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PSEUDO-PERIODIC SOLUTIONS OF A NEW HYPERCHAOTIC SYSTEM USING THE AVERAGING THEORY

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Abstract. We study analytically the zero-Hopf bifurcation at the singular point of a new hyperchaotic system. Using the averaging theory, we find the sufficient conditions such that one pseudo-periodic solution emerges at the bifurcation points. We use the numeric simulations to describe the stability of these orbits.

Key words: hyperchaotic system, zero-Hopf bifurcation, averaging theory, pseudo-periodic solutions.

1. INTRODUCTION

The hyperchaotic Rössler system was first presented by Rössler in 1979, which possesses more than one positive Lyapunov exponent [1]. Due to the characteristics of high capacity, high security, and high efficiency, the hyperchaotic systems have been broadly applied in nonlinear circuits, secure communications, and so on (see, e.g., Refs. [2–8] and references therein). Recently, control and stability of Hopf bifurcation has been researched extensively [9–18]. However, due to the complexity of the higher-dimensional systems, there are few works on the n-dimensional zero-Hopf bifurcation with $n > 3$ [19, 20].

Here we approach a new hyperchaotic system [21, 22], from a dynamical system point of view. In particular, we investigate a four-dimensional zero-Hopf equilibrium (that is, an isolated equilibrium with double zero eigenvalues and a pair of purely imaginary eigenvalues) of the new hyperchaotic system. Since the hyperchaotic system generates multiple positive Lyapunov exponents, its dynamics is hard to predict and control. As far as we know there are no rigorous analytic studies of the existence of periodic solutions for the dynamic system described by Eq. (1). In this paper, we study the periodic orbit that bifurcates in the zero-Hopf bifurcation equilibrium of the differential system (1), using the averaging method [17–19], which is a mature method widely used.

The rest of this paper is arranged as follows. In Section 2, we consider the existence and stability of pseudo-periodic solutions of the new hyperchaotic system. Then numerical simulations of the solutions obtained in Section 2 are given in Section 3. Finally, a brief conclusion is given.

2. PSEUDO-PERIODIC SOLUTIONS

Here we consider a new hyperchaotic system in the form

$$\begin{aligned} \dot{x} &= a(x-y) - yz + w, \\ \dot{y} &= -by + xz, \\ \dot{z} &= -cz + dx + xy \\ \dot{w} &= -e(x+y), \end{aligned} \tag{1}$$

where x, y, z, and w represent dynamical variables, and a, b, c, d, and e are parameters. When $a = 2.45$, $b = 9$, $c = 5$, $d = 0.06$, and $e = 1.4$, the system (1) has the Lyapunov exponents [22]: $\lambda_1 = 0.6916$, $\lambda_2 = 0.1911$, $\lambda_3 = -0.0002$, and $\lambda_4 = -17.5453$. The two positive Lyapunov exponents indicate that system (1) is hyperchaotic. The projections of system (1) are displayed in Fig. 1.

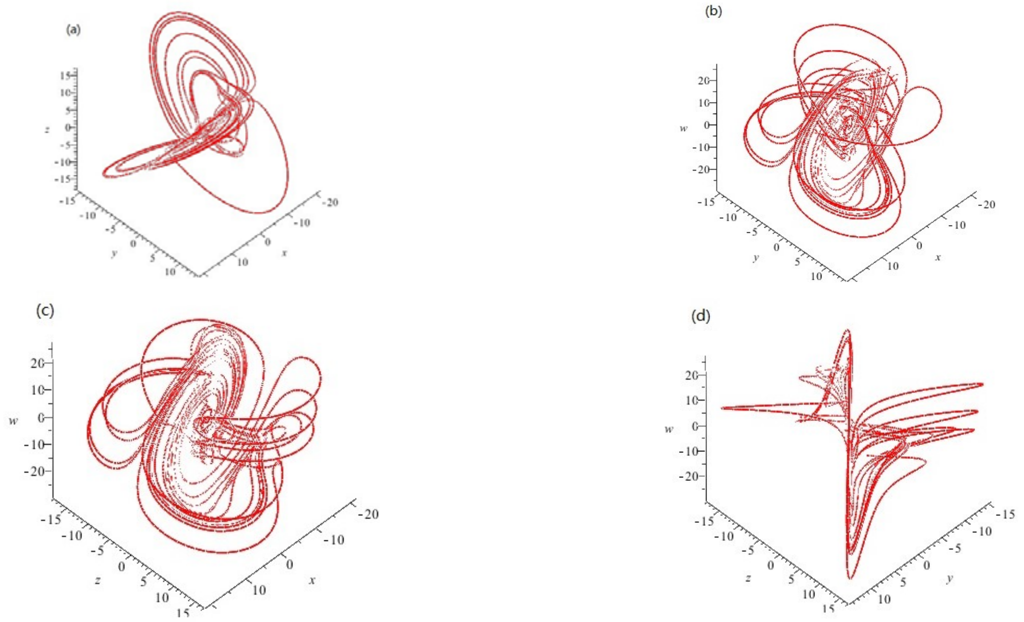

Fig. 1 – Phase portraits of system (1) with the initial value $(x_0, y_0, z_0, w_0) = (0.5, 0.5, 0.5, 0.5)$:
a) the *x*-*y*-*z* space; b) the *x*-*y*-*w* space; c) the *x*-*z*-*w* space; d) the *y*-*z*-*w* space.

Obviously, the origin of coordinates of R^4 is always an equilibrium for the hyperchaotic system (1). The system (1) has other two equilibriums, if $d^2+4bc \geq 0$. Without loss of generality, here we only consider the case that the equilibrium point is (0, 0, 0, 0).

The Jacobian matrix at the origin point (0, 0, 0, 0) is

$$A=\begin{pmatrix} a & -a & 0 & 1 \\ 0 & -b & 0 & 0 \\ d & 0 & -c & 0 \\ -e & -e & 0 & 0 \end{pmatrix}.$$

It is not hard to get that the eigenvalues of A are $\lambda_1 = -b$, $\lambda_2 = -c$, and $\lambda_{3,4} = [a \pm (a^2-4e)^{1/2}]/2$. According the conditions for the origin to be a zero-Hopf equilibrium, the parameters must satisfy $a = b = c = 0$, and $e > 0$. Then the eigenvalues are 0, 0, and $\pm\, \mathrm{i}e^{1/2}$.

Remark 1. When $a = b = c = 0$, and $e > 0$, then there exists a two parameter family of the new hyperchaotic system (1) for which the origin of coordinates is a zero-Hopf equilibrium point. Moreover, the eigenvalues at the origin for this two parameter family are 0, 0, and $\pm\, \mathrm{i}e^{1/2}$.

Based on the averaging theory for periodic orbits [17–19], let $(a, b, c) = (\varepsilon a_1, \varepsilon b_1, \varepsilon c_1)$, where a_1, b_1, and c_1 are real nonzero numbers, and $\varepsilon > 0$ is a sufficiently small parameter. Then the new hyperchaotic system (1) becomes

$$\begin{aligned} \dot{x} &= \varepsilon a_1(x-y) - yz + w, \\ \dot{y} &= -\varepsilon b_1 y + xz, \\ \dot{z} &= -\varepsilon c_1 z + dx + xy \\ \dot{w} &= -e(x+y). \end{aligned} \tag{2}$$

We rescale the variables $(x, y, z, w) = \varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon W)$, and denote the new variables (X, Y, Z, W) by (x, y, z, w). Then the system (2) can be rewritten as follows:

$$\begin{aligned} \dot{x} &= \varepsilon a_1(x-y) - \varepsilon yz + w, \\ \dot{y} &= -\varepsilon b_1 y + \varepsilon xz, \\ \dot{z} &= -\varepsilon c_1 z + dx + \varepsilon xy \\ \dot{w} &= -e(x+y). \end{aligned} \tag{3}$$

Using the averaging method, we study the behavior of system (3), which is rewritten in the following form:

$$s = F_0(t,s) + \varepsilon F_1(t,s) + \varepsilon^2 F_2(t,s,\varepsilon). \tag{4}$$

Here $s = (x, y, z, w)$ is a vector of four variables,

$$F_0(t,s) = \begin{pmatrix} w \\ 0 \\ dx \\ -e(x+y) \end{pmatrix}, \quad F_1(t,s) = \begin{pmatrix} a_1(x-y) - yz \\ -b_1 y + xz \\ -c_1 z + xy \\ 0 \end{pmatrix}$$

and $F_2(t, s, \varepsilon) = 0$. First, we consider the initial value problem of the unperturbed system

$$\begin{aligned} \dot{s} &= F_0(t,s), \\ s(0,\varepsilon) &= (x_0, y_0, z_0, w_0) = z. \end{aligned} \tag{5}$$

So we can get the fundamental matrix $M_z(t)$ of the linearization system (5):

$$M_z(t) = \begin{pmatrix} -\cos(tr) & \cos(tr)-1 & 0 & \sin(tr) \\ 0 & 1 & 0 & 0 \\ \dfrac{d\sin(tr)}{r} & \dfrac{d\sin(tr)-dtr}{r} & 1 & -\dfrac{d\cos(tr)-d}{r^2} \\ -r\sin(tr) & -r\sin(tr) & 0 & \cos(tr) \end{pmatrix},$$

where $r=e^{1/2}$. The solution of the system (5) is $s\,(t, z, \varepsilon) = (x(t), y(t), z(t), w(t))$, presented as following

$$\begin{aligned} x(t) &= x_0\cos(tr) + (\cos(tr)-1)y_0 + \frac{w_0\sin(tr)}{r}, \\ y(t) &= y_0, \\ z(t) &= \frac{dx_0\sin(tr) + d(\sin(tr)-tr)y_0}{r} + z_0 - \frac{d(\cos(tr)-1)w_0}{r^2}, \\ w(t) &= -r\sin(tr)x_0 - r\sin(tr)y_0 + \cos(tr)w_0. \end{aligned} \tag{6}$$

It is easy to find that the solutions $x(t)$, $y(t)$, and $w(t)$ are periodic and admit the same period $T = 2\pi/r$. However, the terms in $z(t)$ are periodic except for the term dy_0t. In the following, we study both the existence and the stability of pseudo-periodic solutions.

Step* 1**. ***The existence of pseudo-periodic solutions

We find the inverse matrix of $M_z(t)$ in the form

$$M_z^{-1}(t) = \begin{pmatrix} -\cos(tr) & \cos(tr)-1 & 0 & -\sin(tr) \\ 0 & 1 & 0 & 0 \\ -\dfrac{d\sin(tr)}{r} & -\dfrac{d\sin(tr)-dtr}{r} & 1 & -\dfrac{d\cos(tr)-d}{r^2} \\ r\sin(tr) & r\sin(tr) & 0 & \cos(tr) \end{pmatrix}.$$

We compute the following integral:

$$\begin{aligned} F(z) &= \frac{1}{T}\int_0^T M_z^{-1}(t,z)F_1(t,s(t,z))dt \\ &= \frac{r}{2\pi}\int_0^{\frac{2\pi}{r}} M_z^{-1}(t,z)F_1(t,s(t,z))dt = (F_1(z), F_2(z), F_3(z), F_4(z)). \end{aligned}$$

Here

$$F_1(z) = \frac{7y_0dw_0 + 6y_0z_0r^2 + 4b_1y_0r^2 + 2dx_0w_0 - 2x_0y_0d\pi - 6y_0^2dr\pi + 2a_1y_0}{4r^2},$$

$$F_2(z) = \frac{y_0(-z_0r + y_0d\pi - b_1r^2)}{r},$$

$$F_3(z)=\frac{12c_1r^3dy_0\pi-12dr^3y_0z_0\pi-6dr^2a_1w_0-12dr^3b_1y_0\pi-15w_0^2d^2-12c_1r^4z_0-12r^4y_0^2}{12r^4}+$$
$$+\frac{18w_0d^2y_0r\pi-18dw_0z_0r^2+16d^2r^2y_0^2\pi^2+18d^2r^2y_0^2-3r^2x_0^2d^2-12c_1r^2dw_0-9d^2r^2y_0x_0}{12r^4},$$

$$F_4(z)=-\frac{-2w_0^2d-2w_0z_0r^2-2r^2a_1w_0+11y_0^2r^2d+3y_0r^2dx_0+2y_0rdw_0\pi}{4r^4}.$$

Solving the nonlinear system given by $F(z)=0$ and assuming $a_1=b_1$ (just only for computing), we can get that the system above has four solutions as follows

$$s_1=(0,0,0,0),\ s_2=(\frac{2r\sqrt{c_1b_1}}{d},0,-b_1,0),\ s_3=(\frac{\sqrt{x_0^2d^2+4dw_0b_1r^2+4c_1r^4b_1}}{rd},0,-\frac{dw_0+r^2b_1}{r^2},w_0),$$
$$s_4=\sqrt{-\frac{9b_1c_1}{d^2+3d^2\pi^2-9r^2}}(\frac{11r}{3},\ r,\ d\pi-b_1/\sqrt{-\frac{9b_1c_1}{d^2+3d^2\pi^2-9r^2}},0).$$

The solution s_1 corresponds to the original equilibrium. However the solutions s_2 and s_3 are nonsingular, $\det(\partial F/\partial z)(s_2)=\det(\partial F/\partial z)(s_3)=0$. For the fourth solution s_4, we note that

$$F(s_4)=0,\ \det(\partial F/\partial z)(s_4)=-\frac{9b_1^2c_1^2d^2}{8(8d^2+3d^2\pi^2-9r^2)}.$$

Remark 2. If $db_1c_1(8d^2+3d^2\pi^2-9e)\neq 0$, and $a_1=b_1$, there exists one T-pseudo-periodic solution $s(t,\square)$ of the system (3) such that $s(t,\varepsilon)$ shrinks to s_4 as $\varepsilon\to 0$, where its T-pseudo period is $T=2\pi/r$.

Step* 2. *The stability of pseudo-periodic solutions

The characteristic polynomial of matrix $\det(\partial F/\partial z)(s_4)$ is

$$f(\lambda)=\lambda^4+c_1\lambda^3-\frac{b_1c_1(871d^2+96d^2\pi^2-288r^2)}{16(8d^2+3d^2\pi^2-9r^2)}\lambda^2+\frac{9b_1c_1d^2(48b_1-47c_1)}{16(8d^2+3d^2\pi^2-9r^2)}\lambda-$$
$$-\frac{9b_1^2c_1^2d^2}{8(8d^2+3d^2\pi^2-9r^2)}.$$

With the stability condition of the periodic solutions from the averaging theory, the coefficients of $f(\lambda)$ should satisfy

$$c_1>0,\qquad \frac{b_1c_1(871d^2+96d^2\pi^2-288r^2)}{16(8d^2+3d^2\pi^2-9r^2)}<0,$$
$$\frac{9b_1c_1d^2(48b_1-47c_1)}{16(8d^2+3d^2\pi^2-9r^2)}>0,\ \frac{9b_1^2c_1^2d^2}{8(8d^2+3d^2\pi^2-9r^2)}>0.$$

Remark 3. The necessary (not sufficient) conditions for the stable pseudo-period orbit are:

$$c_1>0,\quad 0<b_1<\frac{47}{48}c_1,\quad \frac{(3\pi^2+8)d^2}{9}<r^2<\frac{(96\pi^2+871)d^2}{288}.$$

Because the calculation is so complicated due to so many parameters, without loss of generality here we set $c_1=1$ and $b_1=1/2$. With the aid of the Maple software, we can obtain that the range of other parameters is $(3\pi^2+8)d^2/9<r^2<(96\pi^2+871)d^2/288$. When $d=1$, we give a few eigenvalues of $\det(\partial F/\partial z)(s_4)$ according to the parameter r (that is $e^{1/2}$):

$r_1=2.07:$
$-0.32941045+9.4122658\text{i},-0.051165395,-0.29001369,-0.3294045-9.4122658\text{i},$

$r_2=2.23$
$-0.21318776+1.3085306\text{i},-0.047949886,-0.52567457,-0.21318776-1.3085306\text{i},$

$r_3=2.31$
$-0.09192023+0.9000627\text{i},-0.046563546,-0.76959598,-0.09192923-0.9000627\text{i}.$

Obviously, all the imaginary parts of the above-mentioned eigenvalues are negative. It indicates that we can find the following result marked as *Remark* 4.

Remark 4. Let $c_1 = 1$, $b_1 = 1/2$, and $(3\pi^2+8)d^2/9 < r^2 < (96\pi^2+718)\, d^2/329$, then the pseudo-periodic orbit is stable.

Finally, the pseudo-periodic solutions $s = (x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon), w(t, \varepsilon))$ of system (3) generates the pseudo-periodic solutions $\varepsilon s = (\varepsilon x(t, \varepsilon), \varepsilon y(t, \varepsilon), \varepsilon z(t, \varepsilon), \varepsilon w(t, \varepsilon))$ of system (2). Since the pseudo-periodic solutions tend to the equilibrium point (0, 0, 0, 0) as $\varepsilon \to 0$, it thus corresponds to a zero-Hopf bifurcation at the zero-Hopf equilibrium point.

3. NUMERICAL SIMULATIONS OF THE PSEUDO-PERIODIC SOLUTIONS

Without loss of generality, according our main results (see Remarks 1–4), we choose the parameters in system (2) as $(a_1, b_1, c_1, d, e, \varepsilon) = (1, 1, 2, 1, 2.2^2, 10^{-5})$. Figures 2a – 2d illustrate the numerical simulations of the pseudo-period solutions of system (2) in the *x-y-z* space, *x-y-w* space, *x-z-w* space, and *y-z-w* space, respectively.

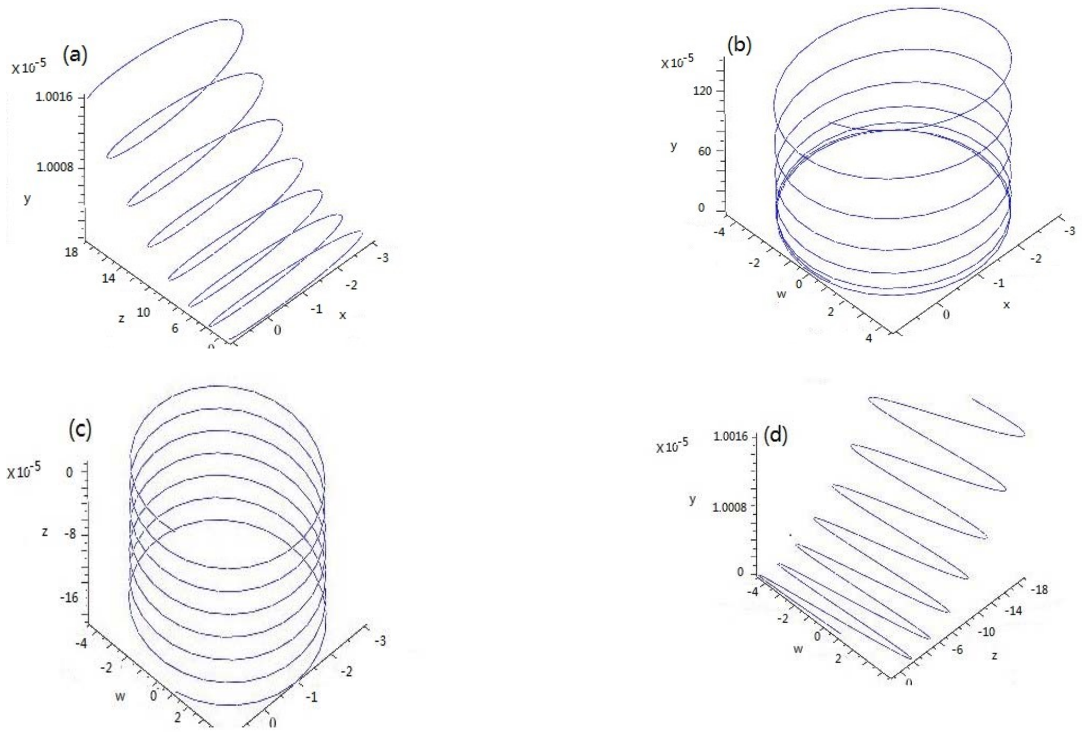

Fig. 2 – Phase portraits of system (2) with initial value $(x_0, y_0, z_0, w_0) = (10^{-5}, 10^{-5}, 10^{-5}, 10^{-5})$: a) the *x-y-z* space; b) the *x-y-w* space; c) the *x-z-w* space; d) the *y-z-w* space. The parameters are $a_1 = 1$, $b_1=1$, $c_1 = 2$, $d = 1$, $e = 2.2^2$, and $\varepsilon = 10^{-5}$.

4. CONCLUSIONS

In conclusion, we have investigated the zero-Hopf bifurcation in a new hyperchaotic system. Based on the averaging theory and symbolic computation, pseudo-periodic solutions are found and parameter conditions (necessary conditions and special sufficient conditions) are given for stable orbits of the new hyperchaotic system. Finally, numerical simulations are used to illustrate the corresponding stable orbits.

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