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A NONLINEAR FOURTH-ORDER DIFFUSION-BASED MODEL FOR IMAGE DENOISING AND RESTORATION

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Abstract. A nonlinear PDE-based image restoration approach is described in this article. The proposed filtering technique is based on a novel fourth-order diffusion model. Unlike many other fourth-order PDE denoising schemes, this nonlinear model provides an optimal trade-off between noise removal, image detail preservation and avoiding of undesired effects. A rigorous mathematical investigation of the well-posdedness of this differential model is also performed. Then, a consistent and fast converging numerical approximation scheme, based on the finite-difference method, is constructed for the fourth-order PDE. The denoising method proposed here outperforms not only the conventional image filters and those based on linear PDEs, but also many state of the art nonlinear second and fourth order diffusion-based techniques, as resulting from the method comparison.

Key words: image denoising and restoration, nonlinear diffusion, fourth-order PDE model, mathematical treatment, numerical approximation scheme, finite-difference method.

1. INTRODUCTION

The nonlinear partial differential equation (PDE) based techniques have been representing the main instrument for digital image denoising and restoration in the last century quarter. Since the conventional two-dimension image filters and the linear PDE-based denoising schemes are generating the blurring effect that destroys the edges and other image details [1], the nonlinear PDE models represent a much better smoothing solution [2]. They produce a directional diffusion that is degenerate along the gradient direction, having the effect of filtering the image along but not across its edges. The boundaries and other important features are thus preserved during the restoration process [2].

Nonlinear diffusion-based image restoration models can be divided into second- and fourth-order PDE-based approaches. The second-order PDE models have been widely used for restoration and scale-space image analysis since Perona and Malik elaborated their influential anisotropic diffusion scheme [3]. Because these nonlinear differential models can be obtained from variational problems, numerous variational denoising schemes have been developed since Rudin, Osher and Fetami introduced their TV Denoising technique [4].

While the second-order PDE-based techniques provide an effective deblurring, they could also produce the undesired staircase (blocky) effect, representing creating of flat zones separated by artifact boundaries [5]. So, a lot of improved PDE and variational models derived from Perona-Malik and TV Denoising schemes, such as the Split Bregman methods, have been proposed to solve this problem [2, 6–8]. We also developed such variational and nonlinear second-order anisotropic diffusion models that reduce considerably the image staircasing, which are disseminated in our past papers [9–12].

However, the nonlinear PDE schemes based on fourth-order diffusion represent a much better solution for staircase effect removing. Such a popular fourth-order PDE restoration model is that developed by Y.L. You and M. Kaveh [13], which approximates the observed image with a piecewise harmonic one. Another influential restoration approach based on fourth order diffusions was proposed by M. Lasaker, A. Lundervold and X. Tai [14].

While these fourth-order PDE denoising methods remove the Gaussian noise successfully and overcome the blocky effect, they may also generate image blurr and speckle noise. Many improved fourth-order PDE schemes that avoid these unintended effects have been elaborated in the last decade [15]. We also developed some fourth-order diffusion-based restoration schemes that improve You-Kaveh model [16, 17].

Here we consider a novel fourth-order nonlinear PDE-based image denoising technique that is totally different from You-Kaveh method. It performs an effective noise removal and overcomes all undesired effects. Also, the proposed approach outperforms some state of the art fourth-order PDE restoration models.

Our nonlinear diffusion approach is detailed in the following section. A rigorous mathematical investigation is performed on the well-posedness of this fourth-order PDE in the third section. Then, a robust finite difference-based numerical approximation scheme developed for this PDE model is described in the fourth section. The fifth section of this paper describes our restoration experiments and method comparison. This article finalizes with a conclusion section, some acknowledgements and a bibliography.

2. NOVEL FOURTH-ORDER DIFFUSION-BASED MODEL

In this section we propose a novel differential model for image denoising and restoration. The proposed nonlinear diffusion-based restoration model is composed of a fourth-order partial differential equation with some boundary conditions. Thus, our PDE-based smoothing technique is expressed as following:

$$\begin{cases} \xi \dfrac{\partial^2 u}{\partial t^2} + \gamma^2 \dfrac{\partial u}{\partial t} + \Delta\left(\varphi_u\left(\|\nabla u\|\right)\cdot \nabla^2 u\right) + \lambda\left(u - u_0\right) = 0 \\ \qquad u(0,x,y) = u_0(x,y) \qquad\qquad (x,y)\in\Omega \\ \qquad u_t(0,x,y) = u_1(x,y) \\ \qquad u(t,x,y) = 0, \quad \forall t \geq 0,\ (x,y)\in\partial\Omega \end{cases} \tag{1}$$

where function u represents an evolving image, parameters $\xi,\gamma,\lambda \in (0,1]$, $\partial\Omega$ is the frontier of domain $\Omega \subseteq R^2$, $\Delta = \nabla^2$ is the Laplacian operator, $\|\nabla u\|$ is the gradient magnitude, u_0 is the initial image corrupted by Gaussian noise and u_1 is a velocity modification of it. The diffusivity function $\varphi_u : [0,\infty) \to (0,\infty)$ has the form:

$$\varphi_u(s) = \delta \sqrt{\frac{\psi(u)}{\alpha \log 10\left(s^2 + \psi(u)\right)^k + \beta}} \tag{2}$$

where

$$\psi(u) = \left|\eta\mu\left(\|\nabla u\|\right) - \varepsilon\right|, \tag{3}$$

coefficients $\delta, \alpha, \eta \in (0,1.5)$, $k, \beta, \varepsilon \in [2,5)$ and $\mu(\cdot)$ returns the average value of the argument.

The diffusivity function given by (2)–(3) is properly chosen for an effective restoration process. Obviously, it is positive: $\varphi_u(s) > 0, \forall s \geq 0$. Also, it is monotonically decreasing, since $\varphi_u(s_1) = \delta\sqrt{\dfrac{\psi(u)}{\alpha \log 10\left(s_1^2 + \psi(u)\right)^k + \beta}} \leq \varphi_u(s_2) = \delta\sqrt{\dfrac{\psi(u)}{\alpha \log 10\left(s_2^2 + \psi(u)\right)^k + \beta}}, \forall s_1 \geq s_2$, and it converges to zero, since $\lim\limits_{s\to\infty} \varphi_u(s) = 0$, which is another denoising-related condition [2, 3]. Also, the function $\varphi_u(s)$ is bounded, since $\exists b_1, b_2 \geq 0 : b_1 \leq \varphi_u(s) \leq b_2, \forall s \geq 0$. It also represents a Lipschitz function, because its derivative $\varphi_u'(s)$ is bounded. These conditions are very important for the mathematical treatment of the proposed model that is performed in the third section and is related to its well-posedness.

The optimally filtered image u is achieved by solving the PDE model expressed by (1), representing its solution. Therefore, we have to investigate the existence and the uniqueness of such a solution for this fourth-order PDE scheme. Thus, the well-posedness of the proposed nonlinear diffusion model will be treated in the next section. So, we demonstrate that this partial differential equation admits a unique and weak solution that corresponds to the restored image. Also, our diffusion scheme has the localization property, its solution propagating with finite speed [18]. In the fourth section, one performs a discretization of the PDE model to approximate that solution.

3. MATHEMATICAL INVESTIGATION OF THE PDE SCHEME

A mathematical treatment of the PDE model (1) is performed in this section. We say that a function $u \in L^2(0,T;H^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$ is a weak solution to the equation (1) if $u_t = \frac{\partial u}{\partial t} \in C(0,T;L^2(\Omega))$ and

$$\forall t \in [0,T]: u_t(t) \in L^2(\Omega) \text{ and } u(t) \in L^2(\Omega)$$

$$\begin{cases} u(t,x,y) = \|\nabla u(t,x,y)\| = \Delta u(t,x,y) = 0, \forall t \in [0,T], (x,y) \in \partial\Omega \\ \xi \int_\Omega [u_t(t_2) - u_t(t_1)]\, fdxdy + \gamma^2 \int_\Omega [u(t_2) - u(t_1)]\, fdxdy + \int_{t_1}^{t_2} \left[\int_\Omega \varphi_u(\|\nabla u\|) \Delta u \Delta f dxdy \right] ds + \lambda \int_{t_1}^{t_2} \left[\int_\Omega (u - u_0)\, fdxdy \right] ds = 0, \\ \forall t_1, t_2 \in [0,T],\ f \in H^2(\Omega) \cap H_0^1(\Omega) \\ u(0,x,y) = u_0(x,y),\ u_t(0,x,y) = u_1(x,y),\ \forall (x,y) \in \Omega \end{cases} \quad (4)$$

Let us consider a bounded positive continuous function $a=a(t, x, y)$ and study the following equation:

$$\begin{cases} \xi \frac{\partial^2 u}{\partial t^2} + \gamma^2 \frac{\partial u}{\partial t} + \Delta(a \cdot \Delta u) + \lambda(u - u_0) = 0 \\ u(0,x,y) = u_0(x,y) \\ u_t(0,x,y) = u_1(x,y) \\ u(t,x,y) = \|\nabla u(t,x,y)\| = \Delta u(t,x,y) = 0, \quad \forall t \geq 0,\ (x,y) \in \partial\Omega \end{cases} \qquad (x,y) \in \Omega \quad (5)$$

Let us denote by $A: D(A) \subset L^2(\Omega) \to [D(A)]'$ the operator $\langle Au, v \rangle = \int_\Omega a \Delta u \Delta v dxdy + \lambda \int_\Omega auvdxdy \forall u, v \in D(A)$, where $D(A) = \{u \in H^2(\Omega); u = \|\nabla u\| = \Delta u = 0 \text{ on } \partial\Omega\}$ and $[D(A)]'$ is the dual space with respect to the pivot space $L^2(\Omega)$. We see that $\langle Au, v \rangle \geq 0, \forall u \in D(A)$, so A is monotone. Also, by the Schwartz inequality, we have: $|\langle A(u_n - u), v \rangle| \leq a|\Delta(u_n - u)|_2 |\nabla v|, \forall u, u_n, v \in D(A)$. It is clear that if $u_n \to u$ in $D(A)$, then $\lim_{n\to\infty} |\Delta(u_n - u)|_2 = 0$ and $\lim_{n\to\infty} |u_n - u|_2 = 0$. It follows that A is demi-continuous [19]. Finally, we see that $\|Au\|_{[D(A)]'} \leq C\|u\|_{D(A)}, \forall u \in D(A)$. We conclude that A is maximal monotone, then, via classical theory, the equation (2) has a unique solution u in $L^2(0,T;D(A))$ such that $u_t \in L^2(0,T;D(A))$ [20]. Then we get:

$$\frac{\xi}{2}\frac{d}{dt}|u|_2^2 + \gamma^2|u|_2^2 + \int_0^t \left(\int_\Omega a(\Delta u)^2 dxdy \right) ds + \lambda \int_0^t |u|_2^2 ds = \xi \int_\Omega u_1 u dxdy + \gamma^2 \int_\Omega u_0 u dxdy + \lambda \int_0^t \left(\int_\Omega u_0 u dxdy \right) ds \quad (6)$$

By $|\bullet|_2$ we have denoted the norm of $L^2(\Omega)$. The equality (6) implies that

$$\frac{\xi}{2}\frac{d}{dt}|u|_2^2 + \gamma^2|u|_2^2 + \lambda \int_0^t |u|_2^2 ds \leq C\left(|u_1|_2^2 + |u_0|_2^2\right) + \gamma^2|u|_2^2 + \lambda \int_0^t |u|_2^2 ds, \quad (7)$$

for some positive constant C. We get $\frac{\xi}{2}\frac{d}{dt}|u|_2^2 \leq C_1, \forall t \in [0,T]$ where the constant C_1 is independent of t, the solution u, and the function a. It depends only on the parameters ξ, γ, λ and the L^2- norm of u_0 and u_1. We also get that $|u(t)|_2^2 \leq C_2$ and the constant C_2 depends only on the parameters of the system and $|u_0|_2$ and $|u_1|_2$. Coming back to (6), we also get that

$$\int_0^t \left(\int_\Omega a(\Delta u)^2 dxdy \right) ds \leq C_3, \forall t \in [0,T] \quad (8)$$

with $C_3 > 0$ depending on ξ, γ, λ, $|u_0|_2$ and $|u_1|_2$. Now, by multiplying equation (5) by u_t, and integrating over [0, t] one obtains:

$$\xi|u_t|_2^2 + \frac{\gamma^2}{2}\frac{d}{dt}|u|_2^2 + \int_0^t\left(\int_\Omega a\Delta u(\Delta u)_s\right)ds + \lambda\frac{1}{2}|u|_2^2 = \xi\int_\Omega u_1 u_t dxdy + \gamma^2\int_\Omega u_0 u_t dxdy + \lambda\frac{1}{2}|u|_2^2 + \lambda\int_0^t\int_\Omega u_0 u_s ds \quad (9)$$

Since $\int_0^t\left(\int_\Omega a\Delta u(\Delta u)_s\right)ds = \frac{1}{2}\int_\Omega a(\Delta u)^2 - a(\Delta u_0)^2 - \int_0^t\int_\Omega a_s(\Delta u)^2 ds$

and $\int_0^t\int_\Omega u_0 u_s ds = \int_0^t \frac{d}{ds}\left(\int_\Omega u_0 u_s\right)ds = \int_\Omega u_0 u - \int_\Omega u_0^2$, we get $|u_t(t)|_2^2 \le C_4, \forall t \in [0,T]$, with the constant C_4 depending on ξ, γ, λ, $|u_0|_2$ and $|u_1|_2$, where we denote $(\Delta u)_s = \Delta u_s$. Now we are ready to construct the weak solution to (1). To this end, let us consider the sequence $(u_n)_n \in L(0,T;D(A))$, defined as:

$$\begin{cases} \xi\dfrac{\partial^2 u_n}{\partial t^2} + \gamma^2\dfrac{\partial u_n}{\partial t} + \Delta\left(\varphi_{u_{n-1}}\left(\|\nabla u_{n-1}\|\right)\Delta u\right) + \lambda\left(u_n - u_0\right) = 0 \\ u_n(0,x,y) = u_0(x,y) \\ (u_n)_t(0,x,y) = u_1(x,y) \\ u_n(t,x,y) = \|\nabla u_n(t,x,y)\| = \Delta u_n(t,x,y), \ (x,y) \in \partial\Omega. \end{cases} \quad (10)$$

Equation (10) has a solution for all $n \in N$, by replacing a to $\varphi_{u_{n-1}}(\|\nabla u_{n-1}\|)$. This follows by standard existence of the results for linear hyperbolic equation with time-dependent coefficients. By using (8), $|u_n(t)|_2^2 \le C_2$ and the Poincare inequality, we see that the sequence u_n is bounded in $u \in L^2(0,T;H^2(\Omega))$ and so, then exists $u \in L^2(0,T;H_0^1(\Omega))$ such that $u_n \to u$ in $L^2(0,T;H_0^1(\Omega))$, from where we also get that $u_n \to u$ in $L^2(0,T;L^2(\Omega))$, $u_n \to u$ in $L^2(0,T;H^2(\Omega))$, $u_n(t) \to u(t)$ in $L^2(\Omega)$ and $(u_n)_t(t) \to u_t(t)$ in $L^2(\Omega)$, $\forall t \in [0,T]$. Let us multiply the PDE in (10) by $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and integrate over $[t_1, t_2]$ to obtain:

$$\begin{aligned} &\xi\int_\Omega\left[(u_n)_t(t_2) - (u_n)_t(t_1)\right]fdxdy + \gamma^2\int_\Omega\left[u_n(t_2) - u_n(t_1)\right]fdxdy + \\ &+\int_{t_1}^{t_2}\left[\int_\Omega \varphi_{u_{n-1}}\left(\|\nabla u_{n-1}\|\right)\Delta u_n \Delta f dxdy\right]ds + \lambda\int_{t_1}^{t_2}\left[\int_\Omega (u_n - u_0)\, fdxdy\right]ds = 0 \end{aligned} \quad (11)$$

Making use of the mentioned convergence relations, the Lipschitz continuity of φ_u, we may pass (11) to the limit ($n \to \infty$) to arrive to the fact that u satisfies the third condition in (4), concluding so that u represents a weak solution to (1).

4. NUMERICAL APPROXIMATION ALGORITHM

Now we intend to approximate that unique and weak solution of the fourth-order PDE, whose existence has been demonstrated in the previous section. Therefore, the continuous nonlinear diffusion-based model is discretized by applying the finite-difference method [21]. A consistent numerical approximation scheme is thus constructed for this mathematical model. So, let us consider a space grid size of h and a time step Δt. The space and time coordinates are quantized as follows:

$$x = ih,\ y = jh,\ t = n\Delta t,\ \forall i \in \{0,..,I\},\ j \in \{0,.,J\}, n \in \{1,..,N\}. \quad (12)$$

The fourth-order diffusion equation given by (1) can be rewritten as:

$$\xi \frac{\partial^2 u}{\partial t^2} + \gamma^2 \frac{\partial u}{\partial t} + \Delta\left(\varphi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right) + \lambda\left(u - u_0\right) = 0. \tag{13}$$

Then, the equation (13) is discretized by using the finite differences [21]. The component $\xi \frac{\partial^2 u}{\partial t^2} + \gamma^2 \frac{\partial u}{\partial t} + \lambda\left(u - u_0\right)$ is approximated first as $\xi \frac{u_{i,j}^{n+\Delta t} + u_{i,j}^{n-\Delta t} - 2u_{i,j}^n}{\Delta t^2} + \gamma^2 \frac{u_{i,j}^{n+\Delta t} - u_{i,j}^{n-\Delta t}}{2\Delta t} + \lambda\left(u_{i,j}^n - u_{i,j}^0\right)$. The second component is discretized next by applying the finite difference-based discretization of the Laplacian operator. Therefore, we compute $\varphi_{i,j}^n = \varphi_u\left(\left\|\nabla u_{i,j}^n\right\|\right)\nabla^2 u_{i,j}^n$ for $n \in \{0,.., N\}$, where

$$\Delta u_{i,j}^n = \frac{u^n(i+h, j) + u^n(i-h, j) + u^n(i, j+h) + u^n(i, j-h) - 4u^n(i, j)}{h^2} \tag{14}$$

and

$$\varphi_u\left(\left\|\nabla u_{i,j}^n\right\|\right) = \varphi_u\left(\sqrt{\frac{\left(u_{i+h,j}^n - u_{i-h,j}^n\right)^2}{4h^2} + \frac{\left(u_{i,j+h}^n - u_{i,j-h}^n\right)^2}{4h^2}}\right) \tag{15}$$

is computed using (2). Then, we apply the Laplacian and get the discretization of the second component:

$$\nabla^2 \varphi_{i,j}^n = \Delta\left(\varphi_u\left(\left\|\nabla u_{i,j}^n\right\|\right)\nabla^2 u_{i,j}^n\right) = \frac{\varphi_{i+1,j}^n + \varphi_{i-1,j}^n + \varphi_{i,j+1}^n + \varphi_{i,j-1}^n - 4\varphi_{i,j}^n}{h^2}. \tag{16}$$

If we consider $h = 1$ and $\Delta t = 1$, the next implicit numerical approximation scheme is obtained for (13):

$$\xi\left(u_{i,j}^{n+1} + u_{i,j}^{n-1} - 2u_{i,j}^n\right) + \gamma^2 \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2} + \Delta\varphi_{i,j}^n + \lambda\left(u_{i,j}^n - u_{i,j}^0\right) = 0. \tag{17}$$

Then, (17) leads to the next explicit numerical approximation algorithm:

$$u^{n+1}(i, j) = \frac{4\xi - 2\lambda}{2\xi + \gamma^2} u^n(i, j) - \frac{2\xi - \gamma^2}{2\xi + \gamma^2} u^{n-1}(i, j) + \frac{2\Delta\varphi_{i,j}^n}{2\xi + \gamma^2} + \frac{2\lambda}{2\xi + \gamma^2} u^0(i, j), \forall n \in [1, N] \tag{18}$$

where $u^n(i, j) = u_{i,j}^n$. This iterative scheme is applied on the evolving image for each $n \in \{1,...,N\}$. The restoration process starts with $u^1 = u^0 = u_0$ representing initial $[I \times J]$ image corrupted by Gaussian noise. Our explicit numerical approximation scheme is consistent to the nonlinear diffusion model (1). For the optimal values of parameters ξ, γ, λ specified in (19), it also converges fast to the approximation of its weak solution, representing the optimal image denoising, u^{N+1}, the number of iterations, *N*, becoming quite low.

5. EXPERIMENTS AND METHOD COMPARISON

We have successfully tested the proposed nonlinear diffusion-based smoothing technique on hundreds of images affected by Gaussian noise. Some important image collections, such as the Volume 3 of the USC - SIPI database have been used in our denoising experiments. We have determined on a trial and error basis, through empirical observation, the next set of parameter values that provide an optimal image enhancement:

$$\delta = 0.5,\ \xi = 0.8,\ \gamma = 0.7,\ \lambda = 0.2,\ \alpha = 1.4,\ \eta = 0.3,\ k = 3,\ \beta = 4,\ \varepsilon = 3,\ N = 14. \tag{19}$$

The performed filtering experiments prove that our PDE-based denoising approach removes successfully the noise, while preserving some essential image details, such as the edges. It overcomes the undesired effects, like blurring, staircasing [5] and speckle noise [13]. The proposed denoising technique executes very fast, given the low value of the iterations number, *N*. Its running time is less than 1 second. The performance of the restoration algorithm has been assessed by using performance measures like Peak Signal-to-Noise Ratio (PSNR), Norm of the Error (NE) and Structural Similarity Image Metric (SSIM) [22].

The described nonlinear fourth-order PDE model outperforms numerous state-of-the-art restoration methods, producing much better values for the performance parameters. Our restoration approach performs

better than classic 2D image filters, such as Average, Median, Gaussian 2D, Wiener [1] and more effective filters like LLMMSE – Lee [23], and the linear PDE-based smoothing approaches, because it overcomes the unintended blurring effect and preserves essential features. Also, unlike those filtering methods, the proposed smoothing technique has the localization property. Also, our diffusion-based denoising method outperforms some influential nonlinear second-order PDE restoration schemes, such as both versions of the Perona-Malik anisotropic diffusion model and other derived algorithms [2,3], and the TV Denoising [4]. Thus, it provides a more effective Gaussian noise removal and executes much faster. Also, unlike these second-order differential models, it can succesfully overcome the staircase effect [5]. The described restoration technique is also more effective than some state of the art nonlinear fourth-order PDE denoising models. Our PDE-based filtering model performs better than some popular nonlinear fourth order PDE schemes, such as the isotropic diffusion model You-Kaveh [13] and the LLT denoising algorithm [14]. Besides avoiding the blurring effect and providing a better edge preservation than those techniques, our filtering algorithm removes successfully the unintended speckle noise [24] and operates much faster than them. The PSNR values achieved by the proposed technique and other restoration approaches mentioned here are registered in the following table. Our fourth-order diffusion model gets higher PSNRs than the other conventional and PDE-based approaches.

Some restoration results generated by these schemes are displayed in Fig. 1. Original $[512 \times 512]$ *Lenna* image is depicted in (a). It is corrupted with an amount of Gaussian noise characterized by μ = 0.21 and *var* = 0.02, the degraded image being displayed in (b). The smoothing results produced by $[3 \times 3]$ 2D filters, such as classic Gaussian filter, Average and Wiener, are displayed in (c) – (e). The LLMMSE-Lee based filtering result is depicted in (f). The restoration results provided by the PDE-based approaches are displayed in (g) – (j): Perona-Malik scheme, TV Denoising, You-Kaveh model and the technique proposed here. One can observe that image (j), marked in red and representing our restoration result looks better than other outputs. Also, the unintended denoising effects, still visible in (c) to (i), are almost completely removed in (j).

Table 1

PSNR values provided by several methods

This PDE scheme	Average	Gaussian	Wiener	LLMMSE-Lee	Perona-Malik	TV Denoising	You-Kaveh
28.05(dB)	25.43(dB)	25.27(dB)	26.51(dB)	27.63(dB)	26.84(dB)	26.79(dB)	27.31(dB)

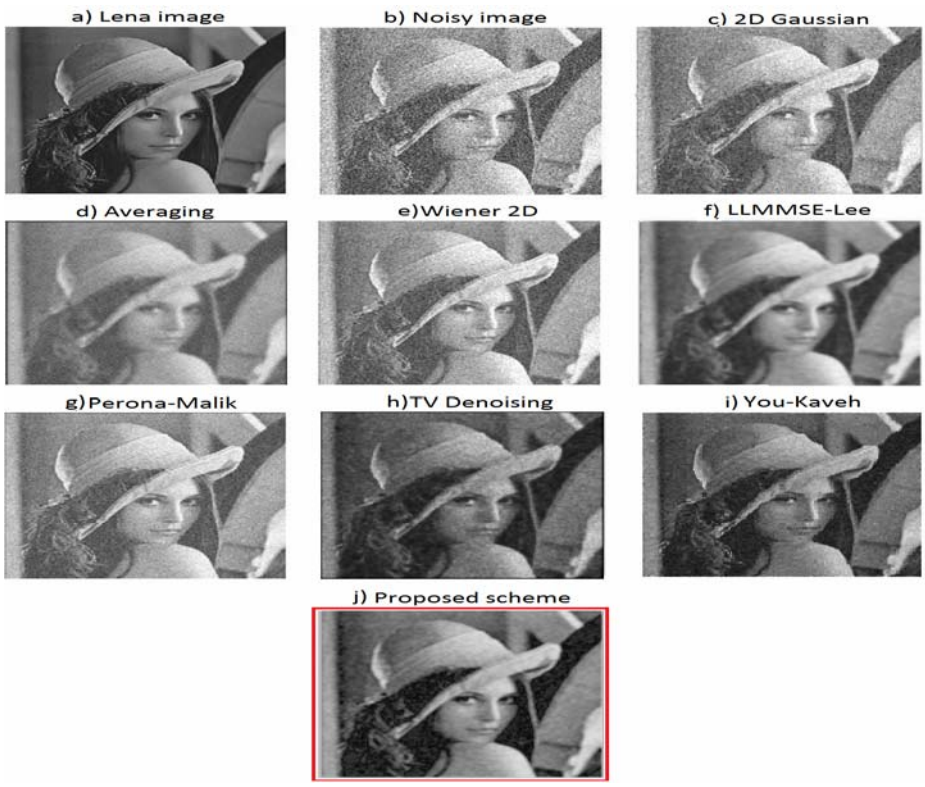

Fig. 1 – Restoration result achieved by various filtering techniques.

6. CONCLUSIONS

A novel PDE-based image restoration technique has been described in this article. The proposed denoising approach is based on a nonlinear fourth-order diffusion-based model that represents our main contribution. This original fourth-order PDE scheme has a much different form than well-known You-Kaveh model and other derived fourth-order PDEs, uses a new diffusivity function and special boundary conditions.

The rigorous mathematical treatment of the differential model represents another contribution of our work. We demonstrate that the diffusivity function is properly selected and the PDE scheme is well-posed, admitting a unique and weak solution. Then, a consistent fast-converging numerical approximation scheme is developed for this fourth-order PDE model. We apply the finite-difference method to get an explicit iterative discretization scheme for the continuous model. Its effectiveness is proved by the results of our experiments and method comparison. It achieves an optimal trade-off between noise reduction, feature preservation and unintended effect removal. Our technique outperforms the conventional filtering solutions, since it avoids the blurring effect. It performs better than many second-order PDE-based algorithms, by running faster and overcoming the staircasing, and outperforms the You-Kaveh like fourth-order PDE denoising models, by providing a better deblurring and speckle removal. We will focus on further improving this denoising technique during our future research. So, novel variants of the diffusivity function will be considered by us. Also we intend to integrate this PDE-based restoration scheme into more complex image enhancement solutions, such as some effective hybrid filtering models incorporating both second- and fourth-order PDEs.

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