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MAGNETIC FIELDS OF QUANTUM ORIGIN IN MAGNETARS' CRUST

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Abstract. The aim of the present paper is to study the behavior of charged particles evolving in the frozen magnetar's crust endowed with an extremely strong magnetic induction and a periodic electric field. The closed-form analytical solution to the Schrödinger equation is used to derive the conserved current density components. It turns out that an additional magnetic induction, of quantum origin, comes into play, its amplitude being given by the absolute value of some Mathieu's functions. In case of parametric resonances, this has a rapidly growing unstable behavior along the Ox axis and might be responsible for intriguing phenomena in magnetar's crust.

Key words: magnetars, Schrödinger equation, current components.

1. INTRODUCTION

In the last years, a tremendous interest has been manifested in studying the behavior of quantum particles evolving in various geometries of fields. In physically important cases with periodic external potentials, the fundamental Schrödinger equation casts into the so-called Mathieu's equation, whose solutions are the celebrated Mathieu functions [1]. In contrast with other categories, these special functions are presenting a high degree of complexity, due to the fact that one has to deal with a double dependence, with respect to the variable and to their parameters [2].

The huge number of important results coming from the Mathieu's equation makes the mission to compile a complete bibliography almost impossible and therefore we are calling the attention to a relatively small number of research papers.

Thus, let us mention the investigation of a free-electron laser (FEL) that uses a medium with a periodically modulated refractive index and is operating with growing modes in the forbidden regions of the Mathieu's equation parameters [3, 4].

In the context of processes concerned with the dynamic stabilization of ions in quadrupole fields, the particle's motion is again governed by the Mathieu's equation [5].

As a more exotic topic, Mohseni *et al*. [6] have focused on the dynamics of massive spinning test particle in plane gravitational waves. Non-trivial solutions characterizing the non-geodesic motion of a particle that has the spin vector orthogonal to the direction of a polarized harmonic gravitational wave have been found in terms of Mathieu functions. It has been suggested that spinning particles may exhibit parametric excitation by gravitational fields in the same manner this excitation takes place for dynamical systems.

Recently, an illustrative environment for the behavior of quantum particles has been represented by the magnetars [7], the most peculiar and violent celestial bodies. As an exotic type of neutron stars, with huge magnetic fields, these astrophysical objects are attractive from the scientific point of view since they allow us to detect and study processes in extreme conditions, not available elsewhere. It is no doubt that one deals with quite spectacular and unusual phenomena and several theoretical models have been proposed to describe the magnetic field structure and the particles behavior in the crust and in the core [8–13].

Following a pedagogical approach, the present paper focuses mainly on some aspects related to charged particles behavior in the frozen crust, pointing out significant differences from the terrestrial experiments. Thus, we start with the closed-form analytical solution to the Schrödinger equation describing the particle moving in a magnetostatic induction $\vec{B}_0 = B_0\vec{u}_z$ and an electric intensity $E_x = E_0 h(x)$. The function $h(x)$ is not unique, but the trigonometric form $h(x) = \sin\kappa x$ is an acceptable choice, since it satisfies the boundary (vacuum) condition $E_x(x=0) = E_x(x = L = \pi/\kappa) = 0$.

2. THE SCHRÖDINGER EQUATION

For a particle of mass m_0, the Schrödinger equation, written in terms of the $U(1)$-gauge covariant derivatives

$$D_\mu \Psi = \Psi,_\mu - \frac{i}{\hbar} q A_\mu \Psi, \quad D_t \Psi = \Psi,_t - \frac{i}{\hbar} q A_4 \Psi\ , \tag{1}$$

has the well-known form

$$\frac{\hbar^2}{2m_0} D_\mu D_\mu \Psi = \frac{\hbar}{i} D_t \Psi, \quad \mu = \overline{1,3}\ . \tag{2}$$

In the followings, let us consider a magnetostatic induction $\vec{B}_0$ parallel to Oz and a periodic electric intensity $E_x = E_0 \sin\kappa x$, where the parameter κ is the wavenumber of the electric field. Such a configuration is assumed to exist in magnetar's crust, soon after the crust forms and the astrophysical object can be treated as being stationary [11–13]. The components of the four-potential being

$$A_x = -B_0 y\ , \quad A_4 = -V_0 \cos\kappa x\ , \quad A_y = A_z = 0\ , \tag{3}$$

satisfying the Lorentz condition $\partial_i A^i = 0$, the general equation (2), with the variables separation

$$\Psi = \mathrm{N}\phi(x)\eta(\mathrm{y})\exp\left[\frac{i}{\hbar}\left(p_z z - Et\right)\right] \tag{4}$$

leads to the following system of decoupled differential equations

$$\begin{aligned} &\text{(a)} \quad \frac{d^2\eta}{dy^2} + \left[(2n+1)\frac{qB_0}{\hbar} - \left(\frac{qB_0}{\hbar}\right)^2 y^2\right]\eta = 0, \\ &\text{(b)} \quad \frac{d^2\phi}{dx^2} + \left[\frac{2m_0E}{\hbar^2} - \left(\frac{p_z}{\hbar}\right)^2 - (2n+1)\frac{qB_0}{\hbar} - \frac{2m_0}{\hbar^2} qV_0 \cos\kappa x\right]\phi = 0. \end{aligned} \tag{5}$$

In terms of the Hermite associated functions

$$\eta_n(\sigma) = \exp\left[-\frac{\sigma^2}{2}\right] H_n(\sigma), \quad \text{with} \quad \sigma = \sqrt{\frac{qB_0}{\hbar}}\, y = \frac{y}{l_B}\ , \quad l_B = \sqrt{\frac{\hbar}{qB_0}}, \tag{6}$$

one may write the general solution to Eq. (5 a) as the superposition

$$\eta(\sigma) = \sum_{n=0}^{\infty} C_n \exp\left[-\frac{\sigma^2}{2}\right] H_n(\sigma)\ ,$$

where C_n are suitable weight factors to make a convergent function. For

$$C_n = \frac{a^n}{n!}, \quad \text{with} \quad \sum_{n=0}^{\infty} C_n = e^a\ ,$$

where a is a real parameter, using the relation [14]

$$\sum_{n=0}^{\infty}\frac{a^n}{n!}H_n(\sigma)=\exp\left[-a^2+2a\sigma\right]$$

one finally ends up with

$$\eta(\sigma)=\exp\left[a^2-\frac{1}{2}(\sigma-2a)^2\right]. \tag{7}$$

In what it concerns the equation (5 b), with the change of variable $\kappa x=2\zeta$, this becomes

$$\frac{d^2\phi}{d\zeta^2}+\left\{\frac{8m_0}{\hbar^2\kappa^2}\left[E-\frac{p_z^2}{2m_0}-\left(n+\frac{1}{2}\right)\hbar\omega_c\right]-\frac{8m_0qV_0}{\hbar^2\kappa^2}\cos(2\zeta)\right\}\phi=0\text{ , where }\omega_c=\frac{qB_0}{m_0}.$$

The above equation can be identified with the Mathieu's equation, in its canonical form [14]

$$u''+\left[\alpha-2\beta\cos 2\zeta\right]u=0, \tag{8}$$

with

$$\begin{aligned}\alpha&=\frac{8m_0}{\hbar^2\kappa^2}\left[E-\frac{p_z^2}{2m_0}-\left(n+\frac{1}{2}\right)\hbar\omega_c\right],\\ \beta&=\frac{4m_0qV_0}{\hbar^2\kappa^2},\end{aligned} \tag{9}$$

whose solutions are the so-called Mathieu's functions

$$\phi=\left\{MC=\text{MathieuC}[\alpha,\beta,\zeta],\ \text{MathieuS}[\alpha,\beta,\zeta]\right\}. \tag{10}$$

Putting everything together, the wave function (4) is given by

$$\Psi=N\phi(x)\eta(\text{y})\exp\left[\frac{i}{\hbar}\left(p_z z-Et\right)\right],$$

with the normalization constant

$$N=\left[\frac{\kappa}{2\pi\hbar\omega_c l_B L_z}\right]^{1/2}. \tag{11}$$

As it is known from the general theory of the Mathieu's functions [14], the solutions of the differential equation (8) are of the form $u\sim e^{i\gamma z}f(z)$, where $f(z)$ is a periodic function, while the Mathieu Characteristic Exponent, γ , depends on the $(\alpha,\beta)-$values and may be real or imaginary.

In order to illustrate the exponentially growing modes, which appear for values of the Mathieu parameters, in Fig. 1, we are representing the absolute value of the MathieuC function, in terms of $w=\kappa x/2$, for β in both the stability and instability ranges.

For given γ and β , one gets an infinite discrete set of values of α , called the eigenvalues or the characteristic values. In the particular case $\alpha>>\beta$ (described by the thick plot in Fig. 1), which is typical for nano-samples, the resonance condition coming from the expansion

$$\alpha_N\approx N^2+\frac{1}{2\left(N^2-1\right)}+\cdots$$

is leading, in view of (9), to the familiar energy quantization law

$$E_{nN}\approx N^2\frac{\hbar^2\kappa^2}{8m_0}+\frac{p_z^2}{2m_0}+\left(n+\frac{1}{2}\right)\hbar\omega_c\,. \tag{12}$$

When the electric field comes into play and

$$\frac{\hbar^2\kappa^2}{8m_0}-\frac{qV_0}{2}<E<\frac{\hbar^2\kappa^2}{8m_0}+\frac{qV_0}{2} \tag{13}$$

the wave functions are exponentially growing as a result of parametric resonance (see, in Fig. 1, the thin plot) and the instability region, of width qV_0, gets larger once the electric field intensity is increasing. For a detailed discussion based on graphical representations of the absolute value of the Mathieu's functions for the parameters inside and outside the stability zones [11–13].

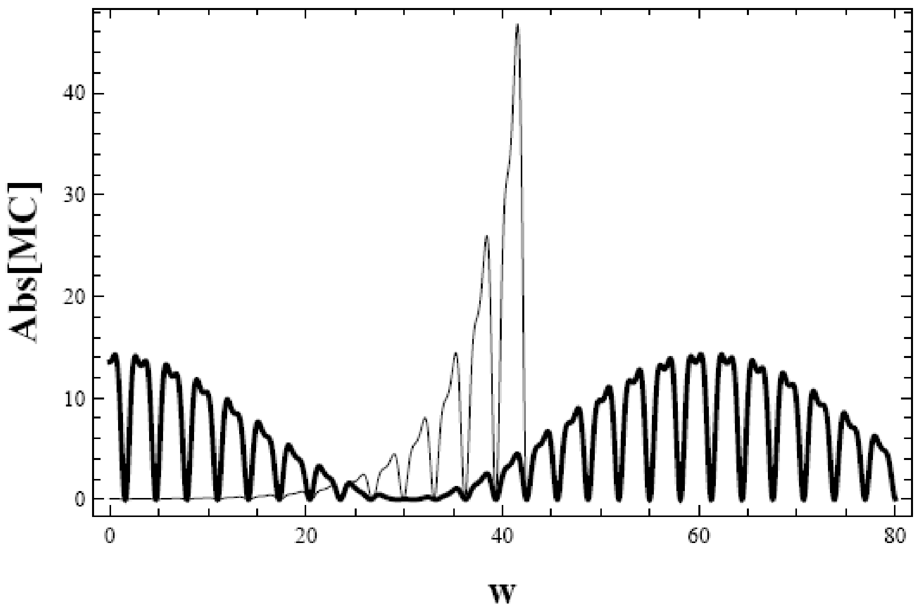

Fig. 1 – The absolute value of MathieuC, as a function of $w=\kappa x/2$, for the stability range (the thick plot) and for the entrance in the instability range (the thin plot).

In the case of a magnetar whose magnetic field is extremely strong, $B_0 \sim 10^{10-12}$ T, one has $l_B=\sqrt{\hbar/(qB_0)} << L=\pi/\kappa$, where L is the crust extension and therefore we can neglect the first term in the right hand side of the relation (12). Moreover, since the distance between the Landau levels is quite large, all particles, situated on the $n=0$ Landau level, have the energy

$$E\approx\frac{p_z^2}{2m_0}+\frac{\hbar\omega_c}{2}\,. \tag{14}$$

In the opposite situation $\beta >> \alpha$ which is the case of particles inside magnetar's crust, one may use the following expansion [15]

$$\alpha\approx -2\beta+4N\sqrt{\beta}\,, \tag{15}$$

that, in view of (9), becomes

$$E_N\approx\frac{p_z^2}{2m_0}+\frac{\hbar\omega_c}{2}-qV_0+N\hbar\kappa\sqrt{\frac{qV_0}{m_0}}\,. \tag{16}$$

Since the imaginary part of the Mathieu Characteristic Exponent has a dominant contribution, the absolute value $|\phi|^2$, with ϕ given in (10), is exponentially growing along *Ox*, as suggested by the thin plot in Fig. 1.

3. THE CONSERVED CURRENT COMPONENTS

Once the full expression of the wave function is known, one is able to compute the four-dimensional conserved current density components defined by

$$j_i = -\frac{\hbar}{2m_0}\left[iq\left(\Psi^* \partial_i \Psi - \Psi \partial_i \Psi^*\right) + \frac{2q^2}{\hbar} |\Psi|^2 A_i \right],$$

whose explicit forms are

$$\text{(a)} \quad j_x = \mathrm{N}^2 \left[q\eta^2 |\phi|^2 \frac{qB_0}{m_0} y - i\frac{q\hbar}{4m_0} \kappa \eta^2 \left(\phi^* \partial_\varsigma \phi - \phi \partial_\varsigma \phi^*\right)\right],$$

$$\text{(b)} \quad j_z = \mathrm{N}^2 q \eta^2 |\phi|^2 \frac{p_z}{m_0},$$

$$\text{(c)} \quad \rho = \mathrm{N}^2 \frac{q}{m_0 c} \eta^2 |\phi|^2 \left[E - qV_0 \cos(\kappa z)\right].$$

Obviously, if the momentum component along the magnetic induction, p_z, is set to zero, the current component j_z vanishes and one is left with the current along the direction of the electric intensity

$$J_x = L_z \int_{-L_y/2}^{L_y/2} j_x dy = \frac{q}{h} \kappa l_B |\phi|^2 \int_{-b/2}^{b/2} \eta^2 \sigma d\sigma , \tag{17}$$

with $b = L_y / l_B$ and $\eta(\sigma)$ given by (7).

In ultra-strong magnetic fields for which $L_y >> l_B$, as in magnetar's crust, one has

$$J_x \approx \frac{q}{h} \kappa l_B |\phi|^2 \int_{-\infty}^{\infty} \eta^2 \sigma d\sigma = \frac{q}{h} \kappa l_B |\phi|^2 a\sqrt{\pi} \exp\left[2a^2\right].$$

For the energy quantization law $E_n \approx n\hbar\omega_c$, the parameter a can be identified with the time-dependent function

$$a = \exp\left[-i\omega_c t\right]$$

so that one gets the following periodic current whose amplitude is expressed in terms of the Mathieu functions,

$$J_x = 2\sqrt{\pi} \frac{q}{h} \kappa l_B |\phi|^2 \cos\left(\omega_c t\right). \tag{18}$$

In the opposite case of nano-samples with $L_y < l_B = 25/\sqrt{B_0}$ (nm), this current component can also be neglected since the first nonzero contribution in the expansion with respect to $b = L_y / l_B$ is

$$J_x \approx \frac{q}{h} \kappa l_B \frac{b^3}{3} \cos\left(\omega_c t\right) \exp\left[-2\cos(2\omega_c t)\right] + O(b^4).$$

Let us remind the reader that the magnetic field evolution in the magnetar's crust of both isolated and accreting neutron stars is described by the basic equation [16]

$$\frac{\partial \vec{b}}{\partial t} = -\nabla \times \left[\frac{\vec{j}}{\sigma_0} + \frac{\vec{j}}{n_0 q} \times \vec{B} \right], \tag{19}$$

which encodes the Ohmic diffusion and the Hall-type contribution.

Thus, the current of quantum origin $j_x = \dfrac{J_x}{L_x L_y}$, with J_x given in (18), may generate, through the Hall-type term in (19), the following (periodic) additional magnetic induction

$$b_x = \sqrt{\pi}\,\frac{m_0}{Qh}\,\kappa^2 L_x l_B \,\frac{d}{d\zeta}\,|\phi|^2 \sin\left(\omega_c t\right), \tag{20}$$

where the total charge is

$$Q = qn_0 L_x L_y L_z = \int \rho dxdydz.$$

In comparison with the background field B_0, the induction b_z given in (20) has rapidly growing unstable modes along *Ox*, for ranges of the parameters where the imaginary part of the Mathieu Characteristic Exponent comes into play (Fig. 2b).

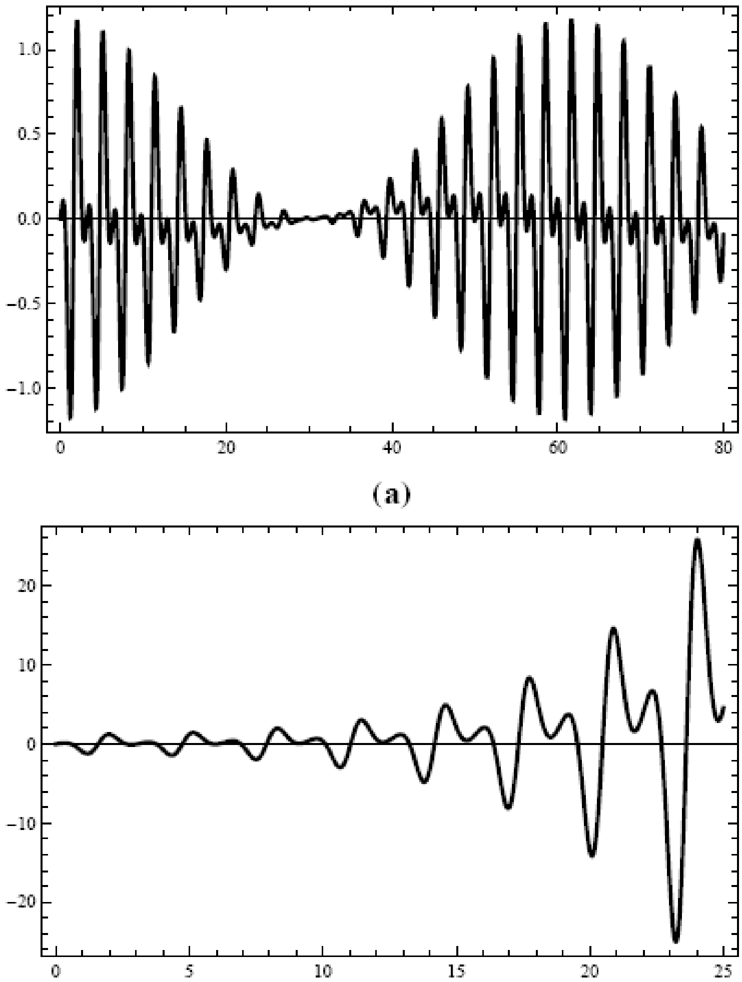

Fig. 2 – Generic representation of the additional magnetic induction (20), as a function of *w*, for:
a) the stability range; b) the instability one.

4. CONCLUSIONS

A motivation to analyze particles evolving in strong magnetic fields comes from the new and interesting results that have been revealed at the interplay between the physics of condensed matter and the physics of compact astrophysical objects. One can expect that, in neutron stars with a wide range of densities, from the density of iron nucleus at the surface to several times the normal nuclear matter density in the core, the existence of different types of particles, which can be studied with Schrödinger equation, has a significant influence on the star properties.

For describing the magnetar's crust, we have considered a background magnetic field parallel to *Oz*, while the electric field along *Ox* is bounded in *x* and can be taken as a periodic function of the form $E_x = E_0 \sin \kappa x$, with $\kappa L_x \approx \pi$. The solutions to the Schrödinger equation describing a (non-relativistic) quantum particle evolving in the crust, expressed in terms of the Hermite and Mathieu's functions, have allowed us to compute the conserved current density components.

The current (18), of quantum origin, is the source of a time-periodic additional magnetic induction (20), whose amplitude is given by the Mathieu's functions and their derivatives. Even though b_z is several orders of magnitude smaller than B_0, in the case of parametric resonances, this is exponentially growing along *Ox* (Fig. 2b).

Within a different approach, Hall drift induced instabilities that are modifying the field decay in compact astrophysical bodies have been assumed to be generated by a density of current $\vec{j} = \nabla \times \vec{b}$, where *b* is a small perturbation of the reference state B_0 [17]. Such instabilities are very important since the Hall timescale being several orders of magnitude faster than the Ohmic one, changes in the field structure (on Hall timescale) might give rise to a turbulent cascade, enhancing the efficiency of the total Ohmic energy decay.

REFERENCES

1. E. MATHIEU, *Memoire sur Le Mouvement Vibratoire d'une Membrane de forme Elliptique*, J. Math. Pures Appl., pp. 137–203, 1868.
2. L. RUBY, *Applications to the Mathieu equation*, Am. J. Phys., **64**, pp. 39–44, 1996.
3. M. V. FEDOROV, E. A. SHAPIRO, *A small-size free-electron LASER using a medium with periodically modulated refractive-index*, Laser Phys., **5**, pp. 735–738, 1995.
4. L. PROCIDA, H.–W. LEE, *Quantum dynamics of electrons in a free electron laser*, Opt. Commun., **49**, pp. 201–204, 1984.
5. W. PAUL, *Electromagnetic traps for charged and neutral particles*, Rev. Mod. Phys., **62**, pp. 53–540, 1990.
6. M. MOHSENI, R. W. TUCKER, C. WANG, *On the motion of spinning test particles in plane gravitational waves*, Class. Quantum Grav., **18**, pp. 3007–3017, 2001.
7. R. C. DUNCAN, C. THOMPSON, *Formation of very strongly magnetized neutron-stars–Implications for Gamma–Ray bursts*, Astrophys. J., **392**, L9, 1992.
8. A. K. HARDING, D. LAI, *Physics of strongly magnetized neutron stars*, Rep. Prog. Phys., **69**, pp. 2631–2708, 2006.
9. J. A. PONS, U. GEPPERT, *Confirmation of the occurrence of the Hall instability in the non-linear regime*, Astron. Astrophys., **513**, L12, 2010.
10. K. PARFREY, A. M. BELOBORODOV, L. HUI, *Twisting, Reconnecting Magnetospheres and Magnetar Spindown*, Astrophys. J., **754**, L12, 2012.
11. M. A. DARIESCU, C. DARIESCU, *Approximative Analytic Study of Fermions in Magnetar's Crust; Ultra-relativistic Plane Waves, Heun and Mathieu Solutions and Beyond*, Astrophys. Space Sci., **341**, pp. 429–435, 2012.
12. C. DARIESCU, M. A. DARIESCU, *A Toy Model for Scalars in Transient Magnetic Fields*, Mod. Phys. Lett. A, **26**, 1245, 2011.
13. O. BUHUCIANU, M. A. DARIESCU, C. DARIESCU, *Relativistic Bosons in Time-Harmonic Electric Fields*, Int. J. Theor. Phys., **51**, pp. 526–535, 2012.
14. I. S. GRADSHTEYN, I. M. RYZHIK, *Table of Integrals, Series, and Products*, 4th Ed., Academic, New York, 1965.
15. W. BECKERT, R. MECKBACHT, H. MITTER, *Motion of relativistic particles in standing wave fields*, J. Phys. A: Math. Gen., **12**, pp. 799–810, 1979.
16. M. RHEINHARDT, D. KONENKOV, U. GEPPERT, *The occurrence of the Hall instability in crusts of isolated neutron stars*, Astron. Astrophys., **420**, pp. 631–645, 2004.
17. M. RHEINHARDT, U. GEPPERT, *A Hall-drift induced magnetic field instability*, Preprint astro-ph/0202225.

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