THE FIRST INTEGRAL METHOD FOR WU-ZHANG NONLINEAR SYSTEM WITH TIME-DEPENDENT COEFFICIENTS

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The first integral method is used to construct traveling wave solutions of Wu-Zhang nonlinear dynamical system with time-dependent coefficients. We obtained different types of exact solutions by using two types of variable transformations. The method is an effective tool to construct the different types of exact solutions of nonlinear partial differential equations having real world applications.

Key words: first integral method, Wu-Zhang system, analytical solutions, traveling wave solutions.

1. INTRODUCTION

Mathematical modeling of dynamical systems is expressed by nonlinear evolution equations, thus, is crucial to reach the general solutions of nonlinear partial differential equations (NPDEs). The solutions of these equations provide much information about the features and structure of the nonlinear evolution equations. Hence, these equations are modeling dynamical systems that have been occurred in mechanics, physics and some other branches of science and engineering. We mention that Wu and Zhang reported three kinds of equations in order to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters having a uniform depth [1].

We recall that many efficient methods have been improved to provide more information for physicists and engineers. Most of these methods, e.g., Tanh [2], G'/G expansion [3], Jacobi elliptic function [4], inverse scattering [5], Hirota bilinear [6], exp-function [7], and first integral [8], use \( \xi = x \mp ct \) wave variable transformation to reduce the NPDEs to ordinary differential equations (ODEs) to acquire the solution. We recall that an accurate Legendre collocation scheme for coupled hyperbolic equations with variable coefficients was presented in [9]. A one-dimensional nonlinear coupled system of equations in the theory of thermoelasticity was investigated in [10]. The dynamics of two-layered shallow water waves with coupled KdV equations was discussed in [11].

The first integral method (FIM) has been initially presented in the literature by Feng [8], by solving Burgers-KdV equation. The prominent aspect of the FIM from other ones is that it could be successfully implemented to NPDEs and some fractional differential equations for obtaining several types of exact solutions. In recent years, many studies on this method have been made. Raslan [12] has used this method for the Fisher equation. Tascan and Bekir [13] have used this method for Cahn-Allen equation. Abbasbandy and Shirzadi [14] have investigated Benjamin Bona-Mohany equation by this method. Jafari et al. [15] has researched for Biswas-Milovic equation.

The rest of this paper is organized as follows. First, we present the first integral method. Second, this method is applied to the Wu-Zhang nonlinear dynamical system [16–17] and after then in the last section we give the conclusions of this paper.
2. THE FIRST INTEGRAL METHOD

The core structure of the FIM is presented below [8].

**Step 1.** We consider a usual NPDE in the form:

\[ Q(q, q_x, q_x x, q_{xx}, \ldots) = 0. \]  \(1\)

Then the Eq. (1) transforms to the ODE with \( \xi = x \mp ct \) as

\[ H(Q, Q', Q'', Q''' \ldots) = 0. \]  \(2\)

**Step 2.** It could be taken of ODE (2) as:

\[ q(x,t) = q(\xi). \]  \(3\)

**Step 3.** A new independent variable is presented by

\[ Q(\xi) = q(\xi), \quad G(\xi) = \frac{\partial q(\xi)}{\partial \xi}. \]  \(4\)

which leads to a new system of ODEs

\[ \begin{cases} \frac{\partial Q(\xi)}{\partial \xi} = G(\xi) \\ \frac{\partial F(\xi)}{\partial \xi} = P(Q(\xi), G(\xi)) \end{cases}. \]  \(5\)

**Step 4.** In accordance with the qualitative theory of ODE [18], if it is possible to get the first integrals for system (5), it could be obtained immediately the solutions of system (5). The division theorem (DT) [19] gives us an idea how to obtain the first integrals.

3. APPLICATIONS

Here we illustrate the FIM for Wu-Zhang nonlinear dynamical system with time-dependent coefficients. Let’s consider the Wu-Zhang system with time-dependent coefficients:

\[ q_t + a(t)qq_x + b(t)p_x = 0 \]  \(6\)

\[ p_t + c(t)(pq)_x + d(t)q_{xxx} = 0. \]  \(7\)

Equations (6) and (7) turn out to the following ODEs by using \( q(x,t) = Q(\xi), \quad p(x,t) = P(\xi) \) and \( \xi = x - \lambda(t) \):

\[ f(t)Q_{\xi} + a(t)QQ_{\xi} + b(t)P_{\xi} = 0 \]  \(8\)

\[ f(t)P_{\xi} + c(t)(QP_{\xi} + PQ_{\xi}) + d(t)Q_{\xi\xi\xi} = 0, \]  \(9\)

where \( f(t) = \frac{\partial \lambda(t)}{\partial t} \). Equations (6) and (7) are integrated once and by replacing Eq. (8) into Eq. (9) we get

\[ \left( \frac{mc(t)}{b(t)} - f(t) \right) Q' + Q^2 \left( \frac{a(t)f(t)}{2b(t)} - \frac{ctf(t)}{b(t)} \right) = \frac{a(t)c(t)}{2b(t)} Q^3 + d(t)Q_{\xi\xi} = n + \frac{mf(t)}{b(t)}, \]  \(10\)

where \( n \) and \( m \) are arbitrary constants of integration.
Then, using (3) and (4) we have
\[
\begin{align*}
\dot{G}(\xi) &= \frac{1}{d(t)} \left[ n + \frac{m f(t)}{b(t)} - \frac{mc(t) - f(t)^2}{b(t)} \right] Q(\xi) - \left( \frac{a(t)f(t)}{b(t)} - \frac{c(t)f(t)}{b(t)} \right) Q^2(\xi) + \frac{a(t)c(t)}{2b(t)} Q^3(\xi),
\end{align*}
\]
(11)

In accordance with the FIM, it is supposed that \( Q(\xi) \) and \( G(\xi) \) are non-trivial solutions of system (11) and \( F(Q,G) = \sum_{i=0}^{r} a_i(Q) G^i \) is an irreducible function in \( C[Q,G] \) such that
\[
F(Q(\xi),G(\xi)) = \sum_{i=0}^{r} a_i(Q) G^i = 0,
\]
(12)

where \( a_r(Q) \neq 0 \). Equation (12) is the first integral for system (12); owing to the DT, there exists \( g(Q) + h(Q) G \) in \( C[Q,G] \) such that:
\[
\frac{dF}{d\xi} = \frac{dF}{dQ} \frac{dQ}{d\xi} + \frac{dF}{dG} \frac{dG}{d\xi} = \left[ g(Q) + h(Q) G \right] \sum_{i=0}^{r} a_i(Q) G^i.
\]
(13)

In this study, we consider \( r = 1 \) and \( r = 2 \) cases in Eq. (12).

Case 1.1. If we equate the coefficients of \( G^i \) (\( i = 0, 1, \ldots, r \)) on both sides of Eq. (12) for \( r = 1 \), we have
\[
a_1(X) = h(Q) a_1(Q)
\]
(14)

\[
\dot{a}_0(X) = a_1(Q) g(Q) + h(Q) a_0(Q)
\]
(15)

\[
a_0(Q) g(Q) = a_1(Q) \left( a_1(Q)(\xi) - (b + c)Q^3(\xi) + n \right).
\]
(16)

From Eq. (14), if we choose \( a_1(Q) \) as a constant, then \( h(Q) = 0 \). For convenience we take \( a_1(Q) = 1 \) and from equalization of the degrees of \( g(Q) \) and \( a_0(Q) \) we conclude that the degree of \( g(Q) \) is equal to zero. Then, we assume that \( g(Q) = A_0 Q + A_1 \) and we obtain from Eq. (15) as follows
\[
a_0(Q) = \frac{A_1}{2} Q^2 + A_0 Q + A_2.
\]
(17)

Introducing \( a_0(Q) \), \( a_1(Q) \), and \( g(Q) \) in Eq. (16) and equating the coefficients of \( Q^i \) to zero, we have:

Family 1:
\[
a(t) = -2c(t), \quad \alpha = n = 0, \quad d(t) = 1, \quad A_0 = \pm \frac{i \sqrt{2} c(t)}{\sqrt{b(t)}}, \quad A_1 = 0, \quad \lambda(t) = \pm C_0 \pm \int_{1}^{t} \sqrt{A_0 A_2 b(s)} ds.
\]
(18)

Setting (18) in (12), we have the following ordinary differential equations
\[
G(\xi) = \pm \frac{i \sqrt{2} c(t)}{2 \sqrt{b(t)}} Q^2(\xi) + \frac{i f^2(t)}{\sqrt{2} \sqrt{b(t) c(t)}}.
\]
(19)
If we solve Eq. (19) by using (3) and (4), respectively, we have the analytical solutions of the Wu-Zhang system:

\[
q(x,t) = \left\{ \begin{array}{ll}
\frac{i\sqrt{f(t)}(\lambda(t)+\lambda(t))}{\sqrt{b(t)}} + e^{2e(t)c(t)} c(t) \\ 
\frac{i\sqrt{f(t)}(\lambda(t)-\lambda(t))}{\sqrt{b(t)}} - e^{2e(t)c(t)} c(t)
\end{array} \right. 
\]

where \( C \) is an arbitrary constant.

**Family 2:**

\[
a(t) = 4c(t), \quad m = n = 0, \quad A_0 = \pm \frac{\sqrt{a(t)c(t)}}{d(t)b(t)}, \quad \lambda(t) = \pm C_0 \pm \int_i A_1 a(s) + \frac{2A_1 c(s)}{3A_0} ds.
\]  

Setting (23) in (12), we have the following ordinary differential equations

\[
G(\xi) = \pm \frac{3A_0 f(t)}{a(t) + 2c(t)} Q(\xi) \mp \sqrt{\frac{a(t)c(t)}{4d(t)b(t)}} Q^2(\xi).
\]  

If we solve Eq. (22) by using (3) and (4), respectively, we have the analytical solutions of the Wu-Zhang system:

\[
q(x,t) = \frac{-6\sqrt{a(t)f(t)}}{e^{\lambda(t)}} c(t) (a(t) - 2c(t)), \quad p(x,t) = \frac{1}{b(t)} \left\{ \lambda q(x,t) - \frac{a(t)}{2} q^2(x,t) \right\} + m,
\]  

where \( C \) is an arbitrary constant.

**Family 3:**

\[
a(t) = -2c(t), \quad b(t) = -2c^2(t), \quad \alpha = n = A_1 = 0, \quad d(t) = 1, A_2 = \frac{2c(t)}{\sqrt{b(t)}}, A_1 = \frac{f^2(t)}{Ab(t)}, \quad \lambda(t) = C_0 \frac{1}{i} \sqrt{b(s)} ds.
\]  

Setting (24) in (3.5), we have Riccati differential equations as follows

\[
G(\xi) = \frac{1}{2} Q^2(\xi) - \frac{1}{2}
\]  

If we solve Eq. (25) by using (3) and (4), we have:

\[
q(x,t) = \tanh \left[ x + \lambda(t) \right] \left\{ \lambda q(x,t) - \frac{a(t)}{2} q^2(x,t) \right\} + m
\]  
or

\[
q(x,t) = \coth \left[ x + \lambda(t) \right] \left\{ \lambda q(x,t) - \frac{a(t)}{2} q^2(x,t) \right\} + m.
\]

**Case 1.2.** If we equate the coefficients of \( G^i \) \((i = 0, 1, 2, \ldots, r)\) on both sides of Eq. (3.6) for \( r = 2 \), we have

\[
\hat{a}_z(Q) = h(Q) a_z(Q),
\]  

where \( h(Q), a_z(Q) \) are constants.
\[ a_1(Q) = a_2(Q) g(Q) + h(Q) a_1(Q), \]  
\[ a_1(Q) g(Q) + h(Q) a_o(Q) = \dot{a}_o(Q) + 2 a_2(Q) \left( a \lambda Q(\xi) + (b + c) Q^2(\xi) + n \right), \]  
\[ a_1(Q) \dot{G} = a_o(Q) g(Q). \]  
If \( a_2(Q) \) is constant then \( h(Q) = 0 \) from (29). For convenience, we take \( a_2(Q) = 1 \) and from equalization of the degrees of \( g(Q), a_1(Q), \) and \( a_2(Q) \) we conclude that the degree of \( g(Q) \) is equal to one. Then, we suppose that \( g(Q) = A_o Q + A_1 \) and we obtain from Eq. (30) as follows

\[ a_1(Q) = \frac{A_o}{2} Q^2 + A_0 Q + A_2, \]  
\[ a_o(Q) = \left\{ -\frac{1}{2} a(t)c(t) + \frac{1}{4} A_o b(t)d(t) \right\} Q^4 + \left\{ \frac{A_o (a(t) + 2c(t))}{2b(t)d(t)} \right\} Q^3 + \]  
\[ \left[ \left( A_1^2 + A_0 A_2 \right) b(t)d(t) - 2 \left( \lambda^2(t) - a c(t) \right) \right] \frac{1}{2b(t)d(t)} Q^2 - \frac{2 h(t) \left( 2n - A_1 A_2 d(t) \right)}{2b(t)d(t)} Q - k, \]  
where \( k \) is an integration constant from Eq. (30). Replacing \( a_o(Q), a_1(Q), \) and \( g(Q) \) in Eq. (30), then equating the coefficients of \( Q^i \) to zero, we have:

\[ A_o = 0, \quad A_1 = 0, \quad A_2 = 0. \]  
Setting (33) and (34) in (12), we have the differential equation as follows

\[ G^2(\xi) = k + \frac{2(n b(t) + a \lambda(t))}{b(t)d(t)} Q(\xi) + \frac{\left( \lambda^2(t) - a c(t) \right)}{b(t)d(t)} Q^2(\xi) - \frac{\lambda(t)(a(t) + 2c(t))}{3b(t)d(t)} Q^3(\xi) + \frac{a(t)c(t)}{4b(t)d(t)} Q^4(\xi). \]  
If we solve the Eq. (28) by using (3) and (4), respectively, we have the analytical solutions of Eq. (3.1). Eq. (28) admits the following rational solution,

\[ q(x,t) = \frac{12 \sqrt{m b(t) d(t)} (a(t) + 2c(t))}{\sqrt{c(t)} ((x + \lambda(t))^3 m a^2(t) + 4(x + \lambda(t))^2 m c(t) + a(t)(4(x + \lambda(t))^2 m c(t) - 9 b(t) d(t)))}, \]  
where \( p(x,t) = \frac{1}{b(t)} \left( \lambda q(x,t) - a(t) \right) + \frac{a(t)}{2}, \) \( m = \frac{\sqrt{m^3 c(t)}}{b(t)} \), and \( \lambda(t) = -\frac{nb(t)}{m} \).

Equation (35) admits the following trigonometric, hyperbolic, cnoidal, and snoidal solutions, respectively

\[ q(x,t) = \sqrt{\frac{2 \left( n^2 b^2(t) - m^2 c(t) \right)}{m^2 c^2(t)}} \sec \left( (x + \lambda(t)) \sqrt{\frac{m^3 c(t) - n^2 b^2(t)}{a^2 b(t) d(t)}} \right) \]  
\[ q(x,t) = \sqrt{\frac{m^3 c(t) - n^2 b^2(t)}{a^2 c(t)^2}} \tan \left( (x + \lambda(t)) \sqrt{\frac{n^2 b^2(t) - m^3 c(t)}{2 m^2 b(t) d(t)}} \right) \]  
\[ q(x,t) = \sqrt{\frac{2 \left( n^2 b^2(t) - m^3 c(t) \right)}{m^2 c^2(t)}} \sech \left( (x + \lambda(t)) \sqrt{\frac{n^2 b^2(t) - m^3 c(t)}{a^2 b(t) d(t)}} \right) \]
where $k$ is the modulus of $cn$ and $sn$, \( p(x,t) = \frac{1}{b(t)} \left( \lambda q(x,t) - \frac{a(t)}{2} q^2(x,t) \right) + m, a(t) = -2c(t), \) and 
\( \lambda(t) = -\frac{nb(t)}{m}. \) Now, we deduce that \( \deg g(Q) = 0, \) only. Then, we suppose that \( g(Q) = A_2. \) For convenience, we take \( a_2(Q) = 1, \) and from the equalization the degrees of \( g(Q), a_1(Q), \) and \( a_2(Q) \) we obtain from Eq. (29) as follows

\[
a_0(Q) = \frac{a(t)c(t)}{4b(t)d(t)}Q^4 + \frac{a(t)+2c(t)\lambda(t)}{3b(t)d(t)}Q^3 + \frac{\alpha c(t) + \frac{1}{2} A_2 b(t)d(t) - \lambda(t)}{b(t)d(t)}Q^2 + \frac{b(t)(-2n + A_2 \lambda d(t)) - 2a\lambda(t)}{b(t)d(t)} Q - k, a_1(Q) = A_2 Q + A_i.
\]

Replacing \( a_0(Q), a_1(Q), a_2(Q), \) and \( g(Q) \) in Eq. (30) then equating all the coefficients of \( Q' \) to zero, we have

\[
A_2 = 0, \quad A_i = 0.
\]

Setting (43) in (12), we have the ordinary differential equation as follows

\[
G^2(\xi) = k + \frac{2nb(t) + m\lambda(t)}{b(t)d(t)} Q^4(\xi) + \frac{\lambda^2(t) - ac(t)}{b(t)d(t)} Q^3(\xi) - \frac{(a(t)+2c(t)\lambda(t))}{3b(t)d(t)} Q^2(\xi) + \frac{a(t)c(t)}{4b(t)d(t)} Q(\xi).
\]

It is seen that (45) is identical to (35) so we have the same solutions as (36)–(42). In the literature, in general the wave variable transformation has been used as \( \xi = x - \lambda(t)t \) with the time-dependent coefficient of wave velocity. If we choose this transformation for the Wu-Zhang equation, it becomes the following ODE

\[
\left( \lambda(t)t + \lambda(t) \right) Q_z - a(t)QQ_z - b(t)P_z = 0
\]

\[
\left( \lambda(t)t + \lambda(t) \right) P_z - c(t)(QP_z + PQ_z) - d(t)Q_{zzz} = 0.
\]

If we take \( z(t) = \frac{\lambda(t)}{a(t)} Q \), each side of Eqs. (6) and (7) is integrated once and then by replacing Eq. (8) into Eq. (9) we get

\[
\left( \frac{mc(t)}{b(t)} - \frac{z^2(t)}{b(t)} \right) U + \left( \frac{a(t)z(t)}{2b(t)} - \frac{c(t)z(t)}{b(t)} \right) Q^2 - \frac{a(t)c(t)}{2b(t)} Q + d(t)Q_{zz} = n + \frac{mz(t)}{b(t)},
\]

where $n$ and $m$ are the constants of integration. Then, using (3) and (4) we have
It is seen that (10) and (11) are the same with (48) and (49), excepting the terms \( f(t) \) and \( z(t) \). Namely, for Family 1, the only changed term in all obtained coefficients in (18) is as follows

\[
\lambda(t) = C_0 + \int A_1 A_2 b(s) ds
\]

instead of \( \lambda(t) = \frac{1}{t} \left( C_0 + \int A_1 A_2 b(s) ds \right) \).

and for Family 2 the only changed term in all obtained coefficients in (21) is as follows

\[
\lambda(t) = C_0 + \int A_1 (a(t) + 2c(t)) \frac{1}{3A_2} ds
\]

instead of \( \lambda(t) = \frac{1}{t} \left( C_0 + \int A_1 (a(t) + 2c(t)) \frac{1}{3A_2} ds \right) \).

4. CONCLUSIONS

We used the FIM for finding some new exact solutions for the Wu-Zhang nonlinear system with time-dependent coefficients. We have obtained different types of traveling wave solutions of this dynamical system. The different types of obtained solutions are denoted in terms of trigonometric, cnoidal, snoidal, and exponential functions. Some of the exact solutions reported in this paper are new, to the best of our knowledge. Consequently, the FIM is a highly effective method to construct different types of exact solutions of NPDEs and systems of NPDEs [20–29].

REFERENCES


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