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# PERTURBATION WITH KERNELS OF MARKOVIAN RESOLVENTS

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**Abstract**. We study perturbations with kernels of the resolvent of a Markov process. We show that the perturbed resolvent is also associated with a Markov process and we present several examples.

***Key words***: resolvent of kernels, right Markov process, subordinate resolvent.

## 1. INTRODUCTION

We study the perturbation with a kernel $P$ of the resolvent family of a Markov process and we show that the perturbed resolvent is also associated with a Markov process. Formally, if $\mathscr{L}$ is the generator of the process $X$ then the generator of the new process is $\mathscr{L} - \mathscr{L} P$; for a more precise statement see assertion (v) in Section 3 below. At the level of potential kernels (formally, the invers operators of the generators), as in [2], we generalize and complete the classical construction of kernels satisfying the complete maximum principle (cf. [25] and [4]). This type of perturbation was studied in [4], in the case when the process $X$ is transient, and we use essentially the technique developed there.

The paper is organized as follows.

The main result (Theorem 2.2) is proved in the next section. It is preceeded by some preparations, formulated in terms of supermedian and excessive functions. Several proofs are presented in the Appendix of the paper, where we also collected basic facts on sub-Markovian resolvents of kernels and an existence result for a Markov process with a prescribed resolvent family. Finally, in Section 3 we present situations where our main result can be applied and we give several complementary comments.

## 2. THE MAIN RESULT

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a right Markov process with state space $E$, a Lusin topological space (i.e., $E$ is homeomorphic to a Borel subset of a compact metric space) and let $\mathscr{U} = (U_\alpha)_{\alpha>0}$ be its resolvent of kernels,

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f \circ X_t \mathrm{d}t, \qquad x \in E,\ f \in p\mathscr{B}.$$

For details see, e.g., [26, 4], and also Appendix A. Here $\mathscr{B}$ denotes the Borel $\sigma$-algebra on $E$ and $p\mathscr{B}$ the set of all positive, real-valued $\mathscr{B}$-measurable functions. For a familiy $\mathscr{G}$ of real-valued functions on $E$ we denote by $b\mathscr{G}$ (resp. $[\mathscr{G}]$) the set of all bounded functions from $\mathscr{G}$ (resp. the linear space spanned by $\mathscr{G}$).

We denote by $\mathscr{E}(\mathscr{U})$ the set of all $\mathscr{B}$-measurable $\mathscr{U}$*-excessive functions*: $u \in \mathscr{E}(\mathscr{U})$ if and only if $u$ is a nonnegative, numerical, $\mathscr{B}$-measurable function such that $\alpha U_\alpha u \le u$ for all $\alpha > 0$ (that is $u$ is an $\mathscr{U}$ - *supermedian function*) and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$. We denote by $\mathscr{S}(\mathscr{U})$ the set of all $\mathscr{U}$ - supermedian functions.

Recall that a kernel $T$ on $E$ is called *proper* provided that there exists $f_0 \in \mathscr{B}$, $0 < f_0 \le 1$, such that $Tf_0$ is a bounded function. The resolvent $\mathscr{U}$ is called *proper* if its initial kernel $U = \sup_{\alpha>0} U_\alpha$ is proper. A proper kernel $T$ is called $\mathscr{U}$*-supermedian* provided that $Tf$ is an $\mathscr{U}$-supermedian function for all $f \in p\mathscr{B}$.

Let further $P$ be an $\mathscr{U}$-supermedian kernel. For $q > 0$ define the kernel $P_q$ on $E$ as

$$P_q f := Pf - qU_q Pf \text{ for all } f \in p\mathscr{B} \text{ with } Pf < \infty.$$

*Remark 2.1.* $P_q$ is a $\mathscr{U}_q$-supermedian kernel.

The assertion follows since, using the resolvent equation, we have: if $w$ is $\mathscr{U}$-supermedian, $w < \infty$, then $w - qU_q w$ is $\mathscr{U}_q$-supermedian.

THEOREM 2.2. *Let $P$ be an $\mathscr{U}$-supermedian kernel such that the function $1 - P1$ is $\mathscr{U}$-excessive (e.g., $P$ is Markovian) and asume that for some q>0 the kernel* $\sum_{n=0}^{\infty} P_q^n$ is bounded. *For every $q > 0$ consider the kernel on $E$*

$$U'_q := \sum_{n=0}^{\infty} P_q^n U_q.$$

*Then the family of kernels $\mathscr{U}' = (U'_q)_{q>0}$ is the resolvent of a right Markov process with state $E$ and moreover*

$$U'_q = U_q + P_q U'_q \text{ for all } q > 0.$$

To prove Theorem 2.2 we need some preparations.

Let further $\mathscr{V} = (V_\alpha)_{\alpha>0}$ be a second resolvent of kernels on $(E, \mathscr{B})$; see $(A.1)$ in Appendix $A$ for equivalent conditions on the existence of a right process having $\mathscr{V} = (V_\alpha)_{\alpha>0}$ as associated resolvent.

The resolvent $\mathscr{U} = (U_\alpha)_{\alpha>0}$ is called *subordinate* to $\mathscr{V} = (V_\alpha)_{\alpha>0}$ provided that $U_\alpha \le V_\alpha$ for all $\alpha > 0$. If in addition the function $V_\alpha f - U_\alpha f$ is $\mathscr{V}_\alpha$-excessive for all bounded function $f \in p\mathscr{B}$ then $\mathscr{U}$ is called *exactly subordinate to* $\mathscr{V}$. Recall that it is the classical subordination in sense of P.A. Meyer (from [24]) we considered here; for the complete theory see e.g. [16], [26], and [4].

Always in the sequel the resolvent will be exactly subordinate but we write simply "subordinate".

If $q > 0$, we denote by $\mathscr{U}_q$ the sub-Markovian resolvent of kernels $\mathscr{U}_q = (U_{q+\alpha})_{\alpha>0}$. Clearly $\mathscr{U}_q$ is subordinate to $\mathscr{U}$.

Remark. The resolvent $\mathscr{U}$ is subordinate to the resolvent $\mathscr{U}'$ from Theorem 2.2.

The following results are versions of Lemma 5.2.1 and Proposition 5.2.2 from [4], for the case when the resolvent $\mathscr{U}$ is not necessary proper. We omit the proofs, since they are similar to those from the situation when $\mathscr{U}$ is proper.

(2.1) If $0 < q < \alpha$ and $n$ is a natural number then

$$P_q^n U_q = P_\alpha^n U_\alpha + (\alpha - q) \sum_{i+j=n} P_q^i U_q P_\alpha^j U_\alpha \text{ and } \sum_{i+j=n} P_\alpha^i U_\alpha P_q^j U_q = \sum_{i+j=n} P_q^i U_q P_\alpha^j U_\alpha.$$

(2.2) The family $\mathscr{U}' = (U'_q)_{q>0}$ from Theorem 2.2 is a resolvent of kernels on $E$ and

$$U'_q = U_q + P_q U'_q \text{ for all } q > 0.$$

In particular, $\mathscr{U}$ is subordinate to $\mathscr{U}'$. Due to this observation and following the terminology from [4], Section 5.2, the new resolvent $\mathscr{U}'$ is called "inverse subordinate" to $\mathscr{U}$.

Assertion (2.2) was also proved in [2], Theorem 1.

PROPOSITION 2.3. *The following assertions hold for any $q = 0$*:

(a) *If* $w \in \mathscr{S}(\mathscr{U}_q') \cap \mathscr{E}(\mathscr{U}_q)$ *then* $w \in \mathscr{E}(\mathscr{U}_q')$.

(b) *Let* $w \in \mathscr{E}(\mathscr{U}_q')$, $w < \infty$, *and suppose that* $\mathscr{U}'$ *is sub-Markovian. Then* $w - P_q w \in \mathscr{S}(\mathscr{U}_q)$.

(c) $P_q f \in \mathscr{S}(\mathscr{U}_q)$ *for all* $f \in p\mathscr{B}$ *such that* $Pf < \infty$.

(d) $1 - P1 \in \mathscr{S}(\mathscr{U})$ *if and only if* $P_q 1 \le 1 - qU_q 1$.

(e) $(P_\alpha)_q = P_{\alpha+q}$ *for all* $\alpha > 0$.

See Appendix $B$ for the proof.

The following two propositions are $q$-level versions, $q > 0$, of Proposition 5.2.3 from [4], in the non-transient case; their proofs are presented in the Appendix $B$. For related results see Theorem 3 from [2].

PROPOSITION 2.4. *The following assertions are equivalent for the resolvent* $\mathscr{U}'$ *from* (2.2).

(a) $1 \in \mathscr{E}(\mathscr{U}')$;

(b) $P1 \in \mathscr{E}(\mathscr{U})$ *and* $\mathscr{U}'$ *is sub-Markovian*;

(c) $1 - P1 \in \mathscr{E}(\mathscr{U})$;

(d) $1 - P_q 1 \in \mathscr{E}(\mathscr{U}_q)$ *for all* $q > 0$.

PROPOSITION 2.5. *If* $1 \in \mathscr{E}(\mathscr{U}')$ *and* $q > 0$ *then the following assertions hold.*

(a) $P_q f \in \mathscr{E}(\mathscr{U}_q)$ *for any* $f \in p\mathscr{B}$ *such that* $Pf < \infty$.

(b) $s \in \mathscr{E}(\mathscr{U}_q')$ *if and only if* $s \in \mathscr{E}(\mathscr{U}_q)$ *and* $s - P_q s \in \mathscr{E}(\mathscr{U}_q)$.

*In particular* $\mathscr{E}(\mathscr{U}_q') \subset \mathscr{E}(\mathscr{U}_q)$ *for all* $q \ge 0$.

*Proof* of Theorem 2.2. By $(A.1)$ in Appendix, we have to prove that $(A.1.b)$ holds for $\mathscr{U}'$.

- *Step* I. We show that condition $(A.1.b)\,(i)$ is verified for $\mathscr{U}'$. Because by hypothesis $1 - P1 \in \mathscr{E}(\mathscr{U})$, by Proposition 2.4 the resolvent $\mathscr{U}'$ is sub-Markovian and by Proposition 2.5 $(b)$ we have $\mathscr{E}(\mathscr{U}_q') \subset \mathscr{E}(\mathscr{U}_q)$. Let further $u, w \in \mathscr{E}(\mathscr{U}_q')$. Since $\inf(u,w)$ is $\mathscr{U}_q'$-supermedian and $\mathscr{U}_q$-excessive, and $U_q \le U_q'$ for all $q > 0$, by Proposition 2.3 $(a)$ it follows that $\inf(u,w) \in \mathscr{E}(\mathscr{U}_q')$. Using the same argument, since $1 \in \mathscr{E}(\mathscr{U}_q) \cap \mathscr{E}(\mathscr{U}_q')$ we get that $1 \in \mathscr{E}(\mathscr{U}_q')$.

We prove that $\mathscr{E}(\mathscr{U}_q')$ generates $\mathscr{B}$. Because $\mathscr{U}$ is the resolvent of a right Markov process, by $(A.1)$ $\mathscr{E}(\mathscr{U}_q)$ generates $\mathscr{B}$ and since $\mathscr{E}(\mathscr{U}_q') \subset \mathscr{E}(\mathscr{U}_q)$, we only have to show that $\mathscr{E}(\mathscr{U}_q) \subset \sigma(\mathscr{E}(\mathscr{U}_q'))$. Let $u \in \mathscr{E}(\mathscr{U}_q)$. By Hunt's Approximation Theorem ([4], Theorem 1.2.8) there exists a sequence $(f_n)_n \in bp\mathscr{B}$ such that $U_q f_n$ is bounded for all $n$ and converges increasingly to $u$. By (2.2) we get $U_q f_n = U_q' f_n - P_q U_q' f_n$ for all $n$ and $q > 0$. Using Proposition 2.5 $(a)$ $P_q U_q' f_n \in \mathscr{E}(\mathscr{U}_q)$ and since $P_q U_q' f_n - P_q P_q U_q' f_n = P_q U_q f_n \in \mathscr{E}(\mathscr{U}_q)$, by Proposition 2.5 $(b)$ we have $P_q U_q' f_n \in \mathscr{E}(\mathscr{U}_q')$. Therefore, $U_q f_n \in [b\,\mathscr{E}(\mathscr{U}_q')]$ for all $n$, $u \in \sigma(\mathscr{E}(\mathscr{U}_q'))$. We conclude that $(A.1.b)\,(i)$ holds for $\mathscr{U}'$.

- *Step* II. We *show* that $(A.1.b)\,(ii)$ also holds for $\mathscr{U}'$. Let $\mu$ be a finite positive measure on $(E, \mathscr{B})$ such that $\mu \circ U_q' \in Exc_{\mathscr{U}_q'}$ and $\xi \in Exc_{\mathscr{U}_q'}$, $\xi \le \mu \circ U_q'$. Consider $(\mu_n)_n$ an increasing sequence of positive measures on $(E, \mathscr{B})$ such that $\sup_n \mu_n \circ U_q' = \xi$ and observe that $\mu_n \circ U_q' \le \mu_{n+1} \circ U_q' \le \mu \circ U_q'$. For $q > 0$ we denote by $Q_q$ the kernel on $(E, \mathscr{B})$ given by

$$Q_q = \sum_{n=1}^{\infty} P_q^n,$$

and by hypothesis we may assume that $Q_q$ is a bounded kernel. Let $\nu_n = \mu_n \circ (I + Q_q)$ and $\nu = \mu \circ (I + Q_q)$. The equality $U'_q = U_q + Q_q U_q$ implies that $\nu_n \circ U_q = \mu_n \circ U'_q$, $\nu \circ U_q = \mu \circ U'_q$ and so $\nu_n \circ U_q \le \nu_{n+1} \circ U_q \le \nu \circ U_q$. Since $(A.1.b)$ $(ii)$ holds with respect to $\mathscr{U}$, there exists a positive finite measure $\theta$ on $(E, \mathscr{B})$ such that $\xi = \sup_n \nu_n \circ U_q = \theta \circ U_q \le \nu \circ U_q$. From $U_q f = U'_q f - P_q U'_q f$ and $0 \le \xi(f) = \theta(U'_q f) - \theta(P_q U'_q f)$ for all $f \in bp\mathscr{B}$, the real-valued functional $\theta'$ on $\mathscr{E}(\mathscr{U}'_q)$ defined as $\theta'(w) := \theta(w) - \theta(P_q w)$, $w \in \mathscr{E}(\mathscr{U}'_q)$, is positive and $\theta'(w) = \sup_n \mu_n(w)$. Consequently, $\theta'$ may be extended to a positive linear functional on the vector lattice $[b\mathscr{E}(\mathscr{U}'_q)]$ and therefore it is given by a positive measure on $(E, \mathscr{B})$ and $\theta = \theta \circ P_q + \theta'$. We conclude that $\xi = \theta' \circ U'_q$.

- *Step* III. We show that $(A.1.b)$ $(iii)$ holds for $\mathscr{U}'$. It is enough to prove that the fine topology with respect to $\mathscr{U}$ is weaker than the fine topology w.r.t. $\mathscr{U}'$. Let $u \in \mathscr{E}(\mathscr{U}_q)$, $u \le U_q 1$. By Hunt's Approximation Theorem, there exists a sequence $(f_n)_n \subset bp\mathscr{B}$ such that $U_q f_n$ is bounded for all $n$ and converges increasingly to $u$. We have $Q_q u \le Q_q U_q 1 \le U'_q 1$, hence $Q_q u$ is a bounded function. Since $U'_q f_n = U_q f_n + Q_q U_q f_n$ and $Q_q U_q f_n$ is bounded, we get that $w = u + Q_q u$, where $w \in \mathscr{E}(\mathscr{U}'_q)$ and $U'_q f_n$ converges increasingly to $w$. By Proposition 2.5 $(b)$ $Q_q U_q f_n \in b\mathscr{E}(\mathscr{U}'_q)$, hence $u = w - Q_q u \in [b\mathscr{E}(\mathscr{U}'_q)]$. The assertion follows using Lemma A from Appendix.

By the previous steps, conditions $(A.1.b)$ $(i)$, $(ii)$, and $(iii)$ hold for the resolvent $\mathscr{U}'$. The resolvent $\mathscr{U}$ is subordinate to $\mathscr{U}'$ since $P_q f \in \mathscr{E}(\mathscr{U}_q)$ and in particular $P_q U'_q f \in \mathscr{E}(\mathscr{U}_q)$ for all $f \in p\mathscr{B}$.

*Remark* 2.6. (*i*) Theorem 2.2 and its proof are extensions to the nontransient case of Proposition 5.2.4 from [4], allowing application to many situations for which the transience hypothesis is restrictive:

(*ii*) There is an alternative method to prove Theorem 2.2, by passing to the the $q$-level of its resolvent, once the result is known in the transient case, as follows. Recall that $\mathscr{U}_q$ is the bounded resolvent associated to the $q$-level subprocess of $X$. Apply the result from [4], the transient case, to the kernel $P_q$ and observe that the perturbed resolvent is $\mathscr{U}'_q$, so, it is bounded. Finally, apply again the perturbing result for the kernel defined as $P'_q = qU'_q$. Note that the function $1 - P'_q 1$ is clearly $\mathscr{U}'_q$-excessive.

(*iii*) If the initial proces $X$ is conservative (i.e. the resolvent $\mathscr{U}$ is Markovian) then by assertion $(d)$ of Proposition 2.3 we have $P_q = 0$, so, the kernel $P$ produces no perturbation in this case.

(*iv*) Perturbations with kernels of Markovian resolvents were studied in [2, 3, 4, 14, 17, 18, 20, 22]. In [21] it is considered a nonlinear type perturbation of the resolvent of the Monge-Ampère operator.

## 3. EXAMPLES AND COMPLEMENTS

(*i*) In Section 3.2 from [15] the process obtained perturbing with a kernel the resolvent of a given right process is studied from the point of view of the path regularity and of its negligible (polar) sets. The existence of this perturbed process is claimed under the boundedness assumption on the kernel $Q_q$, so, the statement of Theorem 2.2 is already stated (without proof) there.

(*ii*) A general result on the perturbation with kernels of the resolvent of a right process was also stated in Proposition 4.5 from [10], in order to specify the base process of a forthcoming measure-valued branching process, the main goal of that article. Note that the corresponding kernel $Q_q$ is bounded in that context. In addition, it is computed there the transition function of the perturbed process, solving an appropriate integral equation; this lead to several applications developed in [8, 9, 11, 13]. By (2.4) from [23] the method from [10] of obtaining the transition function of the perturbed process is applicable in our case here and completes the statement of Theorem 2.2.

(*iii*) The result from Theorem 2.2 was applied twice in [12]. First, in Proposition 4.5, where the kernel $P$ is produced by a (bounded) regular excessive kernel and second, in Theorem 4.10, in order to obtain a branching type Markov process with state space the set of all finite configurations of E. However, in both cases the resovent of the initial process is even bounded and the existence of the process is a consequence of the results from the transient case, proved in [4].

(*iv*) Assume that $\mathscr{L}$ is the generator of the given process $X$. We may regard its resolvent of kernels as a $C_0$-resolvent of sub-Markovian contractions on $L^p(m)$, with $p \in [1,\infty)$, where $m$ is a fixed excessive mesure. Let furher $\mu$ be a positive smooth measure charging no $m$-polar set (see [4] and [5] for details). In [5] and [6] it is solved a Schrödinger equation of the type $\mathscr{L}u + \mu u = 0$, using in particular a convenient Feynman-Kac formula. A perturbing kernel is produced by the regular strongly supermedian kernel associated with the measure $\mu$, using the Revuz correspondence. In the particular case when $\mu$ charges no $m$-semipolar set, under conditions ensuring that the perturbed resolvent is sub-Markovian (given in Proposition 4.3 from [5]), it is possible to apply Theorem 2.2, to show that the operator $\mathscr{L} + \mu$ is still the generator of a right Markov process. This is just an alternative approach for the existence of the process, because it is proved in [5] that the new resolvent is strongly continuous on $L^p(m)$ and thus it is possible to obtain the process from [7], Theorem 2.2.

(*v*) As in the above assertion (*iv*), let $\mathscr{L}$ be the generator of the given process $X$ on $L^p(m)$. Consider a kernel $P$ such that Theorem 2.2 may be applied and furthermore, the perturbed resolvent of kernels is also a $C_0$-resolvent of sub-Markovian contractions on $L^p(m)$ with generator $\mathscr{L}'$; observe that this happens in the example from assertion (iv). If in addition P is an operator on the domain of $\mathscr{L}$, then $\mathscr{L}' = \mathscr{L} - \mathscr{L}P$.

The perturbation with kernels is used in [1] and [14] in order to introduce (finite) jumps in the evolution of a Markov process, by modifying its Levy measure. The perturbing kernel is of the form $P := UN$, where $N$ is a component of the Levy kernel. With the notations from assertion (v) we have formaly $\mathscr{L}' = \mathscr{L} + N$. This equality was proved in Lemma 1.4 from [14].

In a forthcomming paper we intend to investigate further properties of the perturbed processes presented here.

## APPENDIX

### *Appendix A. Sub-Markovian resolvents of kernels*

Assume that $\mathscr{U} = (U_\alpha)_{\alpha>0}$ be s sub-Markovian resolvent of kernels on a Lusin measurable space and let $Exc_{\mathscr{U}}$ be the set of all $\mathscr{U}$*-excessive measures* on $E$: $\xi \in Exc_{\mathscr{U}}$ if and only if it is a $\sigma$-finite measure on $E$ such that $\xi \circ \alpha U_\alpha \le \xi$ for all $\alpha > 0$. Recall that if $\xi \in Exc_{\mathscr{U}}$ then actually $\xi \circ \alpha U_\alpha$ converges increasingly $\xi$ to as $\alpha \to \infty$. We denote by $Pot_{\mathscr{U}}$ the set of all *potential* $\mathscr{U}$-excessive measures $\xi \in Exc_{\mathscr{U}}$ such that $\xi \in Pot_{\mathscr{U}}$ if $\xi = \mu \circ U$, where $\mu$ is a $\sigma$-finite measure on $E$.

We recall now a result on the existence of a right process having a given sub-Markovian resolvent of kernels (see Corollary 1.8.12 in [4] and also [7] for the non-transient case).

($A.1$) The following assertions are equivalent for a sub-Markovian resolvent of kernels $\mathscr{U} = (U_\alpha)_{\alpha>0}$ on a Lusin topological space $E$.

($A.1.a$) $\mathscr{U}$ is the resolvent of a right process with state space $E$.

($A.1.b$) For one (and therefore for all) $q > 0$ we have:

(*i*) The convex cone $\mathscr{E}(\mathscr{U}_q)$ is stable for the pointwise infimum, generates the Borel $\sigma$-algebra $\mathscr{B}$, and $1 \in \mathscr{E}(\mathscr{U}_q)$;

(*ii*) If $\mu \circ U_q$ and $\xi$ are two $\mathscr{U}_q$-excessive measures and $\xi \le \mu \circ U_q$ then $\xi$ is also the potential of a measure; that is there exists a positive measure $\nu$ on $E$ such that $\xi = \nu \circ U_q$;

$(iii)$ Every open subset of $E$ is finely open with respect to $\mathscr{U}_q$; Recall that the *fine topology* (with respect to $\mathscr{U}_q$) is the topology on $E$ generated by all $\mathscr{U}_q$-excessive function, for some $q>0$.

$(A.2)$ If the resolvent $\mathscr{U}$ is proper then the above equivalence also holds for $q=0$.

The next result is a version of Proposition 3.1.1 from [19].

LEMMA A. *Let $u_0$ be a bounded $\mathscr{U}_q$-excessive function, $u_0>0$ and $q>0$. Then the fine topology associated with $\mathscr{U}$ is generated by $\{w\in\mathscr{E}(\mathscr{U}_q): w\le \beta u_0$ for some $\beta>0\}$.*

*Proof.* Let $\tau_0$ be the topology generated by $\{w\in\mathscr{E}(\mathscr{U}_q): w\le \beta u_0$ *for some* $\beta>0\}$ and let $w\in\mathscr{E}(\mathscr{U}_q)$. We may suppose that $w$ is finite, let $w_n:=\inf(w,nu_0)$ and $G_n:=[w<nu_0]$, $n\in\mathrm{N}$.
Since $\sup_n w_n=w$, $G_n=[w_{n+1}<nu_0]$, and $w_n$ is $\tau_0$-continuous, we have $\bigcup_n G_n=E$ and the set $G_n$ is $\tau_0$ open. Because $w=w_n$ on each $G_n$ we conclude that $w$ is $\tau_0$-continuous.

## *Appendix B*

*Proof of Proposition* 2.3. $(a)$ Using (2.2) and $w\in\mathscr{S}(\mathscr{U}_q')$ we have $\alpha U_{q+\alpha}w\le\alpha U'_{q+\alpha}w\le w$ and since $\alpha U_{q+\alpha}w$ converges increasingly to $w$ we get that $\alpha U'_{q+\alpha}w$ converges increasingly to $w$.

$(b)$ Because $w\in\mathscr{E}(\mathscr{U}_q')$ and $\mathscr{U}'$ is sub-Markovian, by Hunt's Theorem there exists a sequence $(f_n)_n\in p\mathscr{B}$ such that $U'_q f_n$ converges increasingly to $w$. By (2.2) we have $U_q f_n=U'_q f_n+P_q U'_q f_n$ for all $q>0$ and therefore there exists $\lim_{n\to\infty}U_q f_n=w-P_q w$. The assertion follows now since $U_q f_n\in\mathscr{S}(\mathscr{U}_q)$ for all $q>0$.

Assertion $(c)$ is clear, see Remark 2.1. Assertion $(d)$ is also clear, while $(e)$ is a consequence of the resolvent equation.

*Proof of Proposition* 2.4. $(c)\Leftrightarrow(d)$. Since $P_q f=Pf-qU_qPf$ for all $f\in p\mathscr{B}$, $Pf<\infty$, we get that $P1$ is $\mathscr{U}$-finely continuous if and only if $P_q1$ is $\mathscr{U}_q$-finely continuous. Thus, it is sufficient to show that $1-P1\in\mathscr{S}(\mathscr{U})$ if and only if $1-P_q1\in\mathscr{S}(\mathscr{U}_q)$ for all $q>0$. If $1-P1\in\mathscr{S}(\mathscr{U})$, then by Proposition 2.3 $(d)$ $P_{q+\alpha}1\le 1-(q+\alpha)U_{q+\alpha}1\le 1-\alpha U_{q+\alpha}1$, hence $(P_q)_\alpha 1\le 1-\alpha U_{q+\alpha}1$ for all $\alpha,q>0$. Therefore $1-P_q1\in\mathscr{S}(\mathscr{U}_q)$. Conversely, if $1-P_q1\in\mathscr{S}(\mathscr{U}_q)$ for all $q>0$, by Proposition 2.3 $(d)$ we have $P_{q+\alpha}1\le 1-\alpha U_{q+\alpha}1$ for all $\alpha>0$. Passing to the limit when $q\to 0$ we get $P_\alpha 1\le 1-\alpha U_\alpha 1$, $1-P1\in\mathscr{S}(\mathscr{U})$.

$(a)\Rightarrow(d)$. By Proposition 2.3 $(b)$ we have $1-P_q1\in\mathscr{S}(\mathscr{U}_q)$ for all $q>0$.

$(d)\Rightarrow(a)$. By Proposition 2.3 $(a)$ we have to show that $1\in\mathscr{S}(\mathscr{U}')$. Therefore, it is sufficient to prove that $\alpha\sum_{k=0}^{n}P_\alpha^k U_\alpha 1\le 1$ for all $\alpha>0$ and $n\in\mathbb{N}$, which is true reasoning by induction using $1\in\mathscr{E}(\mathscr{U}_\alpha)$ and $(d)\Rightarrow(c)$.

$(a)\Leftrightarrow(b)$. Observe first that if the resolvent $\mathscr{U}'$ is sub-Markovian then $1\in\mathscr{E}(\mathscr{U}')$. Indeed, since $1\in\mathscr{E}(\mathscr{U})$ we have $1=\sup_\alpha\alpha U_\alpha 1\le\sup_\alpha\alpha U'_\alpha 1\le 1$, $1\in\mathscr{E}(\mathscr{U}')$. By $(c)$ we have $P1=1-(1-P1)\in[\mathscr{E}(\mathscr{U})]$, hence $P1$ is $\mathscr{U}$-finely continuous and therefore $P1\in\mathscr{E}(\mathscr{U})$.

*Proof of Proposition* 2.5. $(a)$ For $q>0$, using Proposition 2.4 $(d)$ we have $1-P_q1\in\mathscr{E}(\mathscr{U}_q)$, hence $P_q1\in\mathscr{E}(\mathscr{U}_q)$. Since the resolvent $\mathscr{U}_q$ is bounded it turns out that $P_q f\in\mathscr{E}(\mathscr{U}_q)$ for all $f\in p\mathscr{B}$, $Pf<\infty$.

$(b)$ By Proposition 2.4, $(a) \Leftrightarrow (b)$, $\mathscr{U}'$ is sub-Markovian. Recall that the resolvents $\mathscr{U}_q, \mathscr{U}_q'$ are bounded and $\mathscr{U}_q$ is subordinate to $\mathscr{U}_q'$. The assertion follows by Proposition 5.2.3 from [4] (the transient case), applied for $\mathscr{U}_q$ and $\mathscr{U}_q'$.

## REFERENCES


1. R. F. BASS, *Adding and subtracting jumps from Markov processes*, Trans. Amer. Math. Soc., **255**, pp. 363–376, 1979.
2. H. BEN SAAD, *Exemples de noyaux admettant des resolvantes,* Math. Ann., **265**, pp. 149–154, 1983.
3. H. BEN SAAD, *Generalisation des noyaux $V_h$ et applications*, Séminaire de Théorie du Potentiel Paris, No. **7** (Lecture Notes in Math., **1061**), pp. 14–39, 1984.
4. L. BEZNEA and N. BOBOC, *Potential Theory and Right Processes*, Springer Series, Mathematics and its Applications, **572**, Kluwer, Dordrecht, 2004.
5. L. BEZNEA and N. BOBOC, *Feynman-Kac formula for left continuous additive functionals and extended Kato class measures*, Potential Anal., **30**, pp. 139–164, 2009.
6. L. BEZNEA and N. BOBOC, *Measures not charging polar sets and Schrödinger equations in $L^p$*, Acta Math. Sinica (English Ser.), **26**, pp. 249–265, 2010.
7. L. BEZNEA, N. BOBOC, AND M. RÖCKNER, Quasi-regular Dirichlet forms and $L^p$ -resolvents on measurable spaces, Potential Anal., **25**, pp. 269–282, 2006.
8. L. BEZNEA, M. DEACONU, and O. LUPASCU, *Branching processes for the fragmentation equation*, Stochastic Processes and their Applications, **125**, pp. 1861–1885, 2015.
9. L. BEZNEA, M. DEACONU, and O. LUPASCU, *Stochastic equation of fragmentation and branching processes related to avalanches,* J. of Statistical Physics, **162**, pp. 824–841, 2016.
10. L. BEZNEA and O. LUPASCU, *Measure-valued discrete branching Markov processes*, Trans. Amer. Math. Soc., **368**, pp. 5153–5176, 2016.
11. L. BEZNEA, O. LUPASCU, and A.-G. OPRINA, *A unifying construction for measure-valued continuous and discrete branching processes*, Complex Analysis and Potential Theory, CRM Proceedings and Lecture Notes, **55**, Amer. Math. Soc., (Providence, RI), pp. 47–59, 2012.
12. L. BEZNEA and A.-G. OPRINA, *Nonlinear PDEs and measure-valued branching type processes*, J. Math. Anal. Appl., **384**, pp. 16–32, 2011.
13. L. BEZNEA and A.-G. OPRINA, *Bounded and $L^p$-weak solutions for nonlinear equations of measure-valued branching processes*, Nonlinear Analysis: Theory, Methods & Applications, **107**, pp. 34–46, 2014.
14. L. BEZNEA and L. STOICA, *From diffusions to processes with jumps*., Prob. Theory and Math. Stat., Proceedings of the Sixth Vilnius Conference (1993), TEV/VSP, The Netherlands, 1994, pp. 53–74.
15. L. BEZNEA and G. TRUTNAU, *On the quasi-regularity of non-sectorial Dirichlet forms by processes having the same polar sets*, J. Math. Anal. Appl., **384**, pp. 33–48, 2011.
16. R.M. BLUMENTHAL and R.K. GETOOR, *Markov Processes and Potential Theory*, Academic Press, New York, 1968.
17. N. BOBOC and GH. BUCUR, *Perturbation in excessive structures,* in Lecture Notes in Math., **1014**, pp. 155–187, 1983.
18. N. BOBOC and GH. BUCUR, *Excessive and supermedian functions with respect to subordination resolvents of kernels*, Rev. Roum. Math Pures et Appl., **39**, pp. 875–878, 1994.
19. N. BOBOC, GH. BUCUR, and A. CORNEA, *Order and Convexity in Potential Theory: H-Cones*, Lecture Notes in Math., **853** (Springer-Verlag), 1981.
20. R. K. GETOOR, *Measure perturbations of Markovian semigroups*, Potential Analysis, **11**, pp. 101–133, 1999.
21. M. HAIDUCU and C. UDREA, *Perturbation with functions of the Monge-Ampere operator*, Proceedings of the Romanian Academy Series A, **15**, pp. 115–122, 2014.
22. F. HIRSCH, *Conditions nécessaires et suffisantes d'existence de résolvantes*, Z.Wahrsch. Verw. Geb., **29**, pp. 73–85, 1974.
23. O. LUPASCU, *Subordination in the sense of Bochner of $L^p$-semigroups and associated Markov processes*, Acta Math. Sinica., **30**, 2014, pp. 187–196.
24. P.-A. MEYER, *Fonctionnells multiplicatives et additives de Markov*, Ann. Inst. Fourier, Grenoble, **12**, pp. 125–230, 1962.
25. P.-A. MEYER, *Probability and potentials,* Ginn (Blaisdell), Boston, 1966.
26. M. SHARPE, *General Theory of Markov Processes*, Academic Press, Boston, 1988.


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