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SOLITONS AND SHOCK WAVES TO ZAKHAROV-KUZNETSOV EQUATION WITH DUAL-POWER-LAW NONLINEARITY IN PLASMAS

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Abstract. We obtain solitary wave and other solutions to the Zakharov-Kuznetsov equation governed by dual-power-law nonlinearity. The travelling-wave hypothesis is applied to obtain the 1-soliton solution and the solution in series method reveals topological soliton solutions. Constraint conditions are identified in all these methods.

Key words: solitons, travelling-wave, integrability, solution in series method.

1. INTRODUCTION

The Zakharov-Kuznetsov (ZK) equation is one of the most important equations studied in the context of plasma physics and astrophysics [1–18]. This equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasmas in two dimensions [17]. While it is common to analyze the ZK equation with power-law nonlinearity, this paper is going to consider this equation governed by dual-power-law nonlinearity to keep it on a generalized setting. In the past, the *Ansatz* method was only implemented to extract the 1-soliton solution of this equation [2].

The ZK equation falls under the category of nonlinear evolution equations (NLEEs) in the context of mathematics and mathematical physics. In modern times there exists a plethora of integration tools available to integrate NLEEs. A couple of these powerful methods are for example the inverse scattering transform (IST), Hirota's bilinear method and other seemingly rare techniques [19–49]. This paper will adopt a few modern techniques to search for nonlinear wave solutions to this equation. The travelling-wave approach directly reveals a 1-soliton solution and the solution in series will illustrate other solutions, such as *topological solitons*. These are studied sequentially in subsequent sections.

2. MATHEMATICAL ANALYSIS

The ZK equation with dual-power-law nonlinearity is given by [2, 10]

$$q_t + (aq^n + bq^{2n})q_x + c(q_{xx} + q_{yy})_x = 0, \tag{1}$$

which is equivalent to

$$q_t + (aq^n + bq^{2n})q_x + c\nabla^2 q_x = 0. \tag{2}$$

In Eq. (1) the dependent variable $q(x, y, t)$ represents the profile of the wave, while x and y are the spatial variables and t is the temporal variable. The first term is the temporal evolution term, the second and third terms together constitute the nonlinear terms, where n represents the power-law nonlinearity parameter, hence the expression dual-power-law nonlinearity. The last two terms are the dispersion terms.

The solitons are the outcome of a delicate balance between dispersion and nonlinearity. The constants a, b, and c are the coefficients of the nonlinear and dispersion terms.

3. TRAVELLING-WAVE METHOD

For this method, the starting hypothesis is

$$q(x,y,t) = g(B_1 x + B_2 y - vt) = g(s), \tag{3}$$

where

$$s = B_1 x + B_2 y - vt. \tag{4}$$

In Eq. (1) the function $g(s)$ represents the wave profile. The parameters B_1 and B_2 are related to the inverse widths of the solitary wave in the x and y-directions, respectively, that is propagating with a velocity v. We substitute Eq. (3) into Eq. (1) and we integrate once and choose the constant of integration to be zero since the search is for a soliton solution. This results in

$$cB_1(B_1^2 + B_2^2)g'' = vg - \frac{aB_1}{n+1}g^{n+1} - \frac{bB_1}{2n+1}g^{2n+1}, \tag{5}$$

where $g' = dg/ds$ and $g'' = d^2g/ds^2$. Multiplication on both sides of Eq. (3) by g' and a further single integration after we again choose the constant of integration to be zero yields

$$(g')^2 = g^2 \left\{ \begin{array}{l} \dfrac{v}{cB_1(B_1^2+B_2^2)} - \dfrac{2a}{(n+1)(n+2)c(B_1^2+B_2^2)} g^n \\ - \dfrac{b}{(n+1)(2n+1)c(B_1^2+B_2^2)} g^{2n} \end{array} \right\}. \tag{6}$$

Separation of variables in Eq. (6) and integration leads to the 1-soliton solution of Eq. (1) to be

$$q = \frac{A}{\{D + \cosh[B(B_1 x + B_2 y - vt)]\}^{\frac{1}{n}}}, \tag{7}$$

where the amplitude A of the soliton is given by

$$A = \left[\frac{(n+1)^2(n+2)^2(2n+1)^2 v^2}{a^2 B_1^2 (2n+1)^2 + vbB_1(n+1)(n+2)^2(2n+1)} \right]^{\frac{1}{2n}}, \tag{8}$$

while the width B of the soliton is given by

$$B = n\sqrt{\frac{v}{cB_1(B_1^2+B_2^2)}}, \tag{9}$$

and the parameter D of the solution is

$$D = \frac{aB_1}{\{a^2 B_1^2(2n+1)^2 + vbB_1(n+1)(n+2)^2(2n+1)\}^{\frac{1}{2}}}. \tag{10}$$

These parameters introduce the constraint conditions

$$a^2 B_1^2 (2n+1)^2 + vbB_1(n+1)(n+2)^2(2n+1) > 0, \tag{11}$$

and

$$cv > 0. \tag{12}$$

Hence finally to conclude, the 1-soliton solution for Eq. (1) is given by Eq. (7). The amplitude A of the soliton is given by Eq. (8), while the parameters B and D are respectively given by Eqs. (9) and (10). These results introduce the parameter domain restrictions given by Eqs. (11) and (12) that must be valid for the existence of the soliton solution.

4. SOLUTION IN SERIES METHOD

CASE (i): $n = 1$. When $n = 1$, Eq. (5) reduces to

$$-vg + \frac{\alpha}{2} g^2 + \frac{\beta}{3} g^3 + \gamma g'' = 0, \tag{13}$$

where

$$\alpha = aB_1, \beta = bB_1, \gamma = cB_1(B_1^2 + B_2^2). \tag{14}$$

We employ the method of solution in series [19–21] by first finding the solution of the linear part of Eq. (13) in terms of real exponential functions. For example, the linear equation

$$\gamma\psi_{ss} - v\psi = 0,$$

gives the solution

$$\psi = e^{\pm Ks}, K(v) = \sqrt{\frac{v}{\gamma}}.$$

With the scaling $g = \frac{2v}{\alpha}\tilde{g}$, Eq. (13) can be written as

$$-v\tilde{g} + v\tilde{g}^2 + \sigma v\tilde{g}^3 + \gamma\tilde{g}'' = 0, \sigma = (\frac{4\beta}{3\alpha^2})v. \tag{15}$$

Following the solution of the linear part, we set $h(s) = e^{\pm Ks}$ and assume the solution of Eq. (15) in the form

$$\tilde{g}(s) = \sum_{k=1}^{\infty} a_k h^k(s). \tag{16}$$

Substitution of Eq. (16) into Eq. (15) gives rise to the recurrence relation for $k \geq 3$ as

$$(k^2 - 1)a_k + \sum_{l=1}^{k-1} a_{k-l}a_l + \sigma\sum_{m=2}^{k-1}\sum_{l=1}^{m-1} a_{k-m}a_{m-l}a_l = 0. \tag{17}$$

With a_1 arbitrary and $a_2 = -\frac{a_1^2}{3}$, the first few coefficients are

$$a_3 = -\frac{a_1^3}{8}(\sigma - \frac{2}{3}), a_4 = \frac{a_1^4}{12}(\sigma - \frac{2}{9}), \tag{18}$$

$$a_5 = \frac{a_1^5}{64}(\sigma^2 - \frac{20}{9}\sigma + \frac{20}{81}), a_6 = -\frac{a_1^6}{64}(\sigma^2 - \frac{20}{27}\sigma + \frac{20}{81}). \tag{19}$$

With $\sigma = -\frac{2}{9}$, we obtain

$$a_3 = \frac{a_1^3}{9}, a_4 = -\frac{a_1^4}{27}, a_5 = \frac{a_1^5}{81}, a_6 = -\frac{a_1^6}{243}. \tag{20}$$

The general term a_k is given as

$$a_k = \frac{(-1)^{k-1} a_1^k}{3^{k-1}}, k \geq 1. \tag{21}$$

Therefore

$$\tilde{g}(s) = -3\sum_{k=1}^{\infty}(-\frac{a_1 h}{3})^k = \frac{3dh}{1+dh}, d = \frac{a_1}{3}. \tag{22}$$

This provides a closed form for $\tilde{g}$ convergent for $dh < 1$, that is $s > \frac{\ln d}{K}$. By expanding $\tilde{g}$ in a series in powers of e^{Ks}, we obtain a similar closed form for $s < \frac{\ln d}{K}$.

Thus Eq. (22) holds for all $-\infty < s < \infty$.
Then

$$\tilde{g}(s) = \frac{3de^{-Ks}}{1+de^{-Ks}} = \frac{3e^{-Ks/2+\Delta}}{e^{Ks/2-\Delta} + e^{-Ks/2+\Delta}} = \frac{3}{2}\left[1 - \tanh(\frac{Ks}{2} - \Delta)\right], \Delta = \frac{1}{2}\ln d. \tag{23}$$

Thus the *topological soliton solution* of Eq. (3) is given as

$$q = \frac{3v}{\alpha}\left[1 - \tanh\left(\frac{1}{2}\sqrt{\frac{v}{\gamma}}(B_1 x + B_2 y - vt) - \Delta\right)\right]. \tag{24}$$

CASE (ii): $n = 2$. In this case, Eq. (5) reduces to

$$-vg + \frac{\alpha}{3}g^3 + \frac{\beta}{5}g^5 + \gamma g'' = 0, \tag{25}$$

where α, β, γ are given by Eq. (14).

With the scaling $g = \sqrt{\frac{3v}{\alpha}}\tilde{g}$, Eq. (25) can be written as

$$-v\tilde{g} + v\tilde{g}^3 + \tau v\tilde{g}^5 + \gamma\tilde{g}'' = 0, \tau = (\frac{9\beta}{5\alpha^2})v. \tag{26}$$

Again, following the solution of the linear part, we set $h(s) = e^{\pm Ks}$ and assume the solution of Eq. (26) in the same form

$$\tilde{g}(s) = \sum_{k=1}^{\infty} a_k h^k(s). \tag{27}$$

Substitution of Eq. (27) into Eq. (26) yields the recurrence relation for $k \geq 5$ as

$$(k^2 - 1)a_k + \sum_{m=2}^{k-1}\sum_{l=1}^{m-1} a_{k-m}a_{m-l}a_l + \tau\sum_{m=4}^{k-1}\sum_{l=3}^{m-1}\sum_{j=2}^{l-1}\sum_{k=1}^{j-1} a_{k-m}a_{m-l}a_{l-j}a_{j-k}a_k = 0. \tag{28}$$

We find $a_2 = 0$, $a_3 = -\frac{a_1^3}{2^3}$ and $a_4 = 0$. Here a_1 is arbitrary and is assumed to be positive. The first few coefficients are

$$a_5 = -\frac{a_1^5}{192}(8\tau - 3),\ a_6 = 0,\ a_7 = \frac{a_1^7}{512}(8\tau - 1). \tag{29}$$

With $\tau = -\frac{3}{16}$, we get

$$a_5 = \frac{3}{2^7}a_1^5,\ ,\ a_7 = -\frac{15}{3!2^9}a_1^7. \tag{30}$$

Therefore in general

$$a_{2k} = 0,\ ,\ a_{2k+1} = \frac{(-1)^k(2k-1)!!\,a_1^{2k+1}}{k!2^{3k}},\ k = 1, 2, 3, \ldots . \tag{31}$$

Here $(2k-1)!! = (2k-1)(2k-3)\ldots 5.3.1$. This will also hold true for $k = 0$ if we define $(-1)!! = 1$. Substitution of Eq. (31) into Eq. (27) results in

$$\tilde{g}(s) = 2\sum_{k=0}^{\infty}\frac{(-1)^k(2k-1)!!}{k!2^k}\left(\frac{a_1 h(s)}{2}\right)^{2k+1}, \tag{32}$$

which can be written as

$$\tilde{g}(s) = \frac{2dh}{\sqrt{1+d^2h^2}},\ d = \frac{a_1}{2}. \tag{33}$$

As in case (i), Eq. (33) holds for all $-\infty < s < \infty$. Thus

$$\tilde{g}^2(s) = \frac{4d^2h^2}{1+d^2h^2} = \frac{4e^{-Ks-\Delta}}{e^{Ks+\Delta} + e^{-Ks-\Delta}},\ \Delta = -\ln d, \tag{34}$$

and we have

$$\tilde{g}^2(s) = 2[1 - \tanh(Ks\Delta)]. \tag{35}$$

Thus the *topological soliton solution* of Eq. (3) is given as

$$q = \sqrt{\frac{6v}{\alpha}}\left[1 - \tanh\left(\sqrt{\frac{v}{\gamma}}(B_1 x + B_2 y - vt) - \Delta\right)\right]^{1/2}. \tag{36}$$

5. CONCLUSION

The Zakharov-Kuznetsov equation with dual power nonlinearity has been solved using the travelling wave hypothesis to derive 1-soliton solutions. The method of solution in series has been employed to derive topological soliton solutions in the special cases of $n = 1$ and $n = 2$. For other values of n we will investigate exact solutions of the Zakharov-Kuznetsov equation with dual power nonlinearity by using other methods. Foremost among them are the *mapping methods*. They give a variety of solutions in terms of Jacobi elliptic functions (JEFs) by a normal mapping method, a combination of JEFs with their reciprocals using a modified mapping method, and a combination of two different JEFs using an extended mapping method. This will be a topic for future research for special cases of the parameter n.

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