THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 18, Number 3/2017, pp. 199–206

EFFICIENT PSEUDOSPECTRAL SCHEME FOR 3D INTEGRAL EQUATIONS

Mohamed A. ABDELKAWY[1, 3], Eid H. DOHA[2], Ali H. BHRAWY[3], Ahmed Z.A. AMIN[4]

[1] Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), College of Science, Department of Mathematics and Statistics, Riyadh, Saudi Arabia
[2] Cairo University, Faculty of Science, Department of Mathematics, Giza, Egypt
[3] Beni-Suef University, Faculty of Science, Department of Mathematics, Beni-Suef, Egypt
[4] Institute of Engineering, Canadian International College, Department of Basic Science (CIC), Giza, Egypt
Correspondent author, Ali H. Bhrawy, E-mail: alibhrawy@yahoo.co.uk

Abstract. This paper presents a new pseudospectral technique to solve three dimensional integral equations (3D-IEs). The shifted Legendre Gauss-Lobatto collocation method is investigated to approximate the 3D-IEs. The main idea in the novel algorithm is to reduce the 3D-IEs to systems of algebraic equations. The applicability of the present method is examined by several numerical examples. By means of these numerical examples, we ensure that the present technique is a simple and very accurate numerical scheme for solving 3D-IEs.

Key words: three dimensional integral equations, pseudospectral method, Gauss-Lobatto quadrature.

1. INTRODUCTION

Integral equations (IEs) [1–5] play a useful role in many applications in various areas, including computational physics, mathematical chemistry, electrochemistry, semiconductors, seismology, scattering theory, heat conduction, fluid flow, metallurgy, population dynamics, optimal control theory, and mathematical economics [4–6] As the increasing of employing IEs in many social and scientific fields, the main challenge we confront is that of obtaining solutions for them. Unfortunately, for most of these IEs, no one is able to obtain analytic solutions for such problems. Therefore, the creation, improvement, and development of numerical methods for solving IEs have received considerable attention in recently years [7–13].

In many areas of sciences such as biology, engineering, economics, physics and others, several high-order numerical methods have been developed to deal with the related problems. Among these algorithms, *spectral methods* [14–22] are widely applied as powerful techniques in the construction of numerical solutions for differential, fractional differential or IEs. The spectral method is based on expressing the trial solution for the corresponding equation as a finite sum of a certain set of orthogonal polynomials. In the next step, the coefficients of the pproximate solution must be evaluated in order to minimize the error. In the *pseudospectral method* [23–30], the numerical solution must satisfy the proposed problem whenever possible. This means that at certain collocation points, the residuals may be letting to be zero.

This paper develops the shifted Legendre Gauss-Lobatto collocation (SL-GL-C) technique in order to efficiently solve three classes of 3D-IEs, namely 3D-Volterra-Fredholm integral equations (3D-VFIEs), 3D-Fredholm integral equations (3D-FIEs) and 3D-Volterra integral equations (3D-VIEs). For the 3D-IEs, we choose the shifted Legendre Gauss-Lobatto (SL-GL) quadrature points as collocation nodes. The numerical solution of the 3D-IE is approximated as a finite sum of Legendre polynomials, and then we evaluate the residuals of the mentioned problem at the SL-GL quadrature points. Consequently, a system of algebraic equations is obtained. The accuracy of the novel algorithm is confirmed through several numerical examples.

The paper is laid out as follows. We present some mathematical preliminaries in Section 2. In Sections 3, 4, and 5, we report a new numerical technique for the numerical solution of the 3D-VFIEs, 3D-FIEs, and 3D-VIEs, respectively. Section 6 implements the present algorithm on some examples to explore its accuracy and efficiency. Finally, in Section 7 we outline the main conclusions.

2. MATHEMATICAL PRELIMINARIES

Here, we list some approximation results for the SL-GL interpolation, which will be of most importance in the next Sections. The Legendre polynomials $P_k(x)$ $(k=0,1\ldots)$ possess the following Rodrigue's formula

$$P_k(x) \;=\frac{(-1)^k}{2^k\,k!}D^k((1-x^2)^k). \tag{1}$$

Accordingly, $P_i^{(r)}(x)$ (the r th derivative of $P_i(x)$) is evaluated by

$$P_i^{(r)}(x)=\sum_{j=0(j+i=even)}^{i-r} C_r(i,j)P_j(x), \tag{2}$$

where

$$C_r(i,j)=\frac{2^{r-1}(2j+1)\Gamma(\frac{r+i-j}{2})\Gamma(\frac{r+i+j+1}{2})}{\Gamma(r)\Gamma(\frac{2-r+i-j}{2})\Gamma(\frac{3-r+i+k}{2})}.$$

Next, we indicate the norm and the inner product of space $L^2[-1,1]$ by $\|u\|$ and (u,v), respectively. The set of $P_i(x)$ is a complete orthogonal system in $L^2[-1,1]$

$$\left(P_i(x),P_j(x)\right)=\int_{-1}^{1}P_i(x)P_j(x)\mathrm{d}x=h_i\delta_{ij}, \tag{3}$$

where $h_i=\dfrac{2}{2i+1}$ and δ_{ij} is the Dirac function. Thus for any $v\in L^2[-1,1]$, we get

$$v(x)=\sum_{i=0}^{\infty}a_iP_i(x),\; a_i=\frac{1}{h_i}\int_{-1}^{1}v(x)P_i(x)\;\mathrm{d}x. \tag{4}$$

Let $S_N[-1,1]$ be the set of all polynomials of degree N ($N\geq 0$). Thus, we have

$$\int_{-1}^{1}\varphi(x)\mathrm{d}x=\sum_{i=0}^{N}\varpi_{N,i}\varphi(x_{N,i}),\;\forall\varphi\in S_{2N-1}[-1,1], \tag{5}$$

where $x_{N,k}$ ($0\leq k\leq N$) and $\varpi_{N,k}$ ($0\leq k\leq N$) are denoted to the zeros and weights of Legendre Gauss-Lobatto (L-GL) interpolation on $[-1,1]$, respectively. The norm and discrete inner product are defined as

$$\|u\|_N=(u,v)_N^{\frac{1}{2}},\qquad (u,v)_N=\sum_{j=0}^{N}u(x_{N,j})v(x_{N,j})\varpi_{N,j}. \tag{6}$$

Let us denote by $P_{L,k}(x)$ the shifted Legendre polynomials that are defined on $[0,L]$. These polynomials can be engendered from the recurrence relation:

$$(r+1)P_{L,s+1}(x)=(2s+1)(\frac{2x}{L}-1)P_{L,s}(x)-rP_{L,s-1}(x),\; s=1,2,\cdots. \tag{7}$$

The $P_{L,i}(x)$ may be written in the analytic form

$$P_{L,j}(x)=\sum_{k=0}^{j}(-1)^{j+k}\,\frac{(j+k)!}{(j-k)!\,(k!)^2\,L^k}\,x^k. \tag{8}$$

The condition for the orthogonality of these polynomials is given by

$$\int_0^L P_{L,j}(x)P_{L,k}(x)w_L(x)\;\mathrm{d}x=h_k^L\delta_{jk}, \tag{9}$$

where $w_L(x)=1$ and $h_k^L=\dfrac{L}{2k+1}$.

If function $u(t) \in L^2[0,L]$, then one can express it by means of $P_{L,i}(t)$ as

$$u(t) = \sum_{i=0}^{\infty} c_i P_{L,i}(t), \tag{10}$$

where c_i is given by

$$c_i = \frac{1}{h_i^L}\int_0^L u(t)P_{L,i}(t)\mathrm{d}x,\ i = 0,1,2,\cdots. \tag{11}$$

3. 3D VOLTERRA-FREDHOLM INTEGRAL EQUATION

We present two numerical techniques to solve linear and nonlinear 3D-VFIEs by using the SL-GL-C method. In this method, we select the collocation nods as the SL-GL interpolation points. The core of the proposed algorithm consists of discretizing the 3D-VFIEs to obtain an algebraic system of equations.

3.1. Linear Volterra-Fredholm integral equation

Let us consider the linear 3D-IEs:

$$u(x,y,z) = \int_0^x\int_0^L\int_0^L K(x,y,z,t,r,s)u(t,r,s)\mathrm{d}t\mathrm{d}r\mathrm{d}s + f(x,y,z), \tag{12}$$

where $f(x,y,z)$ and $k(x,y,z,t,r,s)$ are given functions. The functions $f(x,y,z)$ and $k(x,y,z,t,r,s)$ are assumed to be continuous and such that Eq. (12) has a unique solution.

The change of variables $s = \frac{x}{L}\eta$ will be used to transform the integrals into the interval $[0,L]$, for the new variable η, to directly implement the SL-GL integration

$$u(x,y,z) = \frac{x}{L}\int_0^L\int_0^L\int_0^L K(x,y,z,t,r,\frac{x}{L}\eta)u(t,r,\frac{x}{L}\eta)\mathrm{d}t\mathrm{d}r\mathrm{d}\eta + f(x,y,z). \tag{13}$$

Now, the methodology of the SL-GL-C method will be outlined for obtaining spectral solutions of linear 3D-VFIEs. We assume the spectral solution of the form

$$u_{N,M,K}(x,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k}P_{L,i}(x)P_{L,j}(y)P_{L,k}(z). \tag{14}$$

Next, employing the SL-GL quadrature, one my express the integral part of 3D-VFIEs (13) as follows

$$\int_0^L\int_0^L\int_0^L K(x,y,z,t,r,\frac{x}{L}\eta)u(t,r,s)\mathrm{d}t\mathrm{d}r\mathrm{d}\eta = \sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\zeta_{\tau,\nu\mu}(x,y,z)u(t_{L,N,\tau},r_{L,M,\mu},\frac{x}{L}\eta_{L,K,\nu}), \tag{15}$$

where $\zeta_{\tau,\nu\mu}(x,y,z) = \varpi_{L,N,\tau}\varpi_{L,M,\mu}\varpi_{L,K,\nu}K(x,y,z,t_{L,N,\tau},r_{L,M,\mu},\frac{x}{L}\eta_{L,K,\nu})$.

In virtue of the Eqs. (14) and (15), one may write Eq. (13) as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k}P_{L,i}(x)P_{L,j}(y)P_{L,k}(z) = f(x,y,z) + \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k}\psi_{i,j,k}(x,y,z), \tag{16}$$

where

$$\psi_{i,j,k}(x,y,z) = \frac{x}{L}\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\zeta_{\tau,\nu\mu}(x,y,z)P_{L,i}(t_{L,N,\tau})P_{L,j}(r_{L,M,\mu})P_{L,k}(\frac{x}{L}\eta_{L,K,\nu}). \tag{17}$$

In the suggested SL-GL-C technique, the residual of (16) is letting to be zero at the collocation nodes. These nodes are $(N+1)\times(M+1)\times(K+1)$ SL-GL points

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k}\Omega_{i,j,k}^{n,m,k} = f(x_{L,N,n}, y_{L,M,m}, z_{L,K,k}), \tag{18}$$

where

$$\Omega_{i,j,k}^{n,m,k} = P_{L,i}(x_{L,N,n})P_{L,j}(y_{L,M,m})P_{L,k}(z_{L,K,k}) - \psi_{i,j,k}(x_{L,N,n}, y_{L,M,m}, z_{L,K,k}).$$

Now, we can approximate $u_{N,M,K}(x,y,z)$ at the point (x,y,z), by means of the following equation

$$u_{N,M,K}(x,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x)P_{L,j}(y)P_{L,k}(z). \tag{19}$$

3.2 Nonlinear Volterra-Fredholm integral equation

This Section extends the previous algorithm to deal with the nonlinear 3D-VFIEs,

$$u(x,y,z) = \int_0^x\int_0^L\int_0^L K\left(x,y,z,t,r,s,u(t,r,s)\right)\mathrm{d}t\mathrm{d}r\mathrm{d}s + f(x,y,z), \tag{20}$$

where $f(x,y,z)$ and $k(x,y,z,t,r,s,u(x,y,z)))$ are given functions, while $u(x,y,z)$ is an unknown function. The change of variables $s = \frac{x}{L}\eta$ will be used to transform the integrals into the interval $[0,L]$, for the variable η, to directly implement the SL-GL integration

$$u(x,y,z) = \frac{x}{L}\int_0^L\int_0^L\int_0^L K(x,y,z,t,r,\frac{x}{L}\eta, u(t,r,\frac{x}{L}\eta))\mathrm{d}t\mathrm{d}r\mathrm{d}\eta + f(x,y,z). \tag{21}$$

Thus the approximate solution can be presented as a truncated shifted Legendre series:

$$u_{N,M,K}(x,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x)P_{L,j}(y)P_{L,k}(z). \tag{22}$$

Similar steps to those discussed in the previous Subsection, allow us to rewrite Eq. (21) as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x)P_{L,j}(y)P_{L,k}(z) = f(x,y,z) + \frac{x}{L}\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K} \xi_{\tau,\nu\mu}(x,y,z), \tag{23}$$

where

$$\xi_{\tau,\nu\mu}(x,y,z) = \varpi_{L,N,\tau}\varpi_{L,M,\mu}\varpi_{L,K,\nu} K(x,y,z,t_{L,N,\tau}, r_{L,M,\mu}, \frac{x}{L}\eta_{L,K,\nu}, \lambda_{\tau,\mu,\nu}),$$

$$\lambda_{\tau,\mu,\nu} = \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(t_{L,N,\tau})P_{L,j}(r_{L,M,\mu})P_{L,k}(\frac{x}{L}\eta_{L,K,\nu}).$$

Therefore, applying the collocation scheme for Eq. (23) at $(N+1)\times(M+1)\times(K+1)$ SL-GL points, yields an algebraic system of the following form

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x_{L,N,n})P_{L,j}(y_{L,M,m})P_{L,k}(z_{L,K,k}) = \chi_{n,m,k} + f(x_{L,N,n}, y_{L,M,m}, z_{L,K,k}), \tag{24}$$

where $\chi_{n,m,k} = \frac{x_{L,N,n}}{L}\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\xi_{\tau,\nu\mu}(x_{L,N,n}, y_{L,M,m}, z_{L,K,k})$.

Finally and after solving the above algebriac system, we can achieve the approximate solution $u_{N,M,K}(x,y,z)$, by means of the following equation

$$u_{N,M,K}(x,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x)P_{L,j}(y)P_{L,k}(z). \tag{25}$$

4. LINEAR 3D FREDHOLM INTEGRAL EQUATION

We apply the above methodology to solve the following linear 3D-FIE:

$$u(x,y,z)=\int_0^L\int_0^L\int_0^L K(x,y,z,t,r,s)u(t,r,s)\mathrm{d}t\mathrm{d}r\mathrm{d}s+f(x,y,z), \tag{26}$$

where $f(x,y,z)$ and $k(x,y,z,t,r,s)$ are given functions.

Let us expand the dependent variable in the form

$$u_{N,M,K}(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K}a_{i,j,k}P_{L,i}(x)P_{L,j}(y)P_{L,k}(z). \tag{27}$$

By means of the SL-GL quadrature, we can express the integral in Eq. (26) by

$$\int_0^L\int_0^L\int_0^L K(x,y,z,t,r,s)u(t,r,s)\mathrm{d}t\mathrm{d}r\mathrm{d}s=\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\rho_{\tau,\nu,\mu}(x,y,z)u\left(t_{L,N,\tau},r_{L,M,\mu},s_{L,K,\nu}\right), \tag{28}$$

where $\rho_{\tau,\nu\mu}(x,y,z)=\varpi_{L,N,\tau}\varpi_{L,M,\mu}\varpi_{L,K,\nu}K(x,y,z,t_{L,N,\tau},r_{L,M,\mu},s_{L,K,\nu})$. Therefore, adapting (27) and (28), one can write (26) as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K}a_{i,j,k}P_{L,i}(x)P_{L,j}(y)P_{L,k}(z)=\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K}a_{i,j,k}\phi_{i,j,k}(x,y,z)+f(x,y,z), \tag{29}$$

where

$$\phi_{i,j,k}(x,y,z)=\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\rho_{\tau,\nu\mu}(x,y,z)P_{L,i}(t_{L,N,\tau})P_{L,j}(r_{L,M,\mu})P_{L,k}(s_{L,K,\nu}). \tag{30}$$

The residual of Eq. (29) is letting to be zero at $(N+1)\times(M+1)\times(K+1)$ of SL-GL points

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K}a_{i,j,k}\Upsilon_{i,j,k}^{n,m,k}=f(x_{L,N,n},y_{L,M,m},z_{L,K,k}), \tag{31}$$

where $\Upsilon_{i,j,k}^{n,m,k}=(P_{L,i}(x_{L,N,n})P_{L,j}(y_{L,M,m})P_{L,k}(z_{L,K,k})-\phi_{i,j,k}(x_{L,N,n},y_{L,M,m},z_{L,K,k}))$.

Finally, we can approximate $u_{N,M,K}(x,y,z)$ at any point (x,y,z), as in Eq. (25).

5. LINEAR 3D VOLTERRA INTEGRAL EQUATION

We apply the above methodology to solve the following linear 3D-VIEs:

$$u(x,y,z)=\int_0^x\int_0^y\int_0^z K(x,y,z,t,r,s)u(t,r,s)\,\mathrm{d}t\mathrm{d}r\mathrm{d}s+f(x,y,z), \tag{32}$$

where $f(x,y,z)$ and $k(x,y,z,t,r,s)$ are given functions.

The change of variables $s=\frac{x}{L}\eta,$, $t=\frac{y}{L}\gamma,$ and $r=\frac{z}{L}\delta$ will be used to transform the integrals into the interval [0, L], for the variable η, to directly implement the SL-GL integration

$$u(x,y,z)=\frac{x}{L}\frac{y}{L}\frac{z}{L}\int_0^L\int_0^L\int_0^L K\left(x,y,z,\frac{z}{L}\delta,\frac{y}{L},\frac{x}{L}\right)\mathrm{d}\delta\mathrm{d}\gamma\mathrm{d}\eta+f(x,y,z). \tag{33}$$

Also we approximate the integral in (33) as

$$\int_0^L\int_0^L\int_0^L K(x,y,z,\frac{z}{L}\delta,\frac{y}{L}\gamma,\frac{x}{L}\eta)u(\frac{z}{L}\delta,\frac{y}{L}\gamma,\frac{x}{L}\eta)\mathrm{d}\delta\mathrm{d}\gamma\mathrm{d}\eta=\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K}\zeta_{\tau,\nu\mu}(x,y,z)u(\frac{z}{L}\delta_{L,N,\tau},\frac{y}{L}\gamma_{L,M,\mu},\frac{x}{L}\eta_{L,K,\nu}), \tag{34}$$

where $\zeta_{\tau,\nu\mu}(x,y,z)=\varpi_{L,N,\tau}\varpi_{L,M,\mu}\varpi_{L,K,\nu}K(x,y,z,\frac{z}{L}\delta_{L,N,\tau},\frac{y}{L}\gamma_{L,M,\mu},\frac{x}{L}\eta_{L,K,\nu})$.

In virtue of the Eqs. (14) and (34), one may write Eq. (33) as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} P_{L,i}(x) P_{L,j}(y) P_{L,k}(z) = f(x,y,z) + \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} \Phi_{i,j,k}(x,y,z), \tag{35}$$

where $\Phi_{i,j,k}(x,y,z) = \frac{x}{L}\frac{y}{L}\frac{z}{L}\sum_{\tau=0}^{N}\sum_{\mu=0}^{M}\sum_{\nu=0}^{K} \zeta_{\tau,\nu\mu}(x,y,z) P_{L,i}(\frac{z}{L}\delta_{L,N,\tau}) P_{L,j}(\frac{y}{L}\gamma_{L,M,\mu}) P_{L,k}(\frac{x}{L}\eta_{L,K,\nu})$.

Therefore, applying the collocation scheme for Eq. (35) at $(N+1)\times(M+1)\times(K+1)$ SL-GL points, yields an algebraic system of the following form

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} \Lambda_{i,j,k}^{n,m,k} = f(x_{L,N,n}, y_{L,M,m}, z_{L,K,k}), \tag{36}$$

where $\Lambda_{i,j,k}^{n,m,k} = P_{L,i}(x_{L,N,n}) P_{L,j}(y_{L,M,m}) P_{L,k}(z_{L,K,k}) - \psi_{i,j,k}(x_{L,N,n}, y_{L,M,m}, z_{L,K,k})$.

Finally, the approximate solution $u_{N,M,K}(x,y,z)$ can be evaluated at any point in the domain.

6. NUMERICAL RESULTS

Based on the previous algorithms, we give in this Section some numerical results. The effectiveness, appropriateness, and high accuracy of our method appeared when we compared it with other methods.

Example 1. We start with the linear 3D-IEs:

$$u(x,y,z) = \frac{1}{72}x^2\left(x\left(4yz\left(-1+\cos 1\right)\right)\right) + 3x\left(\cos 1 - \sin 1\right) + 72y^2 \sin z +$$

$$+\frac{1}{2}\int_0^x\int_0^1\int_0^1 (yz+st)u(t,r,s)\,\mathrm{d}t\mathrm{d}r\mathrm{d}s,$$

whose exact solution is $u(x,y,z) = x^2y^2 \sin z$.

To test the convergence rate of our method, we list the absolute error (AE) in Table 1. The results reveal the effectiveness, appropriateness, and high accuracy of our method. Also, we can observe that our numerical solutions coincide closely with the exact ones.

Table 1

The AEs for Example 1 and for $N = M = K = 4$

(x, y, z)	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{16},\frac{1}{16},\frac{1}{16}\right)$	$\left(\frac{1}{32},\frac{1}{32},\frac{1}{32}\right)$	$\left(\frac{1}{64},\frac{1}{64},\frac{1}{64}\right)$
$E(x, y, z)$	1.2×10^{-11}	6.1×10^{-8}	2.5×10^{-10}	1.1×10^{-11}	4.0×10^{-13}

Example 2. Let us consider the following nonlinear 3D-IEs [7]:

$$u(x,y,z) = yz\sin x - \frac{1}{16}x^3y^3z^3\sin^3 x\left(x+4yz\right) + \frac{1}{2}\int_0^x\int_0^1\int_0^1 x^2 t\left(tz+st\right)u^2\left(s,r,t\right)\mathrm{d}t\mathrm{d}r\mathrm{d}s,$$

where $u(x,y,z) = yz\sin x$ is the exact solution.

Based on the AEs acquired by our method and that achieved in the method in [7], we summarize a comparison in Table 2. It is observed that the obtained results highlight the high accuracy of the proposed scheme.

Table 2

The AEs for Example 2

(x, y, z)	3D-BFs [7]		Our Method
	N = M = K = 2	N = M = K = 3	N = M = K = 3
$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	0.00251040	0.0005329	0.0000618829
$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	0.0108757	0.0001594	2.636×10^{-6}
$\left(\frac{1}{16},\frac{1}{16},\frac{1}{16}\right)$	0.015050594	0.0043430	5.602×10^{-7}
$\left(\frac{1}{32},\frac{1}{32},\frac{1}{32}\right)$	0.0152995	0.0045832	1.985×10^{-8}

Example 3. We consider the following linear 3D-FIEs [3]:

$$u(x, y, z) = xy(z-y) + \frac{1}{180} + \frac{1}{5}\int_0^1\int_0^1\int_0^1 su(s, r, t)\,\mathrm{d}t\mathrm{d}r\mathrm{d}s,$$

$$u(x, y, z) = xy(z-y) + \frac{1}{180} + \frac{1}{5}\int_0^1\int_0^1\int_0^1 su(s, r, t)\,\mathrm{d}t\mathrm{d}r\mathrm{d}s,$$

where $u(x, y, z) = xy(z-y)$ is the exact solution.

Table 3 shows the results for AEs of our method, the best result for AEs acquired using the integral mean value method [3] with $N = M = K = 64$ is 9.1×10^{-5} for all points in $[0,1]^3$. We can observe that our numerical solutions coincide closely with the exact ones and are more accurate than those obtained by using the integral mean value method [3].

Table 3

The AEs for Example 3 and for $N = M = K = 4$

(x, y, z)	$E(x, y, z)$
(0, 0, 0)	7.38×10^{-50}
(0, 1, 0.1)	1.39×10^{-17}
(0.3, 0.3, 0.3)	8.67×10^{-18}
(0.5, 0.5, 0.5)	1.0×10^{-50}
(0.7, 0.7, 0.7)	3.99×10^{-17}
(0.9, 0.9, 0.9)	1.39×10^{-17}

7. CONCLUSIONS

The major goal of our paper is to extend and develop a spectral algorithm for solving 3D-IEs. Our goal has been acquired by means of the SL-GL-C method. The collocation points were selected as the SL-GL interpolation nodes. We provided illustrative examples to check the applicability and validity of this novel algorithm. The achieved results demonstrated that the present pseudospectral method is highly effective. Based on the listed comparisons with other methods, we observed the effectiveness, appropriateness, and high accuracy of our numerical method.

REFERENCES

1. A.M. WAZWAZ, *Linear and Nonlinear Integral Equations Methods and Applications*, Springer, Heidelberg, Dordrecht, London, New York, 2011.
2. F. LIANG, F-R. LIN, *A fast numerical solution method for two dimensional Fredholm integral equations of the second kind based on piecewise polynomial interpolation*, Appl. Math. Comput., **216,** pp. 3073–3088, 2010.
3. M. HEYDARI *et al*., *Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem II. High dimensional problems*, Appl. Math. Modell., **37**, pp. 432–442, 2013.
4. R.T. BAILLIE, *Long memory processes and fractional integration in econometrics*, J. Econometrics, **73**, pp. 5–59, 1996.
5. M.A. ABDELKAWY *et al*., *A Jacobi spectral collocation scheme for solving Abel's integral equations*, Progr. Fract. Differ. Appl., **1**, pp. 187–200, 2015.
6. A.H. BHRAWY *et al*., *Legendre-Gauss-Lobatto collocation method for solving multi-dimensional Fredholm integral equations,* Comput. Math. Appl., Doi: 10.1016/j.camwa.2016.04.011, 2016.
7. F. MIRZAEE *et al*., *Numerical solution for three-dimensional nonlinear mixed Volterra-Fredholm integral equations via three-dimensional block-pulse functions*, Appl. Math. Comput., **237**, pp. 168–175, 2014.
8. F. MIRZAEE, E. HADADIYAN, *Applying the modified block-pulse functions to solve the three-dimensional Volterra-Fredholm integral equations*, Appl. Math. Comput., **265**, pp. 759–767, 2015.
9. A. SAADATMANDI, M. DEHGHAN, *Numerical solution of the higher-order linear Fredholm integro-differential-difference equation with variable coefficients*, Comput. Math. Appl., **59,** pp. 2996–3004, 2010.
10. S. YUZBASI *et al*., *Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases*, Comput. Math. Appl., **61**, pp. 3079–3096, 2011.
11. W.T. CHAOLU, P. JING, *New algorithm for second-order boundary value problems of integro-differential equation*, J. Comput. Appl. Math**.**, **299**, pp. 1–6, 2009.
12. K. MALEKNEJAD *et al*., *A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm integro-differential equations*, Math. Comput. Model., **55**, pp. 1363–1372, 2012.
13. K. MALEKNEJAD *et al*., *Numerical Solution of a Non-Linear Volterra Integral Equation*, Vietnam J. Math., **44**, pp. 5–28, 2016.
14. C. CANUTO, M.Y. HUSSAINI, A. QUARTERONI, T.A. ZANG, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, Berlin, Heidelberg, 2006.
15. A.H. BHRAWY *et al*., *An efficient collocation technique for solving generalized Fokker-Planck type equations with variable coefficients,* Proc. Romanian Acad. A, **15**, pp. 322–330, 2014.
16. A.H. BHRAWY *et al*., *New numerical approximations for space-time fractional Burgers' equations via a Legendre spectral-collocation method*, Rom. Rep. Phys., **67**, pp. 340–349, 2015.
17. E.H. DOHA *et al*., *A Jacobi-Jacobi dual-Petrov-Galerkin method for third- and fifth-order differential equations*, Math. Comput. Modell., **53**, pp. 1820–1832, 2011.
18. E.H. DOHA *et al*., *Efficient spectral-Galerkin algorithms for direct solution for second-order differential equations using Jacobi polynomials*, Numer. Algorithms, 42, pp. 137–164, 2006.
19. E.H. DOHA *et al*., *Jacobi spectral Galerkin method for elliptic Neumann problems*, Numer. Algorithms, **50**, pp. 67–91, 2009.
20. A.H. BHRAWY *et al*., *A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations*, Rom. J. Phys., **59**, pp. 646–657, 2014.
21. J. LIU, G. HOU, *Numerical solutions of the space- and time-fractional coupled Burgers equation by generalized differential transform method,* Appl. Math. Comput., **217**, pp. 7001–7008, 2011.
22. A.H. BHRAWY, *A Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation with time-dependent coefficients*, App. Math. Comput, **222**, pp. 255–264, 2013.
23. A.H. BHRAWY, M.A. ABDELKAWY, *A fully spectral collocation approximation for multi-dimensional fractional Schrödinger equations*, J. Comput. Phys., **294**, pp. 462–483, 2015.
24. A.H. BHRAWY, *A new spectral algorithm for a time-space fractional partial differential equations with subdiffusion and superdiffusion,* Proc. Romanian Acad. A, **17**, pp. 39–47, 2016.
25. R.M. HAFEZ *et al*., *A new collocation scheme for solving hyperbolic equations of second order in a semi-infinite domain*, Rom. Rep. Phys., **68**, pp. 112–127, 2016.
26. A.H. BHRAWY *et al*., *An accurate numerical technique for solving fractional optimal control problems*, Proc. Romanian Acad. A, **16**, pp. 47–54, 2015.
27. M.A. ABDELKAWY *et al*., *Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model*, Rom. Rep. Phys., **67**, pp. 773–791, 2015.
28. E.H. DOHA *et al*., *Numerical treatment of coupled nonlinear hyperbolic Klein-Gordon equations*, Rom. J. Phys., **59**, pp. 247–264, 2014.
29. A.H. BHRAWY *et al*., *A Chebyshev-Laguerre Gauss-Radau collocation scheme for solving time fractional sub-diffusion equation on a semi-infinite domain*, Proc. Romanian Acad. A, **16**, pp. 490–498, 2015.
30. A.H. BHRAWY *et al*., *A novel spectral approximation for the two-dimensional fractional sub-diffusion problems*, Rom. J. Phys., **60**, pp. 344–359, 2015.

Received August 30, 2016