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QUASIRECOGNITION BY PRIME GRAPH OF THE SIMPLE GROUP $B_n(9)$

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In this paper as the main result, we show that if G is a finite group such that the prime graph of G is equal to the prime graph of $B_n(9)$ where $n \geq 8$, then G has a unique nonabelian composition factor which is isomorphic to $B_n(9)$ *or* $C_n(9)$.

Key words: prime graph, simple group, recognition, quasirecognition.

1. INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes p and p' are joined by an edge if and only if G has an element of order pp'. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G), \pi_2(G), \ldots, \pi_s(G)$ be the connected components of $\Gamma(G)$. Sometimes we use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$ we always suppose $2 \in \pi_1(G)$. Let m and n be natural numbers. We write $m \sim n$ if and only if for every prime divisors $r \in \pi(m)$ and $s \in \pi(n)$, r is adjacent to s in $\Gamma(G)$. The *spectrum* of a finite group G, which is denoted by $\pi_e(G)$, is the set of its element orders. A subset X of the vertices of a graph is called an independent set if the induced subgraph on X has no edge. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$, some independent set of vertices in $\Gamma(G)$ with the maximal number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is denoted by $\rho(r,G)$. Let $t(G) = |\rho(G)|$ and $t(r,G) = |\rho(r,G)|$.

A finite nonabelian simple group P is called quasirecognizable by *prime graph* (resp. by *spectrum*), if every finite group G with $\Gamma(G) = \Gamma(P)$ (resp. $\pi_e(G) = \pi_e(P)$) has a unique composition factor isomorphic to P. We denote by $k(\Gamma(G))$ (resp. $h(\pi_e(G))$) the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$ (resp. $\pi_e(G) = \pi_e(H)$). Given a natural number r, a finite group G is called r-*recognizable* by *prime graph* (resp. *by spectrum*) if $k(\Gamma(G)) = r$ (resp. $h(\pi_e(G)) = r$) and *unrecognizable* if $k(\Gamma(G))$ (resp. $h(\pi_e(G))$) is infinite. Usually a 1-*recognizable* group by prime graph (resp. by spectrum) is called a *recognizable* group by prime graph (resp. by spectrum).

In [1, 8], it is proved that if p and $k > 1$ are odd and $q = p^k$ is a prime power, then $PGL(2,q)$ is recognizable by prime graph. In [9, 10, 13, 14] finite groups with the same prime graph as $L_n(2)$ and $U_n(2)$ are obtained. In [3], it is proved that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(5))$ where

$n \geq 6$, then G has a unique nonabelian composition factor isomorphic to $B_n(5)$ or $C_n(5)$, see also [2, 11, 12, 15] and the references of [3].

Shi and Tang proved that $h(\pi_e(B_3(3))) = 2$ in [22]. Lipschutz and Shi proved that the group $B_2(3)$ is unrecognizable by spectrum in [16]. Shi *et al*. in [21], proved that if $p > 3$ is an odd prime and G is a finite group such that $\pi_e(G) = \pi_e(B_p(3))$, then $G \cong B_p(3)$. Then the authors in [19], proved that if G is a finite group such that $\Gamma(G) = \Gamma(B_p(3))$, where $p > 3$ is an odd prime, then $G \cong B_p(3)$ or $C_p(3)$. Also the authors in [20], proved that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(3))$, where $n \geq 6$, then G has a unique nonabelian composition factor isomorphic to $B_n(3)$ or $C_n(3)$. In this paper, we prove that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(9))$, where $n \geq 8$, then G has a unique nonabelian composition factor isomorphic to $B_n(9)$ or $C_n(9)$. For the proof of the main theorem, we use the classification of finite simple groups. Throughout this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [4].

2. PRELIMINARY RESULTS

LEMMA 2.1 ([24, Theorem 1]). *Let G be a finite group with $t(G) \geq 3$ and $t(2,G) \geq 2$. Then the following hold:*

(1) *There exists a finite nonabelian* simple *group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal solvable subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following holds:*

(a) *Every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$; in particular, $t(2,S) \geq t(2,G)$.*

(b) *There exists a prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2,G) = 2$, and $S \cong Alt_7$ or $A_1(q)$ for some odd q.*

Remark 2.2. In Lemma 2.1, for every odd prime $p \in \pi(S)$, we have $t(p,S) \geq t(p,G) - 1$.

LEMMA 2.3 (Zsigmondy's Theorem [27]). *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *There is a primitive prime p' for $p^n - 1$, that is, $p' | (p^n - 1)$ but p' does not divide $p^m - 1$, for every $1 \leq m < n$,*

(ii) $p = 2$, $n = 1$ *or* 6,

(iii) *p is a Mersenne prime and $n = 2$.*

LEMMA 2.4 ([6, Lemmas 2.7 and 2.8]).

(1) *If $G = A_{n-1}(q)$, then G contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/(n, q-1)$.*

(2) *If $G = C_n(q)$, then G contains a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n - 1)/(2, q-1)$.*

(3) *If $G = {}^2D_n(q)$, and there exists a primitive prime divisor r of $q^{2n-2} - 1$, then G contains a Frobenius subgroup with kernel of order q^{2n-2} and cyclic complement of order r.*

(4) *If* $G = B_n(q)$ *or* $D_n(q)$*, and there exists a primitive prime divisor* r_m *of* $q^m - 1$*, where* $m = n$ *or* $n-1$ *such that* m *is odd, then* G *contains a Frobenius subgroup with kernel of order* $q^{m(m-1)/2}$ *and cyclic complement of order* r_m.

LEMMA 2.5 ([18, Lemma 1]). *Let* N *be a normal subgroup of* G. *Assume that* G/N *is a Frobenius group with Frobenius kernel* F *and cyclic Frobenius complement* C. *If* $(|N|,|F|)=1$, *and* F *is not contained in* $NC_G(N)/N$, *then* $p|C| \in \pi_e(G)$, *where* p *is a prime factor of* $|N|$.

Remark 2.6 ([7]). If q is a natural number, r is an odd prime and $(q,r)=1$, then by $e(r,q)$ we denote the smallest natural number m such that $q^m \equiv 1$ (mod r). Given an odd q, put $e(2,q)=1$ if $q \equiv 1$ (mod 4) and put $e(2,q)=2$ if $q \equiv -1$ (mod 4). Using Fermat's little theorem we can see that if r is an odd prime such that $r \mid (q^n - 1)$, then $e(r,q) \mid n$.

Let m be a positive integer and p be a prime number. Then m_p denotes the p-part of m. In other words, $m_p = p^k$ if $p^k \mid m$ but p^{k+1} does not divide m.

LEMMA 2.7 [26, Proposition 2.4]. *Let* G *be one of simple groups of Lie type,* $B_n(q)$ *or* $C_n(q)$*, over a field of characteristic* p. *Define*

$$\eta(m) = \begin{cases} m, & m = 2k+1 \\ m/2, & o.w. \end{cases}$$

Let r *and* s *be odd primes with* $r,s \in \pi(G)\setminus\{p\}$. *Put* $k = e(r,q)$, $l = e(s,q)$ *and suppose that* $1 \le \eta(k) \le \eta(l)$. *Then* r *and* s *are nonadjacent if and only if* $\eta(k)+\eta(l) > n$ *and* l/k *is not an odd natural number.*

3. MAIN RESULTS

Remark 3.1. By [4, 25], we know that $|B_n(q)| = q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$ and $t(B_n(q)) = [(3n+5)/4]$. Throughout this paper, we denote by r_i, a primitive prime divisor of $9^i - 1$. We note that every primitive prime divisor of $3^{\alpha i} - 1$ is a primitive prime divisor of $(3^\alpha)^i - 1$, but the converse does not hold. Also $\rho(2, B_n(9)) = \{2, r_{2n}\}$, where n is an arbitrary natural number. Also throughout the paper, we suppose that p and p' are odd prime numbers.

THEOREM 3.2. *If* G *is a finite group such that* $\Gamma(G) = \Gamma(B_n(9))$*, where* $n \ge 8$*, then* G *has a unique nonabelian composition factor isomorphic to* $B_n(9)$ *or* $C_n(9)$.

Proof. By Lemma 2.1 and Remark 3.1, there exists a finite nonabelian simple group S such that $S \le \overline{G} = G/K \le Aut(S)$ for the maximal normal solvable subgroup K of G. Moreover, $t(S) \ge t(G) - 1$. According to the tables in [25, 26], we consider each possibility for S. If S is a simple group of Lie type over $GF(q)$, where $q = p^\alpha$, then we denote by u_i, a primitive prime divisor of $q^i - 1$.

Step 1. We show that S is not isomorphic to an alternating group.

Let $S \cong A_m$. Since $n \ge 8$, so $t(G) \ge 7$. By Remark 3.1 and Lemma 2.1, $t(S) \ge t(G) - 1 \ge 6$. Therefore $m \ge 43$. Let $t \in \pi(S)$. By [25, Proposition 1.1], t is not adjacent to 19 if and only if $t + 19 > m$. Therefore $t \in [m-18, m]$. There are at least 12 elements in $[m-18, m-1]$ which are divisible by 2 or 3, since $[18/2]+[18/3]-[18/6]=12$. Therefore there are at most 7 prime numbers in $[m-18, m]$. So $t(19, S) \le 8$. On the other hand, $e(19,9)=9$. If $9 = \eta(9) < (n+1)/2$, then by [20, Theorem 3.3],

$t(19,G)\geq 13$. So by Remark 2.2, $12\leq t(19,G)-1\leq t(19,S)\leq 8$, which is a contradiction. Therefore $9=\eta(9)\geq (n+1)/2$ and by [20, Theorem 3.3], $t(19,G)=[(3n+5)/4]$. Then $t(19,G)-1\leq t(19,S)\leq 8$ implies that $n\in\{8, 9, 10, 11\}$. For convenience, we give the details only for $n=8$. Suppose that $n=8$. By Lemma 2.1, $r_{2n}=r_{16}=21523361\in\pi(S)$ and $r_{16}\in[m-3,m]$, since $r_{2n}\in\rho(2,S)$. Therefore $21523361\leq m\leq 21523364$. Hence $21523361\in\pi(S)\setminus\pi(B_8(9))$, which is a contradiction. Similarly $n=9, 10, 11$ are impossible.

Step 2. The simple group S is not isomorphic to any simple exceptional group of Lie type.

Let S be a simple exceptional group of Lie type. Since $t(S)\geq t(G)-1$ and $n\geq 8$, so $t(S)\geq 6$. Therefore $S\cong E_7(q)$ or $E_8(q)$. Let $S\cong E_7(q)$, where $q=p^\alpha$ and $p\in\pi(G)$. Since $t(S)=8$ and $t(S)\geq t(G)-1$, so $n\in\{8, 9, 10, 11\}$. We note that $q(q^{18}-1)\,||\,|E_7(q)|$, so $19\in\pi(E_7(q))$. Therefore $n\in\{9, 10, 11\}$. Since the proofs are similar, so we consider the proof only for $n=9$.

First suppose that $p\neq 3$. We know that $(p^{14}-1)\,|\,|S|$. Since $p\in\pi(B_9(9))$ and for each $p\in\pi(B_9(9))\setminus\{3\}$, $p^{14}-1$ has a prime divisor which is not in $\pi(B_9(9))$, we get a contradiction.

Thus $p=3$. Since $\rho(2,B_9(9))=\{2,r_{18}\}$, so by Lemma 2.1, $r_{18}\in\pi(S)$ and r_{18} is not adjacent to 2. So $r_{18}\in\{u_7,u_9,u_{14},u_{18}\}$, by Table 7 in [25].

Let $r_{18}=u_7$. As we mentioned in Remark 3.1, every primitive prime divisor of $q^{36}-1$ is a primitive prime divisor of $9^{18}-1$. Therefore $36\,|\,7\alpha$ and since $\pi(S)$ is a subset of $\pi(B_9(9))$, it follows that $7\alpha\leq 36$, by Lemma 2.3, which implies that $7\alpha=36$ and this is a contradiction. When $r_{18}=u_{14}$, we get a contradiction similarly.

If $r_{18}=u_9$, then $9\alpha=36$ and $\alpha=4$. Hence $S\cong E_7(81)$ and if t is a primitive prime divisor of $9^{36}-1$, then $t\in\pi(S)\setminus\pi(B_9(9))$, which is a contradiction.

If $r_{18}=u_{18}$, then $18\alpha=36$ and $\alpha=2$. Therefore $S\cong E_7(9)$ and $r_{16}\in\pi(B_9(9))\setminus\pi(E_7(9))$. Since $\pi(Out(S))=\{2\}$, so $r_{16}\in\pi(K)$. We know that $r_4\in\pi(S)$ and r_4 is not adjacent to r_{16} in $\Gamma(B_9(9))$ by Lemma 2.7. By [23], we have $C_4(9)\leq A_7(9)\leq E_7(9)$. Thus by Lemma 2.4, $E_7(9)$ contains a Frobenius subgroup with kernel of order 9^4 and cyclic complement of order $(9^4-1)/2$. Therefore by [20, Lemma 3.1], $r_4\sim r_{16}$ in $\Gamma(G)$, which is a contradiction.

For $S\cong E_8(q)$, we give the result similarly.

Step 3. The simple group S is not isomorphic to a simple classical group over a field of characteristic $p\neq 3$.

Case 1. The simple group S is not isomorphic to ${}^{\varepsilon}A_{m-1}(q)$, $B_m(q)$, $C_m(q)$ and $D_m(q)$, where $q=p^\alpha$.

Let $A=\{r_{2(n-1)},r_{2(n-2)},r_{2(n-3)},r_{2(n-4)},r_{2n}\}$. Since $n\geq 8$, so by [25], A is an independent set in $\Gamma(G)$. Therefore by Lemma 2.1, $|A\cap\pi(S)|\geq 4$. We note that $t(p,S)\leq 3$, for each simple classical group $S\neq {}^2D_m(q)$ and so p does not belong to A. Therefore p is adjacent to at least two elements of A in $\Gamma(G)$. We prove that in each case, $p\in\{2, 5, 7, 13, 17, 41, 73, 193\}$. For instance, let p be adjacent to $r_{2(n-1)}$ and $r_{2(n-3)}$. Let $l=e(p,9)$. So by Lemma 2.7, we have $\eta(l)+(n-1)\leq n$ or $2(n-1)/l$ is an odd natural number. Similarly $\eta(l)+(n-3)\leq n$ or $2(n-3)/l$ is an odd number. Therefore $\eta(l)\leq 3$. Similarly in each case we conclude that $\eta(l)\leq 4$ and so $p\in\{2, 5, 7, 13, 17, 41, 73, 193\}$.

For convenience we give the details of the proof for $S \cong {}^{\varepsilon}A_{m-1}(q)$ in the sequel. Other cases are proved similarly.

Let $S \cong {}^{\varepsilon}A_{m-1}(q)$, where $q = p^{\alpha}$.

Since $n \geq 8$, so $t(G) \geq 7$. Therefore by $t(S) \geq t(G) - 1$, we have $m \geq 11$. Since $m \geq 11$, so $(q^{10} - 1) \mid\mid S \mid$.

Let $p = 2$. We note that $e(31,2) = 5$ and so $31 \in \pi(S)$. Therefore $31 \in \pi(G)$ and since $e(31,9) = 15$, it follows that $n \geq 15$. If $\eta(15) = 15 \geq (n+1)/2$, i.e. $n \leq 29$, then by [20, Theorem 3.3] and Remark 2.2, $t(31,G) - 1 = [(3n+5)/4] - 1 \leq t(31,S) \leq \upsilon(5) = 10$, which implies that $n \leq 14$. But this is a contradiction, since $n \geq 15$. Therefore $\eta(15) < (n+1)/2$, i.e. $n > 29$. Similarly to above, $22 = (3 \cdot 15 - 1)/2 \leq t(31,G) - 1 \leq t(31,S) \leq \upsilon(5) = 10$ by [20, Theorem 3.3], which is a contradiction.

Let $p = 5$. Since $e(521,5) = 10$, so $521 \in \pi(S)$. Also $e(521,9) = 260$, so $n \geq 130$. If $\eta(260) = 130 \geq (n+1)/2$, i.e. $n \leq 259$, then by [20, Theorem 3.3], $t(521,G) - 1 = [(3n+5)/4] - 1 \leq t(521,S) \leq 10$, which is a contradiction, since $n \geq 130$. Therefore $130 < (n+1)/2$, *i.e.* $n > 259$ and by [20, Theorem 3.3],

$$[260/4] + 260/2 - 1 \leq t(521,G) - 1 \leq t(521,S) \leq 10,$$

which is a contradiction.

Let $p = 7$. Since $e(2801,7) = 5$, so $2801 \in \pi(S)$. On the other hand $e(2801,9) = 1400$, which implies that $n \geq 700$. If $n \leq 1399$, then $t(2801,G) = [(3n+5)/4]$, by [20, Theorem 3.3] and in a similar way we get that $n \leq 14$, which is a contradiction. Therefore $n > 1399$. By [20, Theorem 3.3],

$$[1400/4] + 1400/2 - 1 = 1049 \leq t(2801,G) - 1 \leq t(2801,S) \leq 10,$$

which is a contradiction.

For $p = 13, 17, 41, 73, 193$, we get a contradiction similarly by [20, Theorem 3.3].

Case 2. Let $S \cong {}^{2}D_{m}(q)$, where $q = p^{\alpha}$.

Let $n \geq 10$. So by $t(S) \geq t(G) - 1$, we have $[(3m+4)/4] \geq [(3n+5)/4] - 1 \geq 7$. Therefore $m \geq 8$. Let $A' = \{r_{2(n-i)} \mid 0 \leq i \leq 5\}$. By [25, Table 8], A' is an independent set in $\Gamma(G)$. So by Lemma 2.1, $|A' \cap \pi(S)| \geq 5$. Since $t(p,S) \leq 4$, so p does not belong to A'. Therefore p is adjacent to at least two elements of A' in $\Gamma(S)$. Therefore p is adjacent to them in $\Gamma(G)$. Similarly to Case 1 and by [26, Proposition 2.5], we get that $p \in \{2, 5, 7, 13, 17, 41, 73, 193, 1181\}$. If $p \neq 1181$, then exactly similarly to Case 1, we get a contradiction, since $(p^{10} - 1) \mid |S|$. Let $p = 1181$. We have $e(140761,1181) = 5$, so $140761 \in \pi(S)$. Since $e(140761,9) = 11730$, so $n \geq 5865$. If $n \leq 11729$, then $t(140761,G) = [(3n+5)/4]$, by [20, Theorem 3.3]. Now $t(140761,G) - 1 = [(3n+5)/4] - 1 \leq t(140761,S) \leq 10$, which implies that $n \leq 14$, and this is a contradiction. Thus $n > 11729$. Similarly to above,

$$[11730/4] + 11730/2 - 1 = 8796 \leq t(140761,G) - 1 \leq t(140761,S) \leq 10,$$

which is a contradiction.

Let $n \in \{8, 9\}$.

If $n = 9$, then $t(G) = 8$ and so $m \geq 8$. So $(p^{14} - 1) \mid |S|$. On the other hand, $p \in \pi(G)$. But for every $p \in \pi(B_9(9))$, $p^{14} - 1$ has a prime divisor which is not in $\pi(B_9(9))$, and this is a contradiction.

For $n = 8$, we get a contradiction similarly.

Step 4. The simple group S is not isomorphic to a sporadic group.

Let S be a sporadic group. Hence by [25], $t(S) \leq 11$ and since $t(S) \geq t(G)-1$, so $n \leq 15$. We note that $\rho(2,G)=\{2,r_{2n}\}$ and by Lemma 2.1, we must have $\rho(2,G)$ is a subset of $\pi(S)$. But in each case we see that r_{2n} does not belong to $\pi(S)$, which is a contradiction. For instance, $r_{16}=21523361$ does not belong to $\pi(S)$, where $n=8$.

Step 5. Therefore S is a simple classical group of Lie type over $GF(3^{\alpha})$. Now we prove that the simple group S is isomorphic to $B_n(9)$ or $C_n(9)$.

Case 1. Let $S \cong A_{m-1}(3^{\alpha})$. Since $t(S) \geq t(G)-1$, so $[(3n+5)/4]-1 \leq [(m+1)/2]$, which implies that $n<m$. Also since $n \geq 8$, so $m \geq 11$.

By Remark 3.1 and Lemma 2.1, $\rho(2,G)=\{2,r_{2n}\}$ is a subset of $\pi(S)$ and r_{2n} is not adjacent to 2. By [25, Table 6], $r_{2n}=u_m$ or u_{m-1}.

If $r_{2n}=u_m$, then $4n \mid \alpha m$. Also Remark 3.1 and Lemma 2.3 implies that $\alpha m \leq 4n$. So $\alpha m=4n$. Since $m>n$, thus $\alpha<4$. If $\alpha \in \{1,2\}$, then $r_{2n-1} \in \pi(S) \setminus \pi(G)$, which is a contradiction. Therefore $\alpha=3$, *i.e.* $3m=4n$ and so $3 \mid n$. On the other hand, $t(S) \geq t(G)-1$ and so $[(3n+5)/4]-1 \leq [(m+1)/2]$, which implies that $(3n+5)/4-2<(4n+3)/6$, and so $n<15$. Therefore, since $3 \mid n$ and $n \geq 8$, $(n,m) \in \{(9,12),(12,16)\}$. But according to the inequality $[(3n+5)/4]-1 \leq [(m+1)/2]$, this is a contradiction.

If $r_{2n}=u_{m-1}$, then we get a contradiction similarly.

Case 2. Let $S \cong {}^2A_{m-1}(3^{\alpha})$. Since $t(S) \geq t(G)-1$ and $n \geq 8$, so $n<m$ and $m \geq 11$. Similarly to Case 1, by [25, Table 6], $r_{2n}=u_{2m}$ if m is odd and $r_{2n} \in \{u_{m/2},u_m,u_{2m-2}\}$ if m is even.

Let $r_{2n}=u_{m/2}$. So $4n \mid \alpha m/2$. Since m is even, so $(3^{\alpha m}-1) \mid |S|$. Therefore $\alpha m \leq 4n$ and we conclude that $4n \leq \alpha m/2 \leq 2n$, which is a contradiction.

Let $r_{2n}=u_m$. Thus $\alpha m=4n$. If $\alpha \geq 4$, then $\alpha m=4n \geq 4m$ and since $n<m$, so $4n>4n$, which is a contradiction. Therefore $\alpha \in \{1,2,3\}$. If $\alpha=1$, then $r_{2n-1} \in \pi(S) \setminus \pi(G)$ and if $\alpha=2$, then $r_{4n-2} \in \pi(S) \setminus \pi(G)$, which is a contradiction in both cases. Hence $\alpha=3$ and so $3 \mid n$. Since $[(3n+5)/4]-1 \leq [(m+1)/2]$, similarly to above we have $n<15$. Therefore $(n,m)=(9,12)$ or $(12,16)$, since $3 \mid n$. Now we get a contradiction by $[(3n+5)/4]-1 \leq [(m+1)/2]$.

If $r_{2n}=u_{2m-2}$ or u_{2m}, then we get the result similarly.

If $S \cong D_m(3^{\alpha})$ or $S \cong {}^2D_m(3^{\alpha})$, then we get a contradiction similarly.

Case 4. Let $S \cong B_m(3^{\alpha})$. Since $t(S) \geq t(G)-1$, so $3n<3m+7$, which implies that $n-3<m$. By [25, Table 6] and Lemma 2.1, $r_{2n}=u_m$ or u_{2m}, since $\rho(2,G)=\{2,r_{2n}\}$.

Let $r_{2n}=u_m$, where m is odd. So $\alpha m=4n$ and $4 \mid \alpha$. If $\alpha \geq 8$, then $\alpha m=4n \geq 8m>8n-24$, since $n-3<m$. Therefore $n<6$, which is a contradiction. Thus $\alpha=4$ and $S \cong B_n(81)$. But $r_{4n} \in \pi(S) \setminus \pi(G)$, which is a contradiction.

Let $r_{2n}=u_{2m}$. Similarly we get that $\alpha=2$ and so $m=n$. Consequently $S \cong B_n(9)$.

If $S \cong C_m(3^{\alpha})$, then we get that $S \cong C_n(9)$, similarly. Consequently $S \cong B_n(9)$ or $S \cong C_n(9)$.

THEOREM 3.3. Let G be a finite group such that $\Gamma(G)=\Gamma(B_n(9))$, where $n \geq 8$. Then:

(i) *If* n *is odd, then* $G/K \cong B_n(9)$ *or* $B_n(9)\cdot 2$, *the extension of* $B_n(9)$ *by the field automorphism, where* K *is a* 2*-group or* $G/K \cong C_n(9)$ *or* $C_n(9)\cdot 2$, *the extension of* $C_n(9)$ *by the field automorphism, where* K *is an elementary abelian* r_l*-group such that* $l \mid n$.

(ii) *If* n *is even, then* $G/K \cong B_n(9)$ *or* $B_n(9)\cdot 2$, *the extension of* $B_n(9)$ *by the field automorphism, where* K *is a* $\{2,5\}$*-group or* $G/K \cong C_n(9)$ *or* $C_n(9)\cdot 2$, *the extension of* $C_n(9)$ *by the field automorphism, where* K *is an elementary abelian* r_l*-group such that* $\eta(l) \le n/2$ *or* n/l *is odd.*

Proof. We see that by Lemma 2.1 and Theorem 3.2, there exists a nonabelian simple group S such that, $S \le G/K \le Aut(S)$, and K is the maximal normal solvable subgroup of G. Also $S \cong B_n(9)$ or $C_n(9)$. Since $|Out(S)| = 4$, it follows that $G/K = S$, $G/K = S\cdot 2$, the extension of S by the diagonal automorphism of S, $G/K = S\cdot 2$, the extension of S by the field automorphism of S, $G/K = S\cdot 2$, the extension of S by the field-diagonal automorphism of S or $G/K = Aut(S)$. We know that if S is a simple group of Lie type and $\hat{S} = S\cdot d$, where d is the diagonal automorphism of S, then $\hat{S}$ is a group of Lie type in which the maximal tori $\hat{T}$ have order $|T|d$, where $T = \hat{T} \cap S$ by [17]. If $G/K = S\cdot 2$, the extension of S by the diagonal automorphism of S or by the field-diagonal automorphism of S, or $G/K = Aut(S)$, then $2 \sim r_{2n}$ in $\Gamma(G/K)$, which is a contradiction, since 2 is not adjacent to r_{2n} in $\Gamma(G)$. Therefore $G/K = S$ or $G/K = S\cdot 2$, the extension of S by the field automorphism of S. Let there exists p such that $p \mid |K|$. We can assume that K is an elementary abelian p-group by [13]. Since $B_n(3) \le B_n(9)$ and by [5], $B_n(3)$ and $C_n(3)$ acts unisingularly, hence $p \ne 3$.

Let $S \cong B_n(9)$.

Let n be odd and $e(p,9) = l$. Now by Lemma 2.4, G/K contains a Frobenius subgroup with kernel of order $9^{n(n-1)/2}$ and cyclic complement of order r_n. Since $p \ne 3$, so if p is not adjacent to r_n in $\Gamma(G)$, then similarly to the above, we get a contradiction. Therefore $p \sim r_n$ and by Lemma 2.7, n/l is odd. Thus l is odd and so p is not adjacent to r_{2n}. By [23], ${}^2D_n(9) \le B_n(9)$ and so by Lemma 2.4, G/K contains a Frobenius subgroup of the form $9^{2n-2} : r_{2n-2}$. Since p is not adjacent to r_{2n}, so similarly to the previous, $p \sim r_{2n-2}$. Therefore by Lemma 2.7, $\eta(l) = 1$, since l is odd. So $l = 1$ and $p = 2$. Therefore K is a 2-group.

Let n be even. Similarly to above, G/K contains a Frobenius subgroup of the form $9^{(n-1)(n-2)/2} : r_{n-1}$. We claim that p is not adjacent to r_{2n} or p is not adjacent to r_{2n-2}. If $p \sim r_{2n}$ and $p \sim r_{2n-2}$, then similarly to above $l = 2$ and so $p = 5$, which is a contradiction, since 5 is not adjacent to r_{2n} in $\Gamma(G)$. Hence similarly to the above, $p \sim r_{n-1}$, since $p \ne 3$. On the other hand $B_{n-2}(9) \le B_n(9)$ by [23]. So by Lemma 2.4, G/K contains a Frobenius subgroup of the form $9^{(n-3)(n-4)/2} : r_{n-3}$ and $p \sim r_{n-3}$. Since $p \sim r_{n-1}$ and $p \sim r_{n-3}$, so $l \in \{1,2,3\}$ and $p \in \{2,5,7,13\}$.

Let $p = 7$. By [23], ${}^2D_n(9) \le B_n(9)$ and by Lemma 2.4, ${}^2D_n(9)$ contains a Frobenius subgroup of the form $9^{2n-2} : r_{2n-2}$. We know that 7 is not adjacent to r_{2n-2}, since $e(7,9) = 3$. Therefore $7 \sim r_{2n-2}$, which is a contradiction.

If $p = 13$, then we get a contradiction similarly, since $e(13,9) = 3$. Thus K is a $\{2,5\}$-group.

Let $S \cong C_n(9)$. We note that by Lemma 2.4, $C_n(9)$ contains a Frobenius subgroup of the form $9^n : (9^n - 1)/2$. In a similar way, we conclude that $p \sim r_n$. So if n is odd, then n/l is an odd natural number, by Lemma 2.7; and if n is even, then $\eta(l) \le n/2$ or n/l is odd.

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