. HE, *Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method*, Computer Mathematics with Application, **54**, pp. 966–986, 2007.
8. Z.S. FENG, *The first integral method to study the Burgers-Korteweg -de Vries equation*, Journal of Physics A, **35**, pp. 343–349, 2002.
9. E.H. DOHA, A.H. BHRAWY, D. BALEANU, M.A. ABDELKAWY, *An accurate Legendre collocation scheme for coupled hyperbolic equations with variable coefficients,* Romanian Journal of Physics, **59**, pp. 408–420, 2014.
10. A. JAFARIAN, P. GHADERI, A.K. GOLMANKHANEH, D. BALEANU, *On a one-dimensional nonlinear coupled system of equations in the theory of thermoelasticity*, Romanian Journal of Physics, **58**, pp. 694–702, 2013.
11. H. TRIKI, Z. JOVANOSKI, A. BISWAS, *Dynamics of two-layered shallow water waves with coupled KdV equations*, Romanian Reports in Physics, **66**, pp. 251–261, 2014.
12. K.R. RASLAN, *The first integral method for solving some important nonlinear partial differential equations*, Nonlinear Dynamics, **53**, pp. 281–286, 2008.
13. F. TASCAN, A. BEKIR, *Travelling wave solutions of the Cahn Allen equation by using first integral method*, Applied Mathematics and Computation, **207**, pp. 279–282, 2009.
14. S. ABBASBANDY, A. SHIRZADI, *The first integral method for modified Benjamin-Bona-Mahony equation,* Communications in Nonlinear Science and Numerical Simulation, **15**, pp. 1759–1764, 2010.
15. H. JAFARI, A. SOORAKI, C.M. KHALIQUE, *Dark solitons of the Biswas–Milovic equation by the first integral method*, Optik - International Journal for Light and Electron Optics, **124**, pp. 3929–3932, 2013.
16. K. HOSSEINI, R.ANSARI, P. GHOLAMIN, *Exact solutions of some nonlinear systems of partial differential equations by using the first integral method*, Journal of Mathematical Analysis and Applications, **387**, pp. 807–814, 2012.

17. X. ZHENG, Y. CHEN, H. ZHANG, *Generalized extended tanh-function method and its application to (1+1)-dimensional dispersive long wave equation*, Physics Letters A, **311**, pp. 145–157, 2003.
18. T.R. DING, C.Z. LI, *Ordinary Differential Equations*, Peking University Press, Peking, 1996.
19. N. BOURBAKI, *Commutative Algebra*, Addison-Wesley, Paris, 1972.
20. Y. HE, S. LI, Y. LONG, *Exact Solutions of the Modified Benjamin-Bona-Mahoney (mBBM) Equation by Using the First Integral Method*, Differential Equations and Dynamical Systems, **21**, pp. 199–204, 2013.
21. A.M. WAZWAZ, *Multiple soliton solutions for two integrable couplings of the modified Korteweg-de Vries equation*, Proc. Romanian Acad. A, **14**, pp. 219–225, 2013.
22. S. KUMAR, E. ZERRAD, A. YILDIRIM, A. BISWAS, *Topological solitons and Lie symmetry analysis for the Kadomtsev-Petviashvili-Burgers equation with power law nonlinearity in dusty plasmas*, Proc. Romanian Acad. A, **14**, pp. 204–210, 2013.
23. K. YUN-QUAN, Y. JUN, T*he first integral method to study a class of reaction-diffusion equations,* Communications in Theoretical Physics, **43**, pp. 597–600, 2005.
24. B.S. AHMED, E. ZERRAD, A. BISWAS, *Kinks and domain walls of the Zakharov equation in plasmas*, Proc. Romanian Acad. A, **14**, pp. 281–286, 2013.
25. A. Bekir, O. Ünsal, *Analytic treatment of nonlinear evolution equations using first integral method*, Pramana, 79, pp. 3–17, 2012.
26. N. TAGHIZADEH, M. NAJAND, *Comparison of solutions of mKdV equation by using the first integral method and (G'/G)-expansion method*, Mathematica Aeterna, **2**, pp. 309–320, 2012.
27. ZHENGPING YANG, WEI-PING ZHONG, *Analytical solutions to sine-Gordon equation with variable coefficient*, Romanian Reports in Physics, **66**, pp. 262–273, 2014.
28. REZA ABAZARI, *General solution of a special class of a nonlinear BBM-B equation by using the (G'/G)-expansion method*, Romanian Reports in Physics, **66**, pp. 286–295, 2014.
29. H. JAFARI, A. SOORAKI, Y. TALEBIA, A. BISWAS, *The first integral method and traveling wave solutions to Davey–Stewartson equation*, Nonlinear Analysis: Modelling and Control, **17**, pp. 182–193, 2012.

Received September 30, 2014