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A NEW OPERATIONAL MATRIX BASED ON JACOBI WAVELETS FOR A CLASS OF VARIABLE-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The main aim of this paper is to introduce an accurate collocation method for solving a class of variable-order fractional differential equations arising in turbulent fluid dynamics. The proposed approach is based on Jacobi wavelets (JWs) collocation procedure in conjunction with the JWs operational matrix of variable-order fractional derivative. This approach can be seen as a generalization of other wavelet operational approaches, e.g. Chebyshev wavelets of first kind, Chebyshev wavelets of second kind, Legendre wavelets, Gegenbauer wavelets, etc., which are special cases of JWs. The proposed method is implemented for solving the variable-order fractional Basset and Bagley-Torvik equations.

***Key words*:** variable-order fractional derivative, Jacobi wavelets, operational matrix, variable-order Bagley-Torvik equation, variable-order Basset equation.

1. INTRODUCTION

Differential equations with variable-order (fixed) fractional derivatives have important applications in science and engineering [1–13]. The definition of variable-order fractional derivative was proposed by Samko and Ross in Ref. [12]. The variable-order fractional differential equation can be viewed as a natural generalization of the fixed fractional differential equation. Many physical problems are governed by variable-order fractional differential equations, such as Schrödinger equation [14, 15], mobile-immobile advection-dispersion equation [16], Rayleigh-Stokes equation [17], Galilei invariant advection diffusion equations [18], wave equation [19], diffusion equation [20], cable equation [21] and so on. The analytical solutions of most variable-order fractional differential equations are not easy to obtain. Therefore, finding the solutions of these equations has attracted much attention of many researchers [22–24].

Recently, wavelets basis functions and their properties have demonstrated their usefulness in several areas of science and engineering. The main advantages of these functions are given in [25] as: (1) the basis set can be improved in a systematic way, (2) the numerical effort scales linearly with respect to the system size, (3) different resolutions can be used in different regions of space, (4) there are few topological constraints for increased resolution regions, (5) the coupling between different resolution levels is easy, (6) the Laplace operator is diagonally dominant in an appropriate wavelet basis, and (7) the matrix elements of the Laplace operator are very easy to calculate. Wavelets have been widely used in practical applications, such as signal analysis for waveform representations and segmentations and analysis of fast algorithms for easy implementation [26]. Wavelets construct a connection with efficient numerical algorithms [27, 28]. For solving variable-order fractional differential equations, the operational matrices of variable-order fractional derivatives for the Chebyshev wavelets [29] and the Legendre wavelets [30] are calculated.

In this paper, we develop an efficient approach for the variable-order fractional initial value problem [31–33]:

$$\begin{aligned} & y^{(n)}(t) = {}_0^C\mathcal{D}_t^{\nu(t)} y(t) + f\left(t, y(t)\right), \quad n-1 < \nu(t) \le n, \\ & y^{(r)}(0) = y_0^r, \quad r = 0,1,\ldots n-1. \end{aligned} \tag{1}$$

Here, the variable-order fractional derivative ${}_0^C\mathcal{D}_t^{\nu(t)}$ is given by [12]

$${}_0^C\mathcal{D}_t^{\nu(t)} y(t) = \frac{1}{\Gamma\left(n-\nu(t)\right)} \int_0^t \frac{y^{(n)}(t)}{(t-\tau)^{\nu(t)-n+1}} \mathrm{d}\tau, \quad n-1 < \nu(t) \le n, , \tag{2}$$

The outline of this paper is as follows. Section 2 introduces some characteristics of JWs. In Section 3, we derive new operational matrices of JWs. In Section 4, we develop a numerical approach based on JWs collocation technique in conjunction with the operational matrices of the variable-order fractional derivative (2). In Section 5, we report numerical results and comparisons for the variable-order fractional Basset and Bagley-Torvik equations. Section 6 outlines the main conclusions.

2. JACOBI WAVELETS AND THEIR PROPERTIES

The Jacobi polynomials $\mathcal{P}_m^{(\sigma,\varsigma)}(t)$; $\sigma, \varsigma \ge -1$, $t \in [-1,1]$ are defined by the three-term recurrence relation [34, 35]

$$\begin{aligned} & \mathcal{P}_{m+1}^{(\sigma,\varsigma)}(t) = \left(a_m^{(\sigma,\varsigma)} t - b_m^{(\sigma,\varsigma)}\right) \mathcal{P}_m^{(\sigma,\varsigma)}(t) - c_m^{(\sigma,\varsigma)} \mathcal{P}_{m-1}^{(\sigma,\varsigma)}(t), \qquad m \ge 1, \\ & \mathcal{P}_0^{(\sigma,\varsigma)}(t) = 1, \qquad \mathcal{P}_1^{(\sigma,\varsigma)}(t) = \frac{1}{2}(\sigma+\varsigma+2)t + \frac{1}{2}(\sigma-\varsigma), \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_m^{(\sigma,\varsigma)} &= \frac{(2m+\sigma+\beta+1)(2m+\sigma+\varsigma+2)}{2(m+1)(m+\sigma+\varsigma+1)}, \\ b_m^{(\sigma,\varsigma)} &= \frac{(2m+\sigma+\varsigma+1)(\varsigma^2-\sigma^2)}{2(m+1)(m+\sigma+\varsigma+1)(2m+\sigma+\varsigma)}, \\ c_m^{(\sigma,\varsigma)} &= \frac{(2m+\sigma+\varsigma+2)(m+\sigma)(m+\varsigma)}{(m+1)(m+\sigma+\varsigma+1)(2m+\sigma+\varsigma)}. \end{aligned}$$

The explicit forme of $\mathcal{P}_i^{(\sigma,\varsigma)}(t)$ is given by

$$\mathcal{P}_i^{(\sigma,\varsigma)}(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+1+\varsigma)\Gamma(i+1+k+\sigma+\varsigma)}{\Gamma(k+\varsigma+1)\Gamma(i+1+\sigma+\varsigma)(i-k)!k!(2)^k}(t+1)^k. \tag{4}$$

They satisfy the following orthogonality condition

$$\int_{-1}^{1}(1-t)^{\sigma}(1+t)^{\varsigma}\mathcal{P}_m^{(\sigma,\varsigma)}(t)\mathcal{P}_n^{(\sigma,\varsigma)}(t)\mathrm{d}t = \begin{cases} \dfrac{1}{h_m}, & \text{for } m=n, \\ 0, & \text{for } m \ne n. \end{cases} \tag{5}$$

where

$$h_m^{(\sigma,\varsigma)} = \frac{(2m+\sigma+\varsigma+1)\Gamma(m+1)\Gamma(m+\sigma+\varsigma+1)}{2^{(\sigma+\varsigma)}\Gamma(m+\sigma+1)\Gamma(m+\varsigma+1)}. \tag{6}$$

Now, we utilize the Jacobi polynomials to construct the JWs. Wavelets are a family of functions constructed from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter α and the translation parameter β vary continuously, we have the following family of continuous wavelets as

$$\varphi_{\alpha,\beta}(t) = |\alpha|^{-\frac{1}{2}}\varphi\left(\frac{t-\beta}{\alpha}\right), \ \alpha, \ \beta \in \mathbb{R}, \ \alpha \ne 0.$$

The JWs $\varphi_{nm}(t)=\varphi(k,n,m,t)$ have four arguments: $k\in\mathbb{N}$, $n==1,\ldots,2^k$, m is the order of Jacobi polynomials, and t is the normalized time. They are defined on the interval $[0,L]$ by:

$$\varphi_{nm}(t)=\begin{cases}\sqrt{h_m^{(\sigma,\varsigma)}}\,\dfrac{2^{\frac{k}{2}}}{\sqrt{L}}\mathcal{P}_m^{(\sigma,\varsigma)}\left(\dfrac{2^{k+1}}{L}t-2n+1\right), & \dfrac{(n-1)L}{2^k}\le t\le\dfrac{nL}{2^k}\\ 0, & \text{otherwise.}\end{cases}$$

where $m=0,\ldots,M-1$ and $n=1,\ldots,2^k$.

The coefficient $\sqrt{h_m^{(\sigma,\varsigma)}}$ is for orthonormality, the dilation parameter is $\alpha=\dfrac{L}{2^k}$ and the translation parameter is $\beta=\dfrac{nL}{2^k}$.

The JWs form an orthonormal basis for $L^2_{\omega_n}[0,L]$, i.e.:

$$\left(\varphi_{nm}(t),\varphi_{n'm'}\right)=\int_0^L\omega_n(t)\varphi_{nm}(t)\varphi_{n'm'}(t)\,\mathrm{d}t=\begin{cases}1, & (n,m)=(n',m'),\\ 0, & (n,m)\neq(n',m'),\end{cases}\tag{7}$$

where

$$\omega_n=\begin{cases}(2n-\dfrac{2^{k+1}}{L}t)^{\sigma}(\dfrac{2^{k+1}}{L}t-2n+2)^{\varsigma}, & \dfrac{(n-1)L}{2^k}\le t\le\dfrac{nL}{2^k},\\ 0, & \text{otherwise.}\end{cases}$$

A function $y(t)$ that is infinitely differentiable in $[0,L]$ may be expanded by means of JWs as:

$$y(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}a_{nm}\varphi_{nm}(t)\tag{8}$$

where $a_{nm}=\left(y(t),\varphi_{nm}(t)\right)$ and $(.,.)$ denotes the inner product in $L^2_{\omega_n}[0,L]$ with respect to the weight function ω_n. By truncating the expansion (8), we obtain an approximate expression of $y(t)$ as:

$$y(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}a_{nm}\varphi_{nm}(t)=\mathbf{A}^T\Psi(t),\tag{9}$$

where $\mathbf{A}$ and $\Psi(t)$ are column vectors with $\tilde{m}=2^kM$ elements. For simplicity, (9) can be rewritten as:

$$y(t)=\sum_{\ell=1}^{\tilde{m}}a_\ell\varphi_\ell(t)=\mathbf{A}^T\Psi(t),\tag{10}$$

where $a_\ell=a_{nm}$ and $\varphi_\ell=\varphi_{nm}$ and the index ℓ is determined by $\ell=M(n-1)+m+1$. Thus, we have

$$\begin{aligned}\mathbf{A}^T&=[a_1,a_2,\cdots,a_{\tilde{m}}],\\ \Psi(t)&=[\varphi_1(t),\varphi_2(t),\cdots,\varphi_{\tilde{m}}(t)]^T.\end{aligned}\tag{11}$$

For the subsequent discussion, we introduce the following vector of piecewise functions:

$$\mathbf{Z}(t)=[\mathcal{Z}_1(t),\mathcal{Z}_2(t),\cdots,\mathcal{Z}_{\tilde{m}}(t)]^T,\tag{12}$$

where

$$\mathcal{Z}_\ell(t)=\mathcal{Z}_{nm}(t)=\begin{cases}t^m & \dfrac{(n-1)L}{2^k}\le t\le\dfrac{nL}{2^k},\\ 0 & \text{otherwise,}\end{cases}\tag{13}$$

and the index ℓ is determined by $\ell=M(n-1)+m+1$. The connection between these functions and the JWs is given by

$$\mathbf{Z}(t)=\mathbf{P}\Psi(t), \tag{14}$$

where $p_{ij}=(\mathcal{Z}_i,\varphi_j)$ are the matrix entries of $\mathbf{P}$.

3. NEW OPERATIONAL MATRICES

The first-order derivative of the JWs vector $\Psi(t)$ can be expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)=\mathbf{D}^{(1)}\Psi(t), \tag{15}$$

where $\mathbf{D}^{(1)}=\mathbf{P}^{-1}\mathbf{RP}$ is the $\tilde{m}\times\tilde{m}$ operational matrix of the first derivative. Otherwise, we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)=\mathbf{P}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Z}(t)=\mathbf{P}^{-1}\mathbf{RZ}(t)=\mathbf{P}^{-1}\mathbf{RP}\Psi(t)=\mathbf{D}^{(1)}\Psi(t), \tag{16}$$

where $\mathbf{R}=\mathbf{I}\otimes\mathbf{E}$, $\mathbf{I}$ is the $2^k\times 2^k$ identity matrix and

$$\mathbf{E}=(e_{ij})=\begin{cases} j+1, & \text{for } i=j+1,\ j=0,1,\ldots,M-1,\\ 0 & \text{otherwise.}\end{cases} \tag{17}$$

Repeated use of (16), enables one to write

$$\frac{\mathrm{d}^q}{\mathrm{d}t^q}\Psi(t)=(\mathbf{D}^{(1)})^q\Psi(t)=\mathbf{D}^{(q)}\Psi(t),\quad q=1,2,\ldots. \tag{18}$$

where $q\in\mathbb{N}$ and the superscript in $\mathbf{D}^{(q)}$ denotes matrix powers.

The following theorem generalizes the operational matrix of derivatives of JWs given in (16) for the variable-order fractional derivatives.

THEOREM 3.1. *The variable-order fractional derivative of the JWs vector* $\Psi(t)$ *is given by*

$${}_0^C\mathcal{D}_t^{\nu(t)}\Psi(t)=\mathbf{D}_{\nu(t)}\Psi(t), \tag{19}$$

where $n-1<\nu(t)\le n\in\mathbb{N}$ *and* $\mathbf{D}_{\nu(t)}$ *is an* $2^kM\times 2^kM$ *matrix of the following form*

$$\mathbf{D}_{\nu(t)}=\mathbf{P}^{-1}(\mathbf{I}\otimes\mathbf{K})\mathbf{P},$$

and $\mathbf{K}$ *is a* $M\times M$ *matrix and its elements,* $k_{ij};\ 0\le i,j\le M-1$ *are given by*

$$k_{ij}=\begin{cases}\dfrac{t^{-\nu(t)}\Gamma(i+1)}{\Gamma(i+1-\nu(t))}, & \text{for } i=j,\ j=n,n+1,\ldots,M-1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Direct calculations give

$$\begin{aligned}{}_0^C\mathcal{D}_t^{\nu(t)}\Psi(t)&=\mathbf{P}^{-1}\,{}_0^C\mathcal{D}_t^{\nu(t)}\mathbf{Z}(t)=\mathbf{P}^{-1}(\mathbf{I}\otimes\mathbf{K})\mathbf{Z}(t)=\\ &=\mathbf{P}^{-1}(\mathbf{I}\otimes\mathbf{K})\mathbf{P}\Psi(t)=\mathbf{D}_{\nu(t)}\Psi(t).\end{aligned} \tag{20}$$

4. NUMERICAL SCHEME

In the present section, the JWs expansion and its operational matrices are used to derive the collocation scheme for (1).

In virtue of (10), we can write the approximate solution and its derivatives as

$$\begin{aligned} &y(t)=\mathbf{A}^T\Psi(t),\\ &\frac{d^n}{dt^n}y(t)=\mathbf{A}^T\mathbf{D}^{(q)}\Psi(t),\\ &{}_0^C\mathcal{D}_t^{\nu(t)}y(t)=\mathbf{A}^T\mathbf{D}_{\nu(t)}\Psi(t). \end{aligned} \tag{21}$$

Employing (21) in (1), yields

$$\begin{aligned} &\mathbf{A}^T\mathbf{D}^{(n)}\Psi(t)+\mathbf{A}^T\mathbf{D}_{\nu(t)}\Psi(t)=f\left(t,\mathbf{A}^T\Psi(t)\right), \quad n-1<\nu(t)\le n,\\ &\mathbf{A}^T\mathbf{D}^{(r)}\Psi(0)=y_0^r, \quad r=0,1,\dots n-1. \end{aligned} \tag{22}$$

The previous discussion enables us to write the collocation-like discretization associated with (1) as

$$\begin{aligned} &\mathbf{A}^T\mathbf{D}^{(n)}\Psi(t_i)+\mathbf{A}^T\mathbf{D}_{\nu(t_i)}\Psi(t_i)=f\left(t_i,\mathbf{A}^T\Psi(t_i)\right), \quad i=n+1\dots\tilde{m}\\ &\mathbf{A}^T\mathbf{D}^{(r)}\Psi(0)=y_0^r, \quad r=0,1,\dots n-1. \end{aligned} \tag{23}$$

where t_i $(1\le i\le\tilde{m})$ are the shifted Jacobi collocation nodes of $\mathcal{P}_{\tilde{m}}^{(\sigma,\varsigma)}(t)$. This constitutes a system of $\tilde{m}$ algebraic equations in the required coefficients a_i, $i=1,\dots,\tilde{m}$. Consequently, the approximate solution of $y(t)$ given by (21) can be evaluated.

THEOREM 4.1 (Convergence of the Jacobi wavelets expansion). *Let* 2^k, $M\to\infty$. *Then the series solution (21) converges to* $y(t)$.

Proof. Suppose $\tilde{\alpha}=2^k$, $\alpha=2^b$, $\tilde{\beta}=M$, and $\beta=N$, where k and b are the level of resolutions and M, N denotes the order of Jacobi polynomials.

Let $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ be a sequence of partial sums of $a_{ij}\varphi_{ij}(t)$; we prove that $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ is a Cauchy sequence in $L^2_{\omega_n}[0,L]$ and then we show that $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ converges to $y(t)$, when $\tilde{\alpha}$, $\tilde{\beta}\to\infty$

First we show that $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ is a Cauchy sequence. Therefore, let $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ be arbitrary sums of $a_{ij}\varphi_{ij}(t)$ with $\tilde{\alpha}>\alpha$, $\tilde{\beta}>\beta$. Then

$$\begin{aligned} \left\|\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}-\mathcal{S}_{\alpha,\beta}\right\|^2&=\left\|\sum_{i=\alpha+1}^{\tilde{\alpha}}\sum_{j=\beta}^{\tilde{\beta}-1}a_{ij}\varphi_{ij}(t)\right\|^2=\left(\sum_{i=\alpha+1}^{\tilde{\alpha}}\sum_{j=\beta}^{\tilde{\beta}-1}a_{ij}\varphi_{ij}(t),\sum_{r=\alpha+1}^{\tilde{\alpha}}\sum_{s=\beta}^{\tilde{\beta}-1}a_{rs}\varphi_{rs}(t)\right)=\\ &=\sum_{i,r=\alpha+1}^{\tilde{\alpha}}\sum_{j,s=\beta}^{\tilde{\beta}-1}a_{ij}a_{rs}\left(\varphi_{ij}(t),\varphi_{rs}(t)\right)=\sum_{i=\alpha+1}^{\tilde{\alpha}}\sum_{j=\beta}^{\tilde{\beta}-1}\left|a_{ij}\right|^2 \end{aligned} \tag{24}$$

From the Bessel's inequality, $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left|a_{ij}\right|^2$ converges and hence

$$\left\|\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}-\mathcal{S}_{\alpha,\beta}\right\|^2\to 0 \quad \text{as } \tilde{\alpha},\ \tilde{\beta},\ \alpha,\ \beta\to\infty.$$

Thus, $\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}$ is a Cauchy sequence that converges to, say, $g(t)\in L^2_{\omega_n}[0,L]$. We show that $g(t)=y(t)$

$$\begin{aligned} \left(g(t)-y(t),\varphi_{ij}(t)\right)&=\left(g(t),\varphi_{ij}(t)\right)-\left(y(t),\varphi_{ij}(t)\right)\\ &=\lim_{\tilde{\alpha},\tilde{\beta}\to\infty}\left(\mathcal{S}_{\tilde{\alpha},\tilde{\beta}},\varphi_{ij}(t)\right)-\left(a_{ij}\varphi_{ij},\varphi_{ij}(t)\right)\\ &=a_{ij}-a_{ij}=0. \end{aligned}$$

Hence, $\sum_{i=1}^{\tilde{\alpha}}\sum_{j=0}^{\tilde{\beta}-1}a_{ij}\varphi_{ij}(t)\to y(t)$ as $\tilde{\alpha}$, $\tilde{\beta}\to\infty$.

5. NUMERICAL RESULTS

This Section presents two numerical examples to illustrate the efficiency of the proposed method. We shall apply the method in the solution of the variable-order fractional Basset and Bagley-Torvik equations. These equations motivated the attention of mathematicians and physicists, such as Podlubny [36], Moghaddam *et al*. [33], Mainardi *et al*. [37] and Stanek [38]. For all numerical computations we consider $L=5$.

Example 1. Consider the following variable-order fractional Basset equation [22, 33, 39]:

$$y'(t)-2{}_0^C\mathcal{D}_t^{\nu(t)}y(t)+2y(t)=\frac{-4t^{2-\nu(t)}-\Gamma(3-\nu(t))(2(t^2-1)-2t)}{\Gamma(3-\nu(t))},\quad 0<\nu(t)\le 1, \qquad (25)$$

$$y(0)=1.$$

The exact solution is $y(t)=1-t^2$.

The maximum absolute errors (MAEs) of the approximate solution are shown in Table 1.

Example 2. Consider the following variable-order fractional Bagley-Torvik equation [22, 33]:

$$y''(t)+{}_0^C\mathcal{D}_t^{\nu(t)}y(t)+y(t)=\frac{\Gamma(4)}{\Gamma(4-\nu(t))}t^{3-\nu(t)}+t^3+7t+1,\quad 1<\nu(t)\le 2, \qquad (26)$$

$$y(0)=y'(0)=1.$$

The exact solution is $y(t)=t^3+t+1$.

The MAEs of the approximate solution are shown in Table 2.

Table 1

Comparison of the MAEs for problem (25)

with $(2\sigma=2\varsigma=k=1)$ and $t\in[0,\frac{\pi}{2}]$

$\nu(t)$	M-algorithm [22]		IM-algorithm[33]		Our method
0.5	$h=0.02$	8.17×10^{-3}	2.17×10^{-4}	$M=4$	1.37×10^{-14}
	$h=0.01$	4.31×10^{-3}	7.72×10^{-5}	$M=5$	1.34×10^{-14}
	$h=0.005$	2.22×10^{-3}	2.74×10^{-5}	$M=6$	1.33×10^{-15}
$\frac{1-\lvert\cos t\rvert}{4}$	$h=0.02$	5.59×10^{-3}	1.14×10^{-5}	$M=4$	2.14×10^{-14}
	$h=0.01$	2.78×10^{-3}	3.37×10^{-6}	$M=5$	1.02×10^{-14}
	$h=0.005$	1.38×10^{-3}	9.82×10^{-7}	$M=6$	3.33×10^{-15}

Table 2

Comparison of the MAEs for problem (26)

with $(2\sigma=2\varsigma=-k=-1)$ and $t\in[0,\frac{\pi}{2}]$

$\nu(t)$	M-algorithm [22]		IM-algorithm[33]		Our method
1.5	$h=0.02$	8.16×10^{-3}	1.15×10^{-3}	$M=4$	3.55×10^{-15}
	$h=0.01$	4.27×10^{-3}	4.29×10^{-4}	$M=5$	3.55×10^{-15}
	$h=0.005$	2.18×10^{-3}	1.56×10^{-4}	$M=6$	3.12×10^{-15}
$1+0.5\lvert\sin t\rvert$	$h=0.02$	7.22×10^{-3}	1.53×10^{-4}	$M=4$	4.00×10^{-15}
	$h=0.01$	3.70×10^{-3}	5.57×10^{-5}	$M=5$	3.12×10^{-15}
	$h=0.005$	1.86×10^{-3}	1.93×10^{-5}	$M=6$	4.00×10^{-15}

6. CONCLUSION

We have presented an efficient approach based on JWs collocation procedure in conjunction with the operational matrix of variable-order fractional derivative. The proposed method is implemented for solving the variable-order fractional Basset and Bagley-Torvik equations. This method provides a very accurate approximate solution using few terms of the JWs expansion. From the numerical results and comparisons given in Section 5, we may conclude that the obtained results are excellent in terms of accuracy for all tested problems.

REFERENCES

1. G.R. COPER, D.R. COWAN, *Filtering using variable order vertical derivatives*, Comput. Geosci., **30**, pp. 455–459, 2004.
2. M.A. ABDELKAWY, M.A. ZAKY, A.H. BHRAWY, D. BALEANU, *Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model*, Rom. Rep. Phys., **67**, pp. 773–791, 2015.
3. A.H. BHRAWY, M.A. ZAKY, M. ABDEL-ATY, *A fast and precise numerical algorithm for a class of variable-order fractional differential equations*, Proc. Romanian Acad. A, **18**, pp. 17–24, 2017.
4. A.H. BHRAWY, M.A. ZAKY, *A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations*, J. Comput. Phys., **281**, pp. 876–895, 2015.
5. K. PARAND, H. YOUSEFI, M. DELKOSH, *A numerical approach to solve Lane-Emden type equations by the fractional order of rational Bernoulli functions*, Rom. J. Phys., **62**, p. 104, 2017.
6. D. KUMAR, J. SINGH, D. BALEANU, *A fractional model of convective radial fins with temperature-dependent thermal conductivity*, Rom. Rep. Phys., **69**, p. 103, 2017.
7. M.A. SAKER, S.S. EZZ-ELDIEN, A.H. BHRAWY, *A pseudo spectral method for solving the time-fractional generalized Hirota-Satsuma coupled Korteweg-de Vries system,* Rom. J. Phys., **62**, p. 105, 2017.
8. A.H. BHRAWY, M.A. ZAKY, D. BALEANU, *New numerical approximations for space-time fractional Burgers' equations via a Legendre spectral-collocation method*, Rom. Rep. Phys., **67**, pp. 340–349, 2015.
9. W.M. ABD-ELHAMED, Y.H. YOUSSRI, *Spectral solutions for fractional differential equations via a novel Lucas operational matrix of fractional derivatives*, Rom. J. Phys., **61**, pp. 795–813, 2016.
10. A.H. BHRAWY, M.A. ZAKY, *Numerical algorithm for the variable-order Caputo fractional functional differential equation*, Nonlinear Dyn., **85**, pp. 1815–1823, 2016.
11. A.H. BHRAWY, M.A. ZAKY, D. BALEANU, M.A. ABDELKAWY, *A novel spectral approximation for the two-dimensional fractional sub-diffusion problems,* Rom. J. Phys., **60**, pp. 344–359, 2015.
12. S. SAMKO, B. ROSS, *Integration and differentiation to a variable fractional order*, Integr. Transf. Spec. Funct., **1**, pp. 277–300, 1993.
13. J. OROSCO, C.F.M. COMBIRA, *On the control and stability of variable-order mechanical systems*, Nonlinear Dyn., **86**, pp. 695–710, 2016.
14. A.H. BHRAWY, M.A. ZAKY, *Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations*, Comput. Math. Appl., **373**, pp. 1100–1117, 2017.
15. A.H. BHRAWY, M.A. ZAKY, *An improved collocation method for multi-dimensional space-time variable-order fractional Schrödinger equations*, Appl. Numer. Math., **111**, pp. 197–218, 2017.
16. H. ZHANG, F. LIU, M.S. PHANIKUMAR, M.M MEERSCHAERT, *A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model*, Comput. Math. Appl., **66**, pp. 693–701, 2013.
17. A.H. BHRAWY, M.A. ZAKY, J.F. ALZAIDY, *Two shifted Jacobi-Gauss collocation schemes for solving two-dimensional variable-order fractional Rayleigh-Stokes problem,* Advances Diff. Equ., **2016**, pp. 1–17, 2016.
18. M.A. ABDELKAWY, R.T. ALQAHTANI, *Space-time spectral collocation algorithm for the variable-order Galilei invariant advection diffusion equations with a nonlinear source term,* Math. Model. Anal., **22**, pp. 1–20, 2017.
19. A.H. BHRAWY, EH DOHA, J.F. AL-ZAIDI, M.A. ABDELKAWY, *Space-time spectral collocation algorithm for the variable order fractional wave equation,* Springer Plus, **2016**, pp. 1–18, 2016.
20. M.A. ZAKY *et al.*, *An efficient operational matrix technique for multi-dimensional variable-order time fractional diffusion equations*, J. Comput. Nonlinear Dyn., **11**, pp. 1–8, 2016.
21. A.H. BHRAWY, M.A. ZAKY, *Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation*, Nonlinear Dyn., **80**, pp. 101–116, 2015.
22. B.P. MOGHADDAM, J.A. MACHADO, *Extended algorithms for approximating variable order fractional derivatives with applications*, J. Sci. Comput., pp. 1-24, 2016, doi:10. 1007/s10915-016-0343-1.
23. B.P. MOGHADDAM, J.A. MACHADO, *Stable three-level explicit spline finite difference scheme for a class of nonlinear time variable order fractional partial differential equations*, Comput. Math. Appl., **73**, pp. 1262–1269, 2017.
24. X. LI, B. WU, *A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations*, J. Comput. Appl. Math., **311**, pp. 387–393, 2017.
25. S. GOEDECKER, O. IVANOV, *The solution of multi-scale partial differential equations using wavelets*, Comput. Phys., **12**, pp. 548–555, 1998.
26. C.K. CHUI, *Wavelets: A Mathematical Tool for Signal Analysis*, SIAM, Philadelphia, 1997.
27. Y.H. YOUSSRI, W.M. ABD-ELHAMEED, E.H. DOHA, *Ultra spherical wavelets method for solving Lane-Emden type*

equations, Rom. J. Phys., **60**, pp. 1298–1314, 2015.
28. G. BEYLIN, R. COIFMAN, V. ROKHLIN, *Fast wavelet transforms and numerical algorithms*, Commun. Pure. Appl. Math., **44**, pp. 141–183, 1991.
29. M.H. HEYDARI *et al.*, *An optimization wavelet method for multi variable-order fractional differential equations*, Fundamenta Informat., **151**, pp. 255–273, 2017.
30. Y.-M. CHEN *et al.*, *Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets,* Appl. Math. Lett., **46**, pp. 83–88, 2015.
31. H. BRUNNER, T. TANG, *Polynomial spline collocation methods for the nonlinear Basset equation*, Comput. Math. Appl., **18**, pp. 449–457, 1989.
32. K. NOURI, S. ELAHI-MEHR, L. TORKZADEH, *Investigation of the behavior of the fractional Bagley-Torvik and Basset equations via numerical inverse Laplace transform*, Rom. Rep. Phys., **68**, pp. 503–514, 2016.
33. B.P. MOGADDAM, J.A. MACHADO, H. BEHFOROOZ, *An integro quadratic spline approach for a class of variable-order fractional initial value problems*, Chaos, Solitons & Fractals, pp. 1–7, 2017, http://doi.org/10.1016/j.chaos.2017.03.065.
34. A.H. BHRAWY, M.A. ZAKY, *Numerical simulation of multi-dimensional distributed-order generalized Schrödinger equations*, Nonlinear Dyn., pp. 1–18, 2017, DOI: 10.1007/s11071-017-3525-y.
35. A.H. BHRAWY, M.A. ZAKY, *Shifted fractional-order Jacobi orthogonal functions: Application to a system of fractional differential equations*, Appl. Math. Model., **40**, pp. 832–845, 2016.
36. I. PODLUBNY, *Fractional differential equations - an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1999.
37. F. MAINARDI, P. PIRONI, F. TAMPIERI, *On a generalization of Basset problem via fractional calculus*, Proceedings Canadian congress of applied mechanics, **95**, pp. 836–847, 1995.
38. K.S. STANEK, *Periodic problem for the generalized Basset fractional differential equation*, Fract. Calc. Appl. Anal., **18**, 1277–1290, 2015.
39. S. MCKEE, A. STOKESS, *Product integration methods for the nonlinear Basset equation*, SIAM J. Numer. Anal., **20**, 143–160, 1983.

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