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NEW ANALYTICAL SOLUTIONS FOR KLEIN-GORDON AND HELMHOLTZ EQUATIONS IN FRACTAL DIMENSIONAL SPACE

Xiao-Jun YANG [1, 2], Dumitru BALEANU [3, 4], Feng GAO [1, 2]

[1] China University of Mining and Technology, School of Mechanics and Civil Engineering, Xuzhou 221116, China
[2] China University of Mining and Technology, State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, Xuzhou 221116, China
[3] Cankaya University, Faculty of Arts and Sciences, Department of Mathematics, TR-06530 Ankara, Turkey
[4] Institute of Space Sciences, Magurele, RO-077125 Bucharest, Romania
Corresponding author: Dumitru BALEANU, E-mail: dumitru@cankaya.edu.tr

Abstract: We consider the local fractional Klein–Gordon equation and Helmholtz equation in (1+1) fractal dimensional space. The local fractional Laplace series expansion method is used to solve the local fractional partial differential equations in fractal dimensional space. We present the non-differentiable analytical solutions and the corresponding graphs. The obtained results illustrate the accuracy and efficiency of this approach to local fractional partial differential equations.

Key words: Klein-Gordon equation, Helmholtz equation, analytical solution, Laplace transform, series expansion method, local fractional derivative.

1. INTRODUCTION

The partial differential equations (PDEs) in mathematical physics are generalized to describe the anomalous phenomena via fractional calculus [1–7]. Several equations describing the real world phenomena were addressed, e.g. the fractional Burgers equation [1], fractional Schrödinger equation [2], fractional Ginzburg–Landau equation [3], fractional Fokker-Planck equation [4], fractional Poisson equation [5], fractional Helmholtz equation [6] and others [7].

Recently, the local fractional calculus [8] was successfully applied for a better characterization of some mathematical models in science and engineering. We mention, for example, the Klein-Gordon equation (KGE) [8], the Helmholtz equation (HE) [8], the diffusion equation (DE) [9], the heat-conduction equation [10] as well as the Burgers flow [11], the oscillator equation [12] and other types of equations [13–22].

Very recently, the series expansion method was presented for finding the non-differentiable solution for KGE on Cantor sets [13]. We implemented for the first time the local fractional DE with the help of the local fractional Laplace series expansion method (LFLSEM) [14]. Motivated by the results presented in the above mentioned paper, in this work, the main aim is to solve the KGEs and HEs in (1+1) fractal dimensional space (FDS) by the LFLSEM. The manuscript has the following structure. In Section 2, we recall some basics of the theory of the local fractional derivative (LFD) and local fractional integral (LFI). The LFLSEM is analyzed in Section 3. The non-differentiable series solutions for the KGE and HE within LFD are presented in Section 4. The manuscript ends with a conclusion part.

2. PRELIMINARIES

In this Section we begin with the theory of LFD and LFI and the properties of the local fractional Laplace transform (LFLT).

The LFD (or LFD operator) of $\psi(\zeta)$ is defined by the expression (see, for example, [8–16]):

$$\psi^{(\varpi)}(\zeta_0)=\frac{d^\varpi\psi(\zeta)}{d\zeta^\varpi}\Big|_{\zeta=\zeta_0}=\lim_{\zeta\to\zeta_0}\frac{\Delta^\varpi\left(\psi(\zeta)-\psi(\zeta_0)\right)}{(\zeta-\zeta_0)^\varpi}, \tag{1}$$

where $\Delta^\varpi\left(\psi(\zeta)-\psi(\zeta_0)\right)\cong\Gamma(1+\varpi)\Delta\left(\psi(\zeta)-\psi(\zeta_0)\right)$ with the difference value Δ between $\psi(\zeta)$ and $\psi(\zeta_0)$.

The operations of the LFDs of some functions (SFs) [8] are displayed in Table 1.

Table 1
The LFDs of SFs

SFs $\psi(\zeta)$	LFDs $\psi^{(\varpi)}(\zeta)$
$E_\varpi(\zeta^\varpi)=\sum_{i=0}^{n}\zeta^{i\varpi}/\Gamma(1+i\varpi)$	$E_\varpi(\zeta^\varpi)$
$\zeta^\varpi/\Gamma(1+\varpi)$	1
$\zeta^{(n+1)\varpi}/\Gamma\left[1+(n+1)\varpi\right]$	$\zeta^{n\varpi}/\Gamma(1+n\varpi)$

As the inverse operator of the LFD (1), the LFI of $\phi(\mu)$ is defined as follows [8, 13–15]:

$${}_{\mu_a}I_{\mu_b}^{(\varpi)}\phi(\mu)=\frac{1}{\Gamma(1+\varpi)}\int_{\mu_a}^{\mu_b}\phi(\mu)(d\mu)^\varpi=\frac{1}{\Gamma(1+\varpi)}\lim_{\Delta\tau\to0}\sum_{j=0}^{j=N-1}\phi(\mu)(\Delta\mu)^\varpi, \tag{2}$$

where $\Delta\mu=\mu_{j+1}-\mu_j$, $j=0,...,N-1, \mu_0=\mu_a, \mu_N=\mu_b$.

The LFLT of $\psi(\zeta)$ is defined as follows [8, 14]:

$$\begin{aligned}\tilde{\Xi}_\varpi\{\psi(\mu)\}&=\psi_{\varpi,s}(s)\\&=\frac{1}{\Gamma(1+\varpi)}\int_0^\infty E_\varpi(-s^\varpi\mu^\varpi)\psi(\mu)(d\mu)^\varpi,\ 0<\varpi\le1.\end{aligned} \tag{3}$$

The inverse LFLT of $\psi_{\varpi,s}(s)$ is defined as (see [8, 14]):

$$\begin{aligned}\psi(\mu)&=\tilde{\Xi}_\varpi^{-1}\{\psi_{\varpi,s}(s)\}\\&=\frac{1}{(2\pi)^\varpi}\int_{\beta-i\infty}^{\beta+i\infty}E_\varpi(s^\varpi\mu^\varpi)\psi_{\varpi,s}(s)(ds)^\varpi,\end{aligned} \tag{4}$$

where $s^\varpi=\beta^\varpi+i^\varpi\infty^\varpi$ and $\mathrm{Re}_\varpi(S^\varpi)=\beta^\varpi$.

The operations of the LFLTs of SFs [8] are listed in Table 2.

Table 2
The LFLTs of SFs

SFs $\psi(\mu)$	LFLTs $\tilde{\Xi}_\varpi\{\psi(\mu)\}$
$E_\varpi(\mu^\varpi)$	$1/(\mu^\varpi-1)$
$\zeta^{n\varpi}/\Gamma(1+n\varpi)$	$1/\mu^{(n+1)\varpi}$

3. THE APPLIED METHOD

In this section, the LFLSEM for the PDE within the LFD operator is presented (see [14]).

Let us consider the local fractional PDEs in the form of the LFO

$$\varphi_{\tau}^{(2\varpi)} = \Lambda_{\varpi}\varphi\,, \tag{5}$$

where $\varphi_{\tau}^{(2\varpi)} = \partial^{2\varpi}\varphi(\mu,\tau)/\partial\tau^{2\varpi}$ and Λ_{ϖ} is a linear LFD operator with respect to μ.

We consider a function with respect to τ and μ, namely,

$$\varphi(\mu,\tau) = \sum_{i=0}^{\infty} \mathrm{M}_i(\tau)\mathrm{N}_i(\mu)\,, \tag{6}$$

where $\mathrm{M}_i(\tau)$ and $N_i(\mu)$ are two functions of non-differentiable type, and $i = 0,1,\ldots$.

Taking $\mathrm{M}_i(\tau) = \tau^{i\varpi}/\Gamma(1+i\varpi)$, Eq. (6) can be written in the form

$$\varphi(\mu,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\mathrm{N}_i(\mu)\,. \tag{7}$$

With the help of Eq. (7) and taking the LFLT of Eq. (5), we have

$$\tilde{\tilde{\Xi}}_{\varpi}\left\{\varphi_{\tau}^{(2\varpi)}(\mu,\tau)\right\} = \sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma[1+i\varpi]}\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i+2}(\mu)\right\}, \tag{8}$$

$$\tilde{\tilde{\Xi}}_{\varpi}\left\{\Lambda_{\varpi}\varphi(\mu,\tau)\right\} = \Lambda_{\varpi}\left[\sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\tilde{\tilde{\Xi}}_{\varpi}\left(N_i(\mu)\right)\right] = \sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\tilde{\tilde{\Xi}}_{\varpi}\left\{\left(\Lambda_{\varpi}N_i\right)(\mu)\right\}, \tag{9}$$

such that

$$\sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma[1+i\varpi]}\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i+2}(\mu)\right\} = \sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\tilde{\tilde{\Xi}}_{\varpi}\left\{\left(\Lambda_{\varpi}N_i\right)(\mu)\right\}. \tag{10}$$

Thus, we have the following recursion formula

$$\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i+2}(\mu)\right\} = \tilde{\tilde{\Xi}}_{\varpi}\left\{\left(\Lambda_{\varpi}N_i\right)(\mu)\right\}, \tag{11}$$

where

$$\lim_{i\to\infty}\left[\frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\tilde{\tilde{\Xi}}_{\varpi}\left\{N_i(\mu)\right\}\right] = 0\,. \tag{12}$$

Taking the inverse LFLT of Eq. (12) gives

$$\varphi(\mu,\tau) = \tilde{\tilde{\Xi}}_{\varpi}^{-1}\left\{\sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\tilde{\tilde{\Xi}}_{\varpi}\left\{N_i(\mu)\right\}\right\} = \sum_{i=0}^{\infty} \frac{\tau^{i\varpi}}{\Gamma(1+i\varpi)}\left\{\tilde{\tilde{\Xi}}_{\varpi}^{-1}\left(\tilde{\tilde{\Xi}}_{\varpi}\left\{N_i(\mu)\right\}\right)\right\}. \tag{13}$$

4. SOLVING THE KLEIN-GORDON AND HELMHOLTZ EQUATIONS

We now consider the local fractional KGE in (1+1) FDS (see, e.g., [8,13])

$$\frac{\partial^{2\varpi}\Phi(\mu,\tau)}{\partial\tau^{2\varpi}}-\frac{\partial^{2\varpi}\Phi(\mu,\tau)}{\partial\mu^{2\varpi}}-\Phi(\mu,\tau)=0\,, \tag{14}$$

subject to the initial value conditions

$$\Phi(\mu,0)=\frac{\mu^{\varpi}}{\Gamma\left(1+\varpi\right)}, \tag{15}$$

$$\frac{\partial^{\varpi}\Phi(\mu,0)}{\partial\mu^{\varpi}}=0\,. \tag{16}$$

From Eq. (14) we conclude that

$$\Lambda_{\varpi}\Phi(\mu,\tau)=\frac{\partial^{2\varpi}\Phi(\mu,\tau)}{\partial\mu^{2\varpi}}+\Phi(\mu,\tau) \tag{17}$$

and

$$\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i+2}\left(\mu\right)\right\}=\tilde{\tilde{\Xi}}_{\varpi}\left\{\frac{\partial^{2\varpi}N_i\left(\mu\right)}{\partial\mu^{2\varpi}}\right\}+\tilde{\tilde{\Xi}}_{\varpi}\left\{N_i\left(\mu\right)\right\}, \tag{18}$$

where

$$N_0(s)=\frac{1}{s^{2\varpi}}\,. \tag{19}$$

Using Eq. (18) and Eq. (19), we have the non-differentiable series components in the form

$$N_0(s)=\frac{1}{s^{2\varpi}}\,, \tag{20}$$

$$N_1(s)=0\,, \tag{21}$$

$$N_2(s)=\frac{1}{s^{2\varpi}}\,, \tag{22}$$

$$N_3(s)=0\,, \tag{23}$$

$$N_4(s)=\frac{1}{s^{2\varpi}} \tag{24}$$

and so on.

Thus, we obtain the non-differentiable solution

$$\Phi\left(\mu,\tau\right)=\sum_{i=0}^{\infty}\frac{\tau^{2i\varpi}}{\Gamma\left(1+2i\varpi\right)}\tilde{\tilde{\Xi}}_{\varpi}^{-1}\left\{\frac{1}{s^{2\varpi}}\right\}=\frac{\mu^{\varpi}}{\Gamma\left(1+\varpi\right)}\sum_{i=0}^{\infty}\frac{\tau^{2i\varpi}}{\Gamma\left(1+2i\varpi\right)}, \tag{25}$$

which is in accordance with the result from Ref. [13].

We now consider the local fractional KGE in (1+1) FDS (see, for example, [8, 13])

$$\frac{\partial^{2\varpi}\Phi(\mu,\tau)}{\partial\tau^{2\varpi}}-\frac{\partial^{2\varpi}\Phi(\mu,\tau)}{\partial\mu^{2\varpi}}-\Phi(\mu,\tau)=0\,, \tag{26}$$

subject to the initial value conditions

$$\Phi(\mu,0)=E_{\varpi}\left(\mu^{\varpi}\right), \tag{27}$$

$$\frac{\partial^{\varpi}\Phi(\mu,0)}{\partial\mu^{\varpi}}=0\,. \tag{28}$$

Similarly, the iterative formula with the aid of the LFLT can be represented as:

$$\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i+2}\left(\mu\right)\right\}=\tilde{\tilde{\Xi}}_{\varpi}\left\{\frac{\partial^{2\varpi}N_{i}\left(\mu\right)}{\partial\mu^{2\varpi}}\right\}+\tilde{\tilde{\Xi}}_{\varpi}\left\{N_{i}\left(\mu\right)\right\}, \tag{29}$$

where

$$N_0(s)=\frac{1}{s^{\varpi}-1}. \tag{30}$$

Thus, we present the non-differentiable series components in the form

$$N_0(s)=\frac{1}{s^{\varpi}-1}, \tag{31}$$

$$N_1(s)=0\,, \tag{32}$$

$$N_2(s)=\frac{2}{s^{\varpi}-1},, \tag{33}$$

$$N_3(s)=0\,, \tag{34}$$

$$N_4(s)=\frac{4}{s^{\varpi}-1} \tag{35}$$

and so on.

As a result, we obtain the non-differentiable solution

$$\Phi\left(\mu,\tau\right)=\sum_{i=0}^{\infty}\frac{\tau^{2i\varpi}}{\Gamma\left(1+2i\varpi\right)}\tilde{\tilde{\Xi}}_{\varpi}^{-1}\left\{\frac{2^{i}}{s^{\varpi}\text{-}1}\right\}=E_{\varpi}\left(\mu^{\varpi}\right)\sum_{i=0}^{\infty}\frac{2^{i}\tau^{2i\varpi}}{\Gamma\left(1+2i\varpi\right)} \tag{36}$$

and the corresponding graph is displayed in Fig. 1.

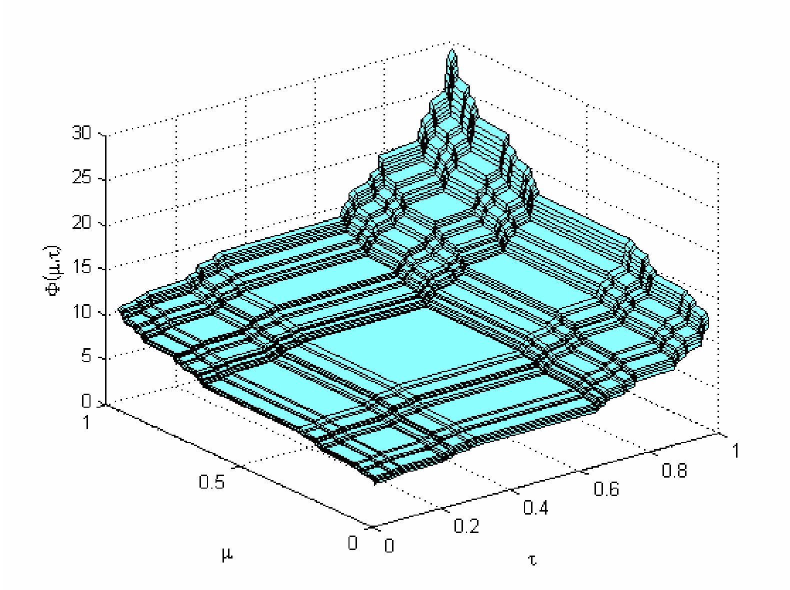

Fig. 1 – The non-differentiable series solution of local fractional KGE in (1+1) FDS.

We now consider the local fractional HE in (1+1) FDS (see, for example, [8])

$$\frac{\partial^{2\varpi}\Phi(\mu,\eta)}{\partial\eta^{2\varpi}}+\frac{\partial^{2\varpi}\Phi(\mu,\eta)}{\partial\mu^{2\varpi}}+\Phi(\mu,\eta)=0, \tag{37}$$

subject to the initial value conditions

$$\Phi(\mu,0)=0, \tag{38}$$

$$\frac{\partial^{\varpi}\Phi(\mu,0)}{\partial\mu^{\varpi}}=E_{\varpi}\left(\mu^{\varpi}\right). \tag{39}$$

With the use of Eq. (37) we have

$$\Lambda_{\varpi}\Phi(\mu,\eta)=-\frac{\partial^{2\varpi}\Phi(\mu,\eta)}{\partial\mu^{2\varpi}}-\Phi(\mu,\eta) \tag{40}$$

such that

$$\tilde{\Xi}_{\varpi}\left\{N_{i+2}\left(\mu\right)\right\}=-\tilde{\Xi}_{\varpi}\left\{\frac{\partial^{2\varpi}N_i\left(\mu\right)}{\partial\mu^{2\varpi}}\right\}-\tilde{\Xi}_{\varpi}\left\{N_i\left(\mu\right)\right\}, \tag{41}$$

where

$$\tilde{\Xi}_{\varpi}\left\{N_1\left(\mu\right)\right\}=\frac{1}{s^{\varpi}-1}. \tag{42}$$

From Eqs. (41) and (42) we obtain the non-differentiable series components in the form

$$N_0(s)=0, \tag{43}$$

$$N_1(s)=\frac{1}{s^{\varpi}-1}, \tag{44}$$

$$N_2(s)=0, \tag{45}$$

$$N_3(s)=-\frac{2}{s^{\varpi}-1}, \tag{46}$$

$$N_4(s)=0, \tag{47}$$

$$N_5(s)=-\frac{4}{s^{\varpi}-1}, \tag{48}$$

$$N_6(s)=0 \tag{49}$$

and so on.

The non-differentiable solution of Eq. (37) is written as follows:

$$\begin{aligned}\Phi(\mu,\eta)&=\tilde{\Xi}_{\varpi}^{-1}\left\{\sum_{i=0}^{\infty}\frac{\eta^{i\varpi}}{\Gamma(1+i\varpi)}N_i(s)\right\}=\\&=\sum_{i=0}^{\infty}\frac{\eta^{(2i+1)\varpi}}{\Gamma\left[1+(2i+1)\varpi\right]}\tilde{\Xi}_{\varpi}^{-1}\left\{\frac{(-2)^i}{s^{\varpi}\text{-}1}\right\}=\\&=E_{\varpi}\left(\mu^{\varpi}\right)\sum_{i=0}^{\infty}\frac{(-2)^i\eta^{(2i+1)\varpi}}{\Gamma\left[1+(2i+1)\varpi\right]}\end{aligned} \tag{50}$$

and the corresponding graph is depicted in Fig. 2 when the fractal dimension (FrD) is changed from $\varpi=\ln 2/\ln 3$ to $\varpi=1$.

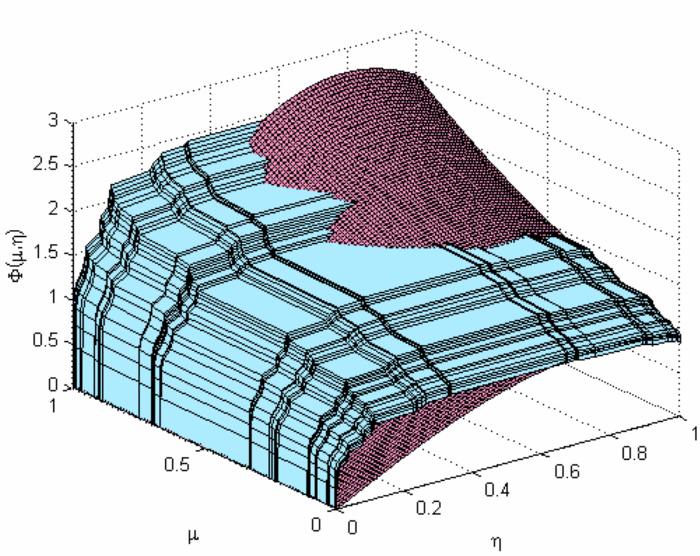

Fig. 2 – The non-differentiable series solution of local fractional HE in (1+1) FDS changing the FrD from $\varpi=\ln 2/\ln 3$ to $\varpi=1$.

5. CONCLUSIONS

In the present work, we considered the local fractional Klein-Gordon equation and Helmholtz equation in (1+1) fractal dimensional space, which appear in the description of many non-differentiable phenomena arising in physics and engineering. With the aid of the local fractional Laplace series expansion method, the non-differentiable-type series solutions for the local fractional Klein-Gordon equation and the Helmholtz equation were presented with the fractal dimension changed from $\varpi = \ln 2 / \ln 3$ to $\varpi = 1$. The implemented method is an efficient technique to determine the series solutions for the local fractional partial differential equations.

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REFERENCES

1. N. SUGIMOTO, *Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves,* J. Fluid Mech., **225**, pp. 631–653, 1991.
2. N. LASKIN, *Fractional Schrödinger equation*, Phys. Rev. E, **66**, *5*, 056108, 2002.
3. V.E. TARASOV, *Psi-series solution of fractional Ginzburg-Landau equation*, J. Phys. A, **39**, *26*, pp. 83–95, 2006.
4. R.M. HAFEZ, S.S. EZZ-ELDIEN, A.H. BHRAWY, E.A. AHMED, D. BALEANU, *A Jacobi Gauss-Lobatto and Gauss-Radau collocation algorithm for solving fractional Fokker-Planck equations*, Nonlinear Dynam., **82**, pp. 1431–1440, 2015.
5. M.S. HASHEMI, D. BALEANU, M. PARTO-HAGHIGHI, *A Lie group approach to solve the fractional Poisson equation*, Rom. J. Phys., **60**, pp. 1289–1297, 2015.
6. M. SAMUEL, A. THOMAS, *On fractional Helmholtz equations*, Fract. Calc. Appl. Anal., **13**, *3*, pp. 295–308, 2010.
7. A. AGILA, D. BALEANU, R. EID, B. IRFANOGLU, *Applications of the extended fractional Euler-Lagrange equations model to freely oscillating dynamical systems*, Rom. J. Phys., **61**, pp. 350–359, 2016.
8. X.J. YANG, D. BALEANU, H.M. SRIVASTAVA, *Local Fractional Integral Transforms and Their Applications*, Academic Press, 2015.
9. X.J. YANG, D. BALEANU, H.M. SRIVASTAVA, *Local fractional similarity solution for the diffusion equation defined on Cantor sets*, Appl. Math. Lett., **47**, pp. 54–60, 2015.
10. X.J. YANG, H.M. SRIVASTAVA, J.H. HE, D. BALEANU, *Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives*, Phys. Lett. A, **377**, *28*, pp. 1696–1700, 2013.
11. X. J. YANG, J. A. T. MACHADO, J. HRISTOV, *Nonlinear dynamics for local fractional Burgers' equation arising in fractal flow*, Nonlinear Dyn., **84**, *1*, pp. 3–7, 2016.
12. X.J. YANG, H.M. SRIVASTAVA, *An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives*, Commun. Nonlinear Sci., **29**, *1*, pp. 499–504, 2015.
13. S.P. YAN, *Local fractional Laplace series expansion method for diffusion equation arising in fractal heat transfer*, Therm. Sci., **19** (S1), pp. 131–135, 2015.
14. D. BALEANU, H.M. SRIVASTAVA, X.J. YANG, *Local fractional variational iteration algorithms for the parabolic Fokker-Planck equation defined on Cantor sets*, Progr. Fract. Differ. Appl, **1**, *1*, pp. 1–11, 2015.
15. X.J. YANG, J.T. MACHADO, D. BALEANU, C. CATTANI, *On exact traveling-wave solutions for local fractional Korteweg-de Vries equation*, Chaos, **26**, *8*, 084312, 2016.
16. Y. ZHANG, D. BALEANU, X.J. YANG, *New solutions of the transport equations in porous media within local fractional derivative*, Proc. Romanian Acad. A, **17**, pp. 230–236, 2016.
17. KAMEL AL-KHALED, *Numerical solution of time-fractional partial differential equations using Sumudu decomposition method*, Rom. J. Phys., **60**, pp. 99–110, 2015.
18. A.H. BHRAWY, M.A. ZAKY, D. BALEANU, M.A. ABDELKAWY, *A novel spectral approximation for the two-dimensional fractional sub-diffusion problems*, Rom. J. Phys., **60**, pp. 344–359, 2015.
19. M. MERDAN, *On the solutions of time-fractional generalized Hirota-Satsuma coupled-KdV equation with modified Riemann-Liouville derivative by an analytical technique*, Proc. Romanian Acad. A, **16**, pp. 3–10, 2015.
20. A.H. BHRAWY, E.H. DOHA, D. BALEANU, S.S. EZZ-ELDIEN, M.A. ABDELKAWY, *An accurate numerical technique for solving fractional optimal control problems*, Proc. Romanian Acad. A, **16**, pp. 47–54, 2015.
21. K. NOURI, S. ELAHI-MEHR, L. TORKZADEH, *Investigation of the behavior of the fractional Bagley-Torvik and Bassetequations via numerical inverse Laplace transform*, Rom. Rep. Phys., **68**, pp. 503–514, 2016.
22. T. BLASZCZYK, *A numerical solution of a fractional oscillator equation in a non-resisting medium with natural boundary conditions*, Rom. Rep. Phys., **67**, pp. 350–358, 2015.

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