THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 16, Number 3/2015, pp. 405–412

ONSET OF A DENSE AVALANCHE ON A PLANE SLOPE

Ioan R. IONESCU[1], Oana LUPAŞCU[2]

[1] LSPM, Institut Galilee, Universite Paris 13, Sorbonne-Paris-Cite, 99, Av. J.B. Clement, 93430 Villetaneuse, France
and "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P. O. Box 1–764, RO 014700 Bucharest, Romania
[2] Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania,
and the Research Institute of the University of Bucharest (ICUB)
E-mail: oana.lupascu@imar.ro

The main goal of this paper is to study the safety factor (of limit load) problem related to the shallow flow avalanche of a visco-plastic fluid/solid with heterogeneous thickness over a plane slope. The first objective is to find the appropriate functional space of the problem and to prove the existence of a onset velocity field. The second objective of the paper is to propose a numerical strategy to solve the limit load problem and to characterize the flow onset of the avalanche. We introduce an optimization problem (called the limit load or safety factor problem) to study the link between the yield limit, the external forces and the thickness distributions for which the shallow flow avalanche of a visco-plastic fluid/solid does, or does not occur. This optimization problem is reconsidered in the space of bounded deformations functions and the velocity boundary conditions are relaxed. We prove that the initial optimization problem is not changed and the reformulated safety factor problem has a least a solution, modelling the onset of the avalanche. We have developed here a DVDS-type numerical technique to solve the safety factor problem through a shape optimization problem. The proposed numerical method makes use of Fourier level set description of the subdomain and of a genetic algorithm in solving the non convex and non-smooth global optimization problem. The proposed numerical approach is illustrated with some numerical simulations of avalanches involving a Bingham circular dome, a Druker-Prager square dome on a plane slope, and a thick Bingham fluid over an obstacle.

Key words: shallow flow, visco-plastic fluid, flow onset, safety factor, limit load, bounded deformation, DVDS, level set, genetic algorithm.

1. INTRODUCTION

The understanding of the physics of the avalanche onset, related to the shallow flow of soils, snow or other geomaterials over an inclined surface, is an important issue in geophysics and engineering (see for instance [1]). Since the problem is three dimensional, and the behaviour of the material is best represented by visco-plastic fluid/solid type models, the mathematical and numerical modelling is very complex and poses many challenges. That is why, reduced 2-D models, called also Saint-Venant models, are generally considered. Such models are able to capture the principal features of the flow (onset, dynamic propagation and arrest).

Very recently, a new Saint-Venant type (shallow flow) model for visco-plastic fluids/solids in frictional contact with a plane slope was obtained in [7]. This model describes the onset of the avalanche through a criterion which relates the yield limit (material resistance) to the external forces distribution. The main problem is to find the maximum multiple of the force distribution that the fluid/solid can withstand without collapsing, and the collapse avalanche flow. In many applications, the strains are localized on some surfaces where the velocity of the collapse flow exhibits discontinuities. That is why, the onset modelling (called also limit analysis) was and remains a difficult mathematical and numerical problem.

The main goal of this paper is to study the safety factor (or limit load) problem related to the shallow flow of a visco-plastic fluid/solid with heterogeneous thickness over a plane slope. The first objective is to find the appropriate functional space of the problem and to prove the existence of a onset velocity field.

The plastic dissipation functional, involved in the limit load problem, is non smooth, and non coercive in classical Sobolev spaces. That is why, we have to consider it in the space of bounded deformation functions (*i.e.*, the space of velocities which have their rate of deformation in the space of bounded measures) introduced in [11, 12].

The second objective of the paper is to propose a numerical strategy to solve the limit load problem and to get the onset flow field of the avalanche. The numerical methods in limit analysis are based on the discretization of the kinematic or static variational principles using the finite element method technics and the convex and linear programming. The first results were obtained in [4, 2] while the literature on FE to limit analysis is very extensive. Despite a great progress in the last decades (X-FEM, re-meshing techniques, ...), the finite element method remains associated to continuous fields and it is not so well adapted for modelling strain localization and velocities discontinuities on unknown surfaces. For that, we will use here the discontinuous velocity domain splitting (DVDS) method, introduced in [6]. DVDS is a mesh free method which does not use a finite element discretization of the solid. It focuses on the strain localization and completely neglect the bulk deformations. The limit load problem is thus reduced to the minimization of a mesh free functional (plastic dissipation power) depending on a domain partition. The avalanche collapse velocity field, which is discontinuous, is associated to the domain partition and to a rigid flow. It has localized deformations only, located at the boundary of the sub-domain.

The main novelty of this paper consists in finding the appropriate functional space of the limit load problem and in obtaining an existence result for the onset velocity field. As far as we know, the use of a mesh free technique (DVDS) for a numerical approach of the safety factor problem is also new.

2. MATHEMATICAL APPROACH OF THE FLOW ONSET

When modelling landslides, or snow avalanches, the fluid/solid is totally at rest (blocked) in its natural configuration and the beginning of a flow can be seen as a "disaster". In a solid mechanics context the onset of the flow is studied through the limit analysis which is based on a very idealized representation of a perfectly plastic material subjected to slowly increasing loads. The main problem in limit analysis is to find the maximum multiple of the force distribution, that the solid/fluid can withstand without flowing (collapsing), and the associated (collapse) flow field. The final result of such an analysis is a non-dimensional number called "limit load" (or "safety factor"). It requires only the yield limit distribution of the material and the geometry of the associated boundary value problem.

The avalanche shallow model used here was derived in [7] under the following asymptotic assumptions: the normal components of the velocity as well as the tangential stresses are small, *i.e.* they are of the same order $\varepsilon << 1$, a small parameter representing the ratio aspect of the thickness.

Following [7] we consider a visco-plastic fluid/solid occupying a domain $\mathcal{D} \subset R^3$ (Fig. 1). For a plane slope of angle α the domain can be described through $\mathcal{D} = \{(x,z); x \in \Omega, 0 \le z \le h(x)\}$, where $\Omega \subset R^2$ is a fixed bounded domain and $h(x) \ge 0$ is the thickness of the fluid and $x = (x_1, x_2)$. All over this paper the space and time coordinates, as well as all mechanical fields, are non dimensional. The boundary $\partial\mathcal{D}$ is divided into three disjoint parts $\partial\mathcal{D} = \Gamma_b \cup \Gamma_s \cup \Gamma_l$. We define by $\Gamma_s = \{(x,z); x \in \Omega, z = h(x) > 0\}$, $\Gamma_b = \{(x,0); x \in \Omega, h(x) > 0\}$, the free and the bottom boundaries while the lateral surface is $\Gamma_l = \{(x,z); x \in \partial\Omega, h(x) > z > 0\}$. We denote by $\boldsymbol{n}$ the outward unit normal on $\partial\mathcal{D}$. On the boundary Γ_b, which corresponds to the bottom part of the fluid, the visco-plastic fluid is in contact with Coulomb friction with the slope plane $z = 0$ (with C_f the friction coefficient) and since the (unknown) boundary Γ_s is a free surface we assume a stress free condition. For the sake of simplicity we will suppose that $h(x) > 0$ for all $x \in \partial\Omega$, and the lateral boundary $\Gamma_l(t)$ is splinted into two parts:

$$\Gamma_l^0 = \{(x,z); x \in \Gamma^0, h(x) > z > 0\} \text{ and } \Gamma_l^1 = \{(x,z); x \in \Gamma^1, h(x) > z > 0\}$$

following a partition of $\partial\Omega = \Gamma_0 \cup \Gamma_1$. We will consider two kinds of boundary conditions: adherence on Γ_l^1 and stress free on Γ_l^0.

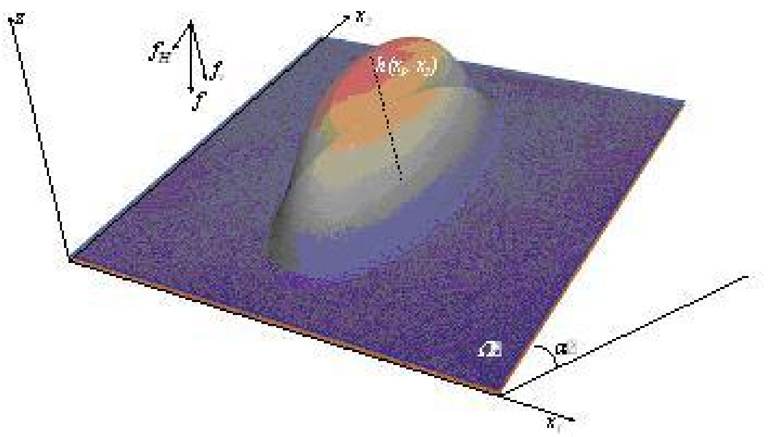

Fig. 1 – 3D representation of the fluid domain $\mathcal{D}$ flowing on a plane slope of angle α, the bottom part Γ_b and the Ox_1z section described through the thickness function $(x_1, x_2) \to z = h(x_1, x_2)$.

We introduce here the "safety factor" (or limit load) to study the link between the yield limit distribution and the external forces distribution for which the shallow flow avalanche of the visco-plastic fluid does or does not occur.

In order to get the characterization of the fact that the fluid is totally at rest in its initial configuration we have to check whenever $h(t) \equiv h,\ v(t) \equiv 0$ is a solution of the time evolution variational inequality derived in [7]. We get the following stationary inequality

$$\int_\Omega h\kappa_0^{\text{shallow}} \sqrt{\frac{1}{2}\left[|D(\Psi)|^2 + (\text{div}\Psi)^2\right]}\text{d}x + \frac{\gamma\cos(\alpha)}{Fr^2}\int_\Omega \rho h C_f\,|\Psi|\text{d}x \geq \frac{\gamma}{Fr^2}\int_\Omega h\rho\left[\sin(\alpha)\psi_1 + \frac{1}{2}h\cos(\alpha)\text{div}\Psi\right]\text{d}x \quad (1)$$

for all $\Psi \in V =: \{\Psi : \Omega \to R^2; \Psi = 0 \text{ on } \Gamma_0\}$, where ρ is the mass density, γ is the gravitational acceleration. $\kappa_0^{\text{shallow}}$ is the distribution of the shallow yield limit in the rest configuration given by $\kappa_0^{\text{shallow}} = \kappa^0 + \frac{\mu}{2Fr^2}\rho h\gamma\cos\alpha$, where κ^0 the cohesion, μ the internal frictional coefficient and Fr^2 is the Froude number.

Note that the problem (1) is rate independent which means that the viscous effects, related to the rate dependency, will disappear in the optimization problem modelling the avalanche onset. This fact has important mathematical consequences for the regularity of the solution. As we can see in the next, the solutions of the rigid-plastic shallow flow problem (1) have spatial discontinuities.

Let write now the variational inequality (1) to get the safety factor (or limit load) problem. We denote by $\phi : R_S^{2\times 2} \to R_+$

$$\phi(D) =: \sqrt{\frac{1}{2}\left[|D|^2 + (\text{trace}D)^2\right]},\ G(\Psi) =: \int_\Omega g\phi(D(\Psi))\text{d}x + \int_\Omega q|\Psi|\text{d}x,$$

the shallow plastic strain rate potential and by G the total dissipation power (plastic and frictional dissipation), where $g =: h\kappa_0^{\text{shallow}}$ and $q =: \frac{\gamma\cos\alpha}{\text{Fr}^2}\rho h C_f$. Denoting by $F_1 =: \frac{\gamma}{\text{Fr}^2}h\sin\alpha$, $b =: \frac{\gamma\rho}{2\text{Fr}^2}h^2\cos\alpha$, the external forces dissipation power reads $L(\Psi) =: \int_{\Omega} (F_1\psi_1 + b\,\mathrm{div}(\Psi))\mathrm{d}x$. If we introduce the *safety factor (or limit load)* λ^* by

$$\lambda^* = \inf_{\Psi \in V, L(\Psi)=1} G(\Psi), \tag{2}$$

then we can see that (1) is verified (*i.e.*, the solid/fluid is at rest) if and only if $\lambda^* \geq 1$. For geological structures which are at rest in their natural configuration we can formulate the following flow/no flow criterion of an avalanche onset:

The shallow flow (avalanche) of the visco-plastic fluid/solid starts if and only if $\lambda^* < 1$. The plastic dissipation functional involved in (2) is non-smooth, and non coercive in the classical Sobolev spaces. Moreover, the above formula of the plastic dissipation power G is valid only for smooth velocity fields from the Sobolev space V. For non-smooth (discontinuous) velocities the gradient operator involved in the definition of the rate of deformation tensor has to be understood in the sense of distributions. As it follows from [8] the strain rate $D(\Psi)$ belongs to the space of bounded measures $M^1(\Omega)$ and the associated functional space is the space of bounded deformations functions

$$BD(\Omega) =: \left\{ \Psi : \Omega \to R^2; \Psi \in L^1(\Omega)^2, D(\Psi) \in M^1(\Omega)^{2\times 2} \right\},$$

introduced and discussed in [11, 12, 13]. We assume the following regularity conditions:

$$b, g \in C^0(\bar{\Omega}), \quad g(x) \geq g_0 > 0, \; q, F_1 \in L^\infty(\Omega) \qquad q \geq 0.$$

Since $D \to \phi(D)$ is an equivalent norm on $R^{2\times 2}$ and satisfies the conditions of Theorem 4.1, Chapter 2 from [12], $\phi(D(\Psi))$ is a bounded positive measure on Ω. We can use this to extend the functionals G and L for all $\Psi \in BD(\Omega)$ through the formula

$$G(\Psi) = \int_{\Omega} g d\phi(D(\Psi)) + \int_{\Omega} q|\Psi|\mathrm{d}x, \;\; L(\Psi) = \int_{\Omega} F_1\psi_1 \mathrm{d}x + \int_{\Omega} bd\;\, \mathrm{div}(\Psi).$$

In order to handle the velocity boundary conditions on Γ_0 for non-smooth velocity fields, we have to add it as additional terms in the functionals G and L. These terms are modelling a discontinuity surface of a non-smooth velocity field located at the boundary Γ_0. To do that we introduce

$$G_0(\Psi) =: \int_{\Gamma_0} \frac{1}{2} g \sqrt{\left[|\Psi|^2 + 3(\Psi\cdot n)^2\right]}\mathrm{d}S, \quad \tilde{G}(\Psi) =: G(\Psi) + G_0(\Psi), \quad L_0(\Psi) =: -\int_{\Gamma_0} b\Psi\cdot n\mathrm{d}S,$$

$$\tilde{L}(\Psi) =: L(\Psi) + L_0(\Psi),$$

and we remark that $\tilde{G}(\Psi) = G(\Psi)$ and $\tilde{L}(\Psi) = L(\Psi)$ for all $\Psi \in V$, which means that

$$\inf_{\Psi \in BD(\Omega), \tilde{L}(\Psi)=1} \tilde{G}(\Psi) \leq \lambda^*.$$

We proved that the above relaxation of the boundary conditions does not change the initial optimization problem.

Theorem 2.1. *We have*

$$\inf_{\Psi \in BD(\Omega), \tilde{L}(\Psi)=1} \tilde{G}(\Psi) = \lambda^* = \inf_{\Psi \in V, L(\Psi)=1} G(\Psi). \tag{3}$$

Moreover, we also have the following existence result for the relaxed optimization problem in $BD(\Omega)$.

Theorem 2.2. *Let suppose that* $meas(\Gamma_0) > 0$. *Then there exists an onset velocity field* $v^* \in BD(\Omega)$ with $\tilde{L}(v^*) = 1$ *solution of the relaxed optimization problem*

$$\tilde{G}(v^*) = \lambda^* = \min_{\Psi \in BD(\Omega), \tilde{L}(\Psi)=1} \tilde{G}(\Psi).$$

3. FLOW ONSET NUMERICAL APPROACH AND SIMULATIONS

Almost all nontrivial known solutions of the limit load problems have spatial discontinuities. This is not so surprising if we have in mind that the extremal problem (2) models phenomena as ductile fracture or strain localization. For that, to solve the limit load extremal problem (3), involving $\tilde{G}$, we make use here of a mesh free method which does not use a finite element discretization of the solid. This new limit analysis method is called *discontinuous velocity domain splitting* (DVDS). Even if a detailed description of DVDS can be found in [6], we shall briefly recall it here.

Let us define $\mathcal{V}$ the set of DVDS velocity fields $\mathcal{V}$: = { $\boldsymbol{r}1_\omega$; $\omega \subset \Omega$, $\boldsymbol{r} \in \mathcal{R}$ }, where 1_ω is the characteristic function of a subdomain $\omega \subset \Omega$ (*i.e.* $1_\omega(x) = 1$ if $x \in \omega$ and $1_\omega(x) = 0$ if $x \notin \omega$). Since the space of bounded deformation $BD(\Omega)$ include functions with spatial discontinuities, we have $\mathcal{V} \subset BD(\Omega)$, and we can take test functions from $\mathcal{V}$ in (3) to get an upper-bound λ^*_1 of λ^*:

$$\lambda^*_1 = \inf_{r1_\omega \in V, \tilde{L}(r1_\omega)=1} \tilde{G}(\Psi) \geq \lambda^* = \inf_{\Psi \in BD(\Omega), \tilde{L}(\Psi)=1} \tilde{G}(\Psi).$$

For DVDS velocity fields the plastic dissipation involved above can be computed explicitly (we use the same arguments as in the proof of Theorem 3.1) from ω and $\boldsymbol{r}$. Denote by $J(\omega, r)$ the following shape dependent functional

$$J(\omega, r) := \frac{\tilde{G}(r1_\omega)}{\left[\tilde{L}(r1_\omega)\right]_+} = \frac{\int_{\partial\omega\setminus\Gamma_1} \frac{1}{2} g \sqrt{\left[|r|^2 + 3(r \cdot n)^2\right]} \mathrm{d}S + \int_\omega q|r| \mathrm{d}x}{\left[\int_\omega F_1 r_1 \mathrm{d}x + \int_{\partial\omega\setminus\Gamma_1} br \cdot n \mathrm{d}S\right]_+},$$

where $[s]_+ = (s + |s|)/2$ is the positive part, we get $\lambda^*_1 =: \inf_{\omega \subset \Omega, r \in R} J(\omega, r) \geq \lambda^*$. The DVDS optimal value λ^*_1 is very close to the limit load λ^*. For the anti-plane flow DVDS gives an exact evaluation of the safety factor, i.e. $\lambda_1^* = \lambda^*$ [5]. Moreover, in all in-plane flows problems where the safety factor is known, the above upper-bound λ_1^* is very close (less than 2–5 %) to the global minimum λ^* [6].

From the optimal set ω^* and the optimal rigid flow r^*

$$J(\omega^*, r^*) =: \min_{\omega \subset \Omega, r \in R} J(\omega, r)$$

one can construct the avalanche onset velocity filed $v^* =: r^* 1_{\omega^*}$. The boundary of ω^*, delimiting the flow zone from the non-flow zone, where the onset velocity v^* is discontinuous, represents the collapse fracture surface. The study of the existence and of the uniqueness of the optimal set ω^* and the optimal rigid flow r^* is beyond of the scope of the present paper.

For the numerical minimization of the functional *J*, which depends on the subdomain ω and on the rigid motion r, our approach consists in the following principal ingredients: the description of the subdomain ω with a small number of parameters; the description of the vector field *r*; the reconstruction of the topology of ω and the computation of the cost function *J*.

Finally, for the global minimization of the cost functional J we have used standard genetic algorithm techniques (see [10] for details on stochastic optimization methods).

For all the following numerical simulations we have chosen the domain $\Omega=[0,1]\times[0,1]$, the slope angle $\alpha=45^{\circ}$, the density $\rho=1$ and the gravitational acceleration $\gamma=10$. All the integrals involved in the cost function J were done on 50×50 points grid. In the first example we have considered a circular dome, given by the following thickness distribution: $h(x_1,x_2)=h_D\left(1+\cos\frac{\pi}{\delta}\sqrt{(x_1-x_{01})^2+(x_2-x_{02})^2}\right)+h_e$ if $(x_1,x_2)\in B(x_0,\delta)$, and $h(x_1,x_2)=h_e$ else with $x_0=(0.5,0.5),\delta=0.25$ and $h_D=0.125, h_e=0.01$.

For a Bingham fluid ($\mu=0,\kappa_0=10$) with no friction ($C_f=0$) we have obtained the safety factor $\lambda^*=0.6464$.

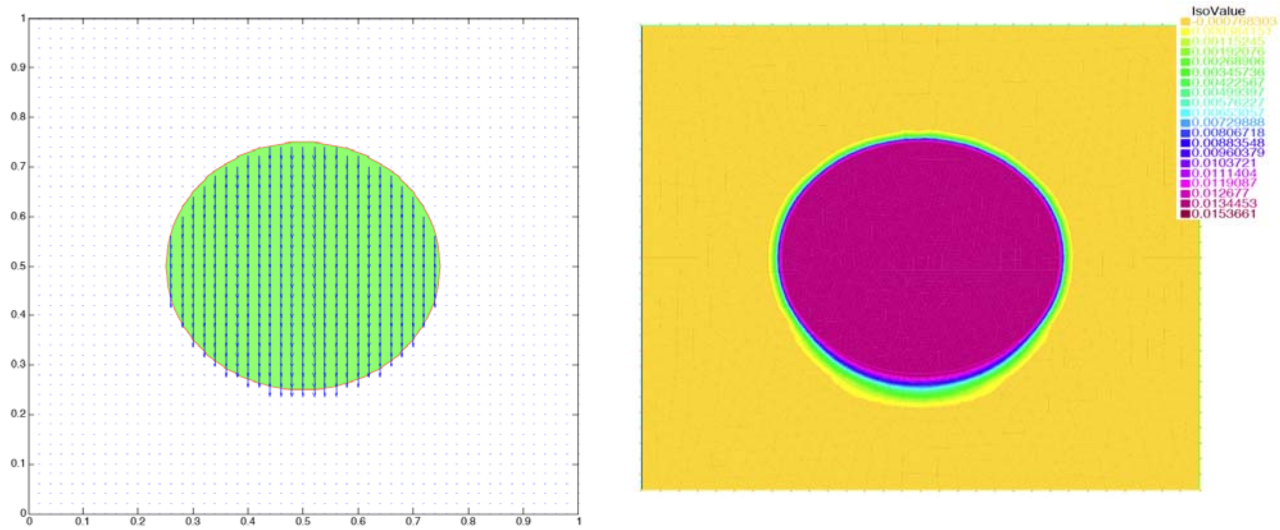

Fig. 2 – Left: the distribution of the thickness function for the square dome geometry; right: the velocity onset obtained with our approach.

The avalanche onset velocity v^*, plotted in Fig. 2 (left), shows that the fracture is circular and occurs at the base of the dome. In order to compare this result with another method we have computed the dynamic onset of the flow for a visco-plastic fluid using the finite element/finite volume numerical method described in [7]. In Fig. 2 (right), we have plotted the norm of the velocity field computed at the beginning of the flow. We can see that the onset region described by our approach is the same as for the dynamic computations. In the second simulation we have considered a square dome (Fig. 3 (left)), given by:

$$h(x_1,x_2)=h_D\left(1+\cos\left(\frac{\pi}{\delta}(x_1-x_{01})\right)\right)\left(1+\cos\left(\frac{\pi}{\delta}(x_2-x_{02})\right)\right)+h_e\text{ , if } |x_1-x_{01}|, |x_2-x_{02}|<\delta$$

and $h(x_1,x_2)=h_e$ else, for a Drucker-Prager fluid ($\mu=\tan 30^{\circ},\kappa_0=0.1$) with no friction ($C_f=0$). We have obtained the safety factor $\lambda^*=0.3053$ and the onset velocity is plotted is Fig. 3 right. We remark that the avalanche fracture, which is a rounded rectangle, occurs at the base of the dome. In the third simulation we have considered a circular dome over a thick uniform fluid in the presence of a circular obstacle $B=B(x_0^o,\delta^o)$ Fig. 4 (left). The obstacle is located at $x_0^0=(0.5,0)$ and has a radius of $\delta^0=0.2$, while the thickness function h is the same as in the first simulation, but with $h_e=0.1$, ten times larger. It is more suitable to model the obstacle by the penalization of the yield limit κ_0. For that, we have considered the same domain $\Omega=[0,1]\times[0,1]$, as before, but with $\kappa_0=1$ on $\Omega\setminus B$ outside the obstacle and $\kappa_0=50$ inside the obstacle $\Omega\cap B$. We have found the safety factor $\lambda^*=3.4713$ and the onset velocity v^* is plotted is Fig. 4 (right). We remark that the avalanche fracture is rather different form the dome over a thin film, computed in the first simulation. The fracture is close to the corners of the domain and avoids the circular obstacle B.

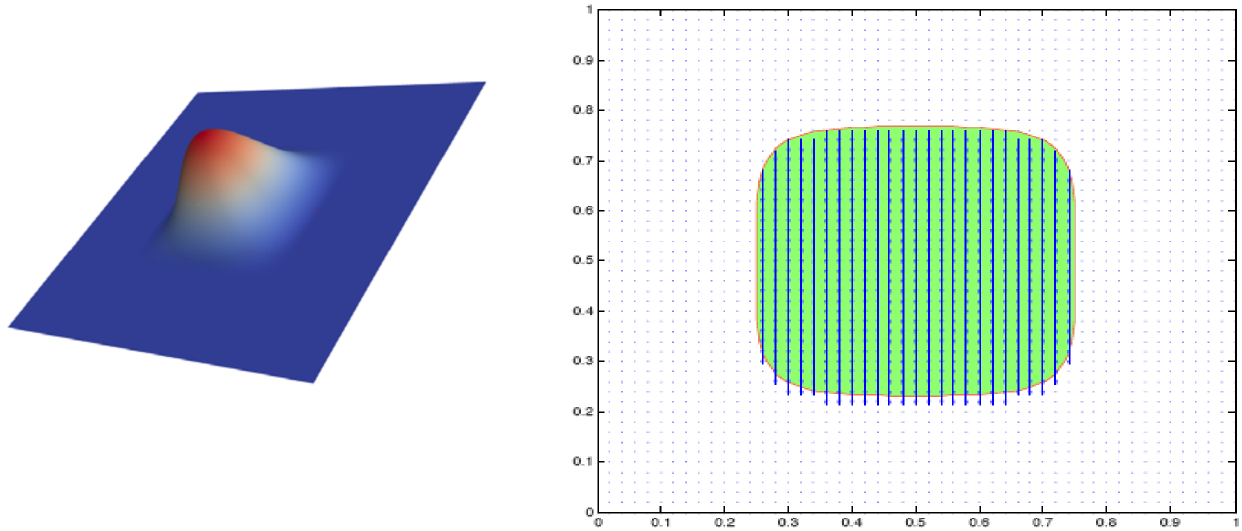

Fig. 3 – Left: the distribution of the thickness function for the square dome geometry; right: the velocity onset obtained with our approach.

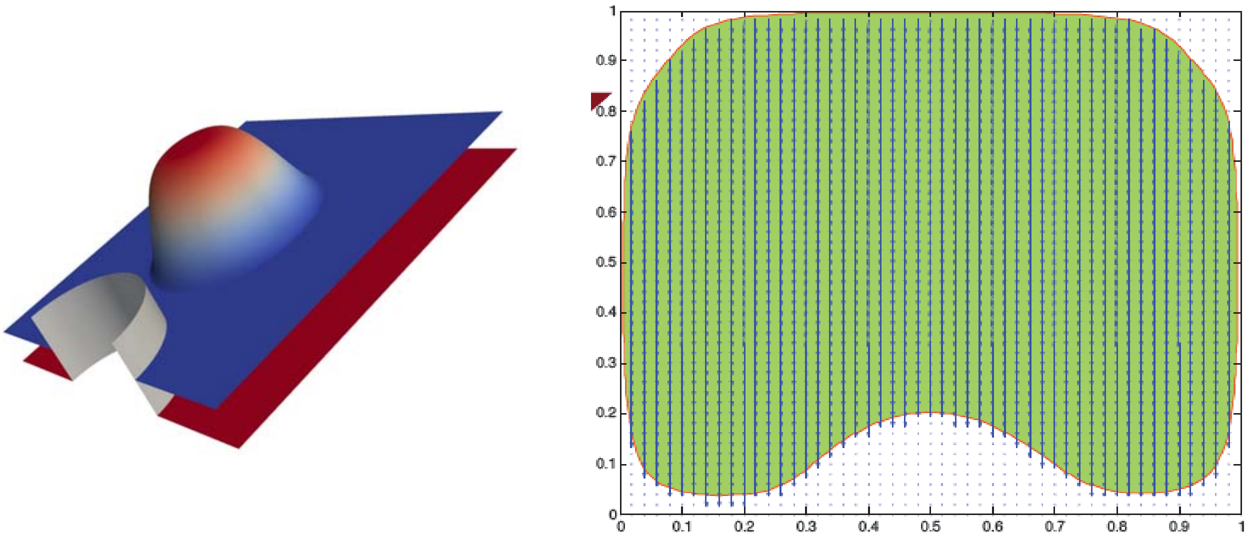

Fig. 4 – Left: the distribution of the thickness function for a thick fluid with an obstacle; right: the velocity onset obtained with our approach

ACKNOWLEDGEMENTS

This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0045. Support from POSDRU Project 82514 is also acknowledged. For the second author the research was financed through the project "Excellence Research Fellowships for Young Researchers", the 2015 Competition, founded by the Research Institute of the University of Bucharest (ICUB).

Note added in proof. The article [8] studies the modelling of the onset of a shallow avalanche (soils, snow or other geomaterials) over a general basal topography. To distinguish if an avalanche occurs or not, it is used a criterion similar with that from the present paper. The numerical DVDS method is adapted accordingly.

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Received January 27, 2015