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PROPAGATION OF COUPLED POROSITY-TEMPERATURE WAVES IN ISOTROPIC POROUS MEDIA

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Abstract. In this paper an application of a non-equilibrium thermodynamic theory developed in previous papers for porous media filled by a fluid flow is given. A problem of propagation of coupled porosity and temperature waves is explored in a special case. For perfect isotropic media and in the one dimensional case the dispersion relation is worked out and three modes of propagation are derived. The diagrams of the wave propagation speeds as functions of the wave number are represented. The carried out results have applications in several fields of science, as seismic waves, biology, medical sciences, nanotechnology and geology.
Key words: non-equilibrium thermodynamics, internal variables, porous media, waves in porous media.

1. INTRODUCTION

In previous articles [1,2] the propagation of coupled porosity-fluid concentration waves and porosity-fluid concentration flux-temperature waves was studied, using a non-equilibrium thermodynamic model (see [3–7]) for porous media satured by a fluid flow, formulated following the standard methods of the non-equilibrium thermodynamics with internal variables (see [8–10]). In this contribution we apply this theory to a problem of coupled porosity-temperature waves in perfect isotropic porous media in the one dimensional case. In Section 2 we present the equations of the thermodynamical model used to describe the behaviour of an anisotropic porous medium filled by a fluid flow. In Section 3 we specialize the transport equation for the temperature field and the rate equations for the porosity field and its flux (see [4] and [3]) in a special case. In Section 4 we treat the perfect isotropic case. In Section 5, assuming that the medium occupies the whole space, in the one-dimesional case, we derive the dispersion relation and the velocities of the three modes of propagation of coupled porosity-temperature plane harmonic waves. We represent the diagrams of these speeds as functions of the wave number. The worked out results can be applied in several technological sectors as medical sciences, biology, geology and nanotechnology. In the nanostructures the volume element size L along a direction is comparable or smaller than the free mean path l of the heat carriers, the phonons, i.e. $\frac{l}{L} \geq 1$, the rate of variation of their properties is faster than the time scale of the relaxation times of the fluxes to their values of equilibrium and situations of propagation of high-frequency waves are present. In [11] a thermodynamic theory for erosion and/or deposition in elastic porous media was developed by the authors.

2. GOVERNING EQUATIONS

In the following we use the standard Cartesian tensor notation in rectangular coordinate systems and we refer to a current configuration K_t at the time t. We consider a theory for porous media filled by a fluid flow, formulated in [3–6] in the framework of rational extended thermodynamics, where we have assumed that there are present the elastic field described by the symmetric stress tensor τ_{ij} and the small-strain tensor ε_{ij}; the thermal field described by the temperature T, its gradient $T_{,i}$ and the heat flux q_i; the field of the fluid concentration c, its gradient $c_{,i}$ and its flux j_i^c; the porosity field, whose geometric description is given by a structural permeability tensor r_{ij} à la Kubik (see [1,5]); its gradient $r_{ij,k}$ and its flux $\mathscr{V}_{ijk}$, because the thin porous channels sometimes can self propagate because of some surrounding favorable conditions. Thus, we choose the following thermodynamic state vector: $C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, c_{,i}, T_{,i}, r_{ij,k}, \mathscr{V}_{ijk}\}$, where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with u_i the displacement field.

We suppose that the porous skeleton and the fluid, flowing inside it, constitute two-components mixture, ρ_1 being the fluid mass density and ρ_2 the density of the elastic porous skeleton, so that we have $\rho = \rho_1 + \rho_2$, with ρ the density of the mixture as a whole. In the following we suppose the mass density ρ is constant.

We consider *the continuity equation* (see [4, 9])

$$\rho \dot{c} + j_{i,i}^c = 0, \tag{1}$$

where the source term is neglected, a superimposed dot indicates the material derivative (i.e. $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + x_\gamma \frac{\partial}{\partial x_\gamma}$), a comma in lower indices indicates the spatial derivation, Einstein convention for repeated indices is used. In (1) the concentration of the fluid is defined by $c = \frac{\rho_1}{\rho}$ and its flux j_i^c is given by $j_i^c = \rho_1(v_{1i} - v_i)$, where v_{1i} is the fluid velocity and v_i the barycentric velocity of the mixture, defined by $\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}$, v_{2i} being the velocity of the porous skeleton.

In [4] *the rate equations* for r_{ij} and the fluxes j_i^c, q_i and $\mathscr{V}_{ijk}$, obeying the objectivity and frame indifference principles [12], were obtained in the form

$$\dot{r}_{ij} + \mathscr{V}_{ijk,k} = \beta^1_{ijkl}\varepsilon_{kl} + \beta^2_{ijkl}r_{kl} + \beta^3_{ijk}j_k^c + \beta^4_{ijk}q_k + \beta^5_{ijklm}\mathscr{V}_{klm} + \beta^6_{ijk}c_{,k} + \beta^7_{ijk}T_{,k} + \beta^8_{ijklm}r_{kl,m}, \tag{2}$$

$$\tau^q \dot{q}_i = \chi^1_{ij}j_j^c - q_i + \chi^3_{ijkl}\mathscr{V}_{jkl} + \chi^4_{ij}c_{,j} - \chi^5_{ij}T_{,j} + \chi^6_{ijkl}r_{jk,l}, \tag{3}$$

$$\tau^{j^c} \dot{j}_i^c = -j_i^c + \xi^2_{ij}q_j + \xi^3_{ijkl}\mathscr{V}_{jkl} - \xi^4_{ij}c_{,j} + \xi^5_{ij}T_{,j} + \xi^6_{ijkl}r_{jk,l}, \tag{4}$$

$$\dot{\mathscr{V}}_{ijk} = \gamma^1_{ijkl}j_l^c + \gamma^2_{ijkl}q_l + \gamma^3_{ijklmn}\mathscr{V}_{lmn} + \gamma^4_{ijkl}c_{,l} + \gamma^5_{ijkl}T_{,l} + \gamma^6_{ijklmn}r_{lm,n}, \tag{5}$$

where the phenomenological tensors of different order $\boldsymbol{\beta}^s$ ($s = 1, 2, \ldots, 8$), $\boldsymbol{\chi}^m$ ($m = 1, 3, 4, 5, 6$), $\boldsymbol{\xi}^n$ ($n = 2, 3, 4, 5, 6$) and $\boldsymbol{\gamma}^q$ ($q = 1, 2, \ldots, 6$) are assumed constant. Furthermore, in (3) τ^q is the relaxation time of the heat flux q_i and in (4) τ^{j^c} is the relaxation time of the fluid concentration flux j_i^c. In [4] (see also [3]) the constitutive equations were worked out to close the system of balance equations for the media under consideration, and in [3] *the generalized telegraph temperature equation* was derived in the form

$$\tau^q \ddot{T} + \dot{T} = -\gamma_{ij}(\tau^q \ddot{\varepsilon}_{ij} + \dot{\varepsilon}_{ij}) + \varphi(\tau^q \ddot{c} + \dot{c}) + \eta_{ij}(\tau^q \ddot{r}_{ij} + \dot{r}_{ij}) + \mathscr{K}_{ij}T_{,ji} - \nu^1_{ij}j^c_{j,i} - \nu^3_{ijkl}\mathscr{V}_{jkl,i} - \nu^4_{ij}c_{,ji} - \nu^6_{ijkl}r_{jk,li}, \tag{6}$$

where $\mathscr{K}_{ij}$ is the thermal diffusivity tensor and the other phenomenological coefficients $\boldsymbol{\gamma}$, φ, $\boldsymbol{\eta}$, $\boldsymbol{\nu}^p$ ($p = 1, 3, 4, 6$) are supposed constant. Equations (2)–(6) describe disturbances, having finite velocity of propagation and own relaxation time to reach the respective thermodynamic equilibrium values, and show that the porous channels influence mechanical, thermal and other properties of the media taken into account. Equations (2), (5) and (6) describe the evolution of the porosity field, of its flux and of the temperature field, and in their right hand sides the source terms represent the contributions of the fields occurring inside these media. Equations (3) and (4) are the anisotropic generalized Maxwell-Vernotte-Cattaneo equation for the heat flux and the anisotropic Fick-Nonnenmacher equation for the fluid concentration flux, respectively (see [4] and [3]).

3. EVOLUTION EQUATIONS FOR THE POROSITY, ITS FLUX AND TEMPERATURE FIELDS IN A SPECIAL CASE

Let us consider the system of equations (2), (5) and (6). We assume the following:
– the porous medium into consideration is at rest,
– in equation (2) the influence of the small deformations field ε_{ij} and the porosity field r_{ij} can be disregarded,
– in the rate equation (5) the contribution of the fluid concentration gradient $c_{,k}$, the fluid concentration flux j_i^c and the heat flux q_i can be neglected,
– in equation (6) the influence of the first and second partial time derivative of the small deformations field ε_{ij}, of the concentration field c and of the porosity field r_{ij}, the gradient of the fluid concentration flux, $j^c_{j,i}$, the gradient of the porosity field flux, $\mathscr{V}_{ijk,i}$, and the gradient of $c_{,j}$, can be disregarded.
Thus, we have

$$\frac{\partial r_{ij}}{\partial t} + \mathscr{V}_{ijk,k} = \beta^3_{ijk} j^c_k + \beta^4_{ijk} q_k + \beta^5_{ijklm} \mathscr{V}_{klm} + \beta^6_{ijk} c_{,k} + \beta^7_{ijk} T_{,k} + \beta^8_{ijklm} r_{kl,m}, \tag{7}$$

$$\frac{\partial \mathscr{V}_{ijk}}{\partial t} = \gamma^3_{ijklmn} \mathscr{V}_{lmn} + \gamma^5_{ijkl} T_{,l} + \gamma^6_{ijklmn} r_{lm,n}, \tag{8}$$

$$\tau^q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathscr{K}_{ij} T_{,ji} - \nu^6_{ijkl} r_{jk,li}. \tag{9}$$

In the rate equation (7), because of the symmetry of r_{ij}, $r_{ij} = r_{ji}$, the phenomenological coefficients $\boldsymbol{\beta}^s$ ($s = 3\ldots,8$) have the following symmetries

$$\beta^p_{ijk} = \beta^p_{jik} \quad (p = 3,4,6,7), \quad \beta^5_{ijklm} = \beta^5_{jiklm}, \quad \beta^8_{ijklm} = \beta^8_{jiklm} = \beta^8_{ijlkm} = \beta^8_{jilkm}. \tag{10}$$

From (7), the symmetry property of r_{ij} and (10), the divergence of the porosity field flux $\mathscr{V}_{ijk,k}$ is symmetric in the indexes $\{i, j\}$, i. e. $\mathscr{V}_{ijk,k} = \mathscr{V}_{jik,k}$. Also, in equations (8) and (9) we have for the phenomenological tensors γ^6_{ijklmn}, $\mathscr{K}_{ij}$ and ν^6_{ijkl} the following symmetries

$$\gamma^6_{ijklmn} = \gamma^6_{ijkmln}, \qquad \nu^6_{ijkl} = \nu^6_{ikjl} = \nu^6_{ljki} = \nu^6_{lkji}, \quad \mathscr{K}_{ij} = \mathscr{K}_{ji}. \tag{11}$$

The symmetry relations (10) and (11) reduce the number of the significant components of the considered phenomenological tensors in equations (7)–(9). The number of these significant components has a further reduction if we establish some other assumptions. Furthermore, we introduce the deviator tensor, $\tilde{r}_{ij}$, and the scalar (or spherical) part, $r\delta_{ij}$, of r_{ij} in the following way

$$r_{ij} = \tilde{r}_{ij} + r\delta_{ij}, \quad \tilde{r}_{ij} = r_{ij} - r\delta_{ij}, \quad \tilde{r}_{kk} = 0, \quad r = \frac{1}{3} r_{kk}, \quad (i,j,k = 1,2,3), \tag{12}$$

(r_{ij} being symmetric also $\tilde{r}_{ij}$ is symmetric) and we decompose $\mathscr{V}_{ijk}$ in its following three symmetric contributions $\mathscr{V}_{ijk} = \mathscr{V}_k\delta_{ij} + \mathscr{V}_i\delta_{jk} + \mathscr{V}_j\delta_{ik}$. For the sake of simplicity in the following we consider for the porosity field only its scalar (spherical) part and for its flux only its first contribution, i.e.

$$r_{ij} = r\delta_{ij}, \quad \mathscr{V}_{ijk} = \mathscr{V}_k\delta_{ij}. \tag{13}$$

Thus, by virtue of assumptions (13), the rate equations (7)–(9) keep the form

$$\frac{\partial r}{\partial t}\delta_{ij} + \mathscr{V}_{k,k}\delta_{ij} = \beta^3_{ijk} j^c_k + \beta^4_{ijk} q_k + \beta^5_{ijklm} \mathscr{V}_m \delta_{kl} + \beta^6_{ijk} c_{,k} + \beta^7_{ijk} T_{,k} + \beta^8_{ijklm} r_{,m}\delta_{kl}, \tag{14}$$

$$\frac{\partial \mathscr{V}_k}{\partial t}\delta_{ij} = \gamma^3_{ijklmn} \mathscr{V}_n \delta_{lm} + \gamma^5_{ijkl} T_{,l} + \gamma^6_{ijklmn} r_{,n}\delta_{lm}, \tag{15}$$

$$\tau^q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathscr{K}_{ij} T_{,ji} - \nu^6_{ijkl} r_{,li}\delta_{jk}. \tag{16}$$

From (15) the following symmetries are valid

$$\gamma^5_{ijkl} = \gamma^5_{jikl}, \quad \gamma^3_{ijklmn} = \gamma^3_{jiklmn} = \gamma^3_{ijkmln} = \gamma^3_{jikmln}, \quad \gamma^6_{ijklmn} = \gamma^6_{jiklmn} = \gamma^6_{ijkmln} = \gamma^6_{jikmln}. \tag{17}$$

The properties $(17)_1$–$(17)_3$ come from the symmetry of $r\delta_{ij}$ and $\mathscr{V}_k\delta_{ij}$ in the indexes $\{i,j\}$ and from the symmetry of $\mathscr{V}_n\delta_{lm}$ and $r_{,n}\delta_{lm}$ in the indexes $\{l,m\}$.

4. SYSTEM OF EQUATIONS DESCRIBING THE PROPAGATION OF COUPLED POROSITY-TEMPERATURE WAVES IN PERFECT ISOTROPIC MEDIA

The number of the significant Cartesian components of the phenomenological tensors present in (14)–(16) have a further reduction in the special case of perfect isotropic media, when their geometric, transport and thermal properties are invariant respect to all rotations and inversions of the frame axes (under orthogonal transformations). In this case we have [13]:
the tensors of odd order vanish, i.e.

$$L_{ijk} = 0, \qquad L_{ijklm} = 0, \quad \text{thus} \quad \beta^3_{ijk} = \beta^4_{ijk} = \beta^5_{ijklm} = \beta^6_{ijk} = \beta^7_{ijk} = \beta^8_{ijklm} = 0; \tag{18}$$

the second order tensors keep the form

$$L_{ij} = L\delta_{ij}, \quad \text{thus} \quad \mathscr{K}_{ij} = \mathscr{K}\delta_{ij}; \tag{19}$$

the fourth order tensors must have the form

$$L_{ijkl} = L_1\delta_{ij}\delta_{kl} + L_2\delta_{ik}\delta_{jl} + L_3\delta_{il}\delta_{jk}, \tag{20}$$

where L_r $(r = 1,2,3)$ are the 3 significant components of L_{ijkl}, so that in equations (15) and (16) γ^5_{ijkl} and ν^6_{ijkl} have only three independent components. But, when fourth order perfect isotropic tensors have special symmetry properties (as γ^5_{ijkl} and ν^6_{ijkl}) in [1] it was seen that these tensors can be expressed only by two significant *independent* components. In particular, when a perfect isotropic tensor L_{ijkl} has the symmetry

$$L_{ijkl} = L_{jikl}, \tag{21}$$

(valid for the tensor γ^5_{ijkl} present in (8)) from relation (20) we have (see [1])

$$L_{ijkl} = A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \text{with} \quad A_1 = L_1, \quad A_2 = (L_2+L_3)/2, \quad \text{so that} \tag{22}$$

$$\gamma^5_{ijkl} = \gamma^5_1\delta_{ij}\delta_{kl} + \gamma^5_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{23}$$

Also, the perfect isotropic tensor ν^6_{ijkl}, present in (16), keeps the form (see [1])

$$\nu^6_{ijkl} = \nu^6_1\delta_{il}\delta_{jk} + \nu^6_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \quad \text{by virtue of its symmetries (see } (11)_2). \tag{24}$$

The sixth order tensors L_{ijklmn} (see γ^3_{jikmln} and γ^6_{ijklmn} present in (15)) assume the following form [13]

$$\begin{aligned} L_{ijklmn} = {} & L_1\delta_{ij}\delta_{kl}\delta_{mn} + L_2\delta_{ij}\delta_{km}\delta_{ln} + L_3\delta_{ij}\delta_{kn}\delta_{lm} + L_4\delta_{ik}\delta_{jl}\delta_{mn} + L_5\delta_{ik}\delta_{jm}\delta_{ln} \\ & + L_6\delta_{ik}\delta_{jn}\delta_{lm} + L_7\delta_{il}\delta_{jk}\delta_{mn} + L_8\delta_{il}\delta_{jm}\delta_{kn} + L_9\delta_{il}\delta_{jn}\delta_{km} + L_{10}\delta_{im}\delta_{jk}\delta_{ln} \\ & + L_{11}\delta_{im}\delta_{jl}\delta_{kn} + L_{12}\delta_{im}\delta_{jn}\delta_{kl} + L_{13}\delta_{in}\delta_{jk}\delta_{lm} + L_{14}\delta_{in}\delta_{jl}\delta_{km} + L_{15}\delta_{in}\delta_{jm}\delta_{kl}, \end{aligned} \tag{25}$$

where L_r $(r = 1,2,\dots,15)$ are the 15 significant components of L_{ijklmn}. But, when a sixth order perfect isotropic tensor L_{ijklmn} has the two symmetries

$$L_{ijklmn} = L_{jiklmn}, \quad L_{ijklmn} = L_{ijkmln}, \quad \text{equivalent to} \quad L_{ijklmn} = L_{jiklmn} = L_{ijkmln} = L_{jikmln}, \tag{26}$$

(valid for the tensors γ^3_{ijklmn} and γ^6_{ijklmn} in equation (8)), in [1] it was shown that *the significant independent components of this tensor reduce from* 15 *to* 6, i. e.

$$L_{ijklmn} = D_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + D_2\delta_{ij}\delta_{kn}\delta_{lm} + D_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] +$$

$$D_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} + D_5(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} + D_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}], \tag{27}$$

$$\text{with } D_1 = L_1 = L_2;\ D_2 = L_3;\ D_3 = L_4 = L_5 = L_7 = L_{10};\ D_4 = L_6 = L_{13};\ D_5 = L_8 = L_{11};\ D_6 = L_9 = L_{12} = L_{14} = L_{15}. \tag{28}$$

Taking into account (11), (17), (18), (19), (23), (24) and (27), we derive from (14)–(16) the following simplified system of equations governing the evolution of the porosity, its flux, temperature fields. In particular, from (14), using relations (18), when $i = j$ we obtain

$$\frac{\partial r}{\partial t} + \mathcal{V}_{k,k} = 0, \tag{29}$$

from (15), using the special forms (23) and (27), assumed by the fourth order tensor γ^5_{ijkl} and the sixth order tensors γ^r_{ijklmn} $(r = 3, 6)$, we have

$$\begin{aligned}
\delta_{ij}\frac{\partial \mathcal{V}_k}{\partial t} = {} & \{\gamma^3_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + \gamma^3_2\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^3_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] \\
& + \gamma^3_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} + \gamma^3_5(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} + \gamma^3_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} \\
& + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}]\}\mathcal{V}_n\delta_{lm} + [\gamma^5_1\delta_{ij}\delta_{kl} + \gamma^5_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]T_{,l} \\
& + \{\gamma^6_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + \gamma^6_2\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^6_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] \\
& + \gamma^6_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} + \gamma^6_5(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} + \gamma^6_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} \\
& + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}]\}r_{,n}\delta_{lm},
\end{aligned} \tag{30}$$

where γ^3_s and γ^6_s $(s = 1, \ldots 6)$ are the 6 independent significant components of the tensors γ^3_{ijklmn} and γ^6_{ijklmn}, respectively, and γ^5_1, γ^5_2 are the 2 independent significant components of the tensor γ^5_{ijkl}.

Thus, from (30) we get

$$\delta_{ij}\frac{\partial \mathcal{V}_k}{\partial t} = \left[\left(2\gamma^3_1 + 3\gamma^3_3 + 2\gamma^3_5\right)\mathcal{V}_k + \left(2\gamma^6_1 + 3\gamma^6_3 + 2\gamma^6_5\right)r_{,k} + \gamma^5_1 T_{,k}\right]\delta_{ij} \quad +$$

$$\left[\left(3\gamma^3_2 + 2\gamma^3_4 + 2\gamma^3_6\right)\mathcal{V}_j + \left(3\gamma^6_2 + 2\gamma^6_4 + 2\gamma^6_6\right)r_{,j} + \gamma^5_2 T_{,j}\right]\delta_{ik} + \left[\left(3\gamma^3_2 + 2\gamma^3_4 + 2\gamma^3_6\right)\mathcal{V}_i + \left(3\gamma^6_2 + 2\gamma^6_4 + 2\gamma^6_6\right)r_{,i} + \gamma^5_2 T_{,i}\right]\delta_{jk}. \tag{31}$$

When $i = j$ we have

$$\frac{\partial \mathcal{V}_k}{\partial t} = \left(2\gamma^3_1 + 6\gamma^3_2 + 3\gamma^3_3 + 4\gamma^3_4 + 2\gamma^3_5 + 4\gamma^3_6\right)\mathcal{V}_k + \left(\gamma^5_1 + 2\gamma^5_2\right)T_{,k} + \left(2\gamma^6_1 + 6\gamma^6_2 + 3\gamma^6_3 + 4\gamma^6_4 + 2\gamma^6_5 + 4\gamma^6_6\right)r_{,k}, \quad \text{i.e.} \tag{32}$$

$$\tau^\nu\frac{\partial \mathcal{V}_k}{\partial t} = -\mathcal{V}_k - D_\nu r_{,k} + \beta_\nu T_{,k}, \tag{33}$$

where we have defined

$$2\gamma^3_1 + 6\gamma^3_2 + 3\gamma^3_3 + 4\gamma^3_4 + 2\gamma^3_5 + 4\gamma^3_6 = -\left(\tau^\nu\right)^{-1};\ \ \beta_\nu = \tau^\nu\left(\gamma^5_1 + 2\gamma^5_2\right);\ \ D_\nu = -\tau^\nu\left(2\gamma^6_1 + 6\gamma^6_2 + 3\gamma^6_3 + 4\gamma^6_4 + 2\gamma^6_5 + 4\gamma^6_6\right), \tag{34}$$

τ^ν being the relaxation time of the field $\mathcal{V}_k$, D_ν a diffusion coefficient and β_ν a coefficient describing the influence of the temperature gradient on the time partial derivative of the field $\mathcal{V}_k$. From equation (16), using the special forms $(19)_2$ and (24) of the tensors $\mathscr{K}_{ij}$ and ν^6_{ijkl}, respectively, we derive

$$\tau^q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathscr{K} T_{,ii} - \left[\nu^6_1 \delta_{il} \delta_{jk} + \nu^6_2 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl})\right] r_{,li} \delta_{jk}, \tag{35}$$

where ν^6_1, ν^6_2 are the 2 significant independent components of the fourth order tensor ν^6_{ijkl} and $\mathscr{K}$ is the only one significant component of the second order tensor $\mathscr{K}_{ij}$. Therefore, equation (35) keeps the form

$$\tau^q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \mathscr{K} T_{,ii} + \alpha_T r_{,ii} = 0, \quad \text{with} \quad \alpha_T = 3\nu^6_1 + 2\nu^6_2 \quad \text{a coupling coefficient.} \tag{36}$$

From equation (29), its time partial derivative and the divergence of equation (33) we have

$$\tau^\nu \frac{\partial^2 r}{\partial t^2} + \frac{\partial r}{\partial t} - D_\nu r_{,ii} + \beta_\nu T_{,ii} = 0. \tag{37}$$

5. PROPAGATION OF THE COUPLED POROSITY-TEMPERATURE PLANE HARMONIC WAVES

The aim of this Section is to study the propagation of the coupled waves of porosity and temperature fields and therefore to find the dispersion relation for the propagation velocities of these waves as functions of the wave number and to represent these speeds in diagrams. We confine our considerations to one-dimensional plane harmonic waves propagating along the x direction and we suppose that the porous medium under consideration occupies the whole space. Thus, we assume that the solutions of the set of equations (36) and (37) have the form

$$r(x,t) = \widehat{r} e^{ik(x-vt)}, \quad T(x,t) = \widehat{T} e^{ik(x-vt)}, \tag{38}$$

with $\widehat{r}$ and $\widehat{T}$ the amplitudes of the waves $r(x,t)$ and $T(x,t)$, k the wave number, v the wave velocity, defined by $v = \frac{\omega}{k}\,[\mathrm{m\,s^{-1}}]$, with ω the angular frequency, $\omega = 2\pi f\,[\mathrm{s^{-1}}]$, f being the wave frequency and $k = \frac{2\pi}{\lambda}\,[\mathrm{m^{-1}}]$, with λ the wave length. Thus, using the relations (38) and their derivatives into (36), (37) we obtain the following system of equations

$$\left(\mathscr{K} k - \tau^q k v^2 - iv\right) \widehat{T} - \alpha_T k \widehat{r} = 0, \tag{39}$$

$$\beta_\nu k^2 \widehat{T} + \left(\tau^\nu k^2 v^2 - D_\nu k^2 + ikv\right) \widehat{r} = 0, \tag{40}$$

that has non-trivial solutions only if its determinant vanishes, i.e.

$$\mathscr{D} = \begin{vmatrix} \mathscr{K} k - \tau^q k v^2 - iv & -\alpha_T k \\ \beta_\nu k^2 & \tau^\nu k^2 v^2 - D_\nu k^2 + ikv \end{vmatrix} = 0. \tag{41}$$

Developing $\mathscr{D}$ we obtain the following *dispersion relation* for the wave propagation velocities v

$$\tau^q \tau^\nu k^2 v^4 + ik\left(\tau^q + \tau^\nu\right) v^3 - \left[\left(\mathscr{K} \tau^\nu + D_\nu \tau^q\right) k^2 + 1\right] v^2 - ik\left(\mathscr{K} + D_\nu\right) v + k^2 \left(\mathscr{K} D_\nu - \alpha_T \beta_\nu\right) = 0. \tag{42}$$

From the imaginary part of the dispersion relation (42), we derive

$$k\left(\tau^q + \tau^\nu\right) v^3 - k\left(\mathscr{K} + D_\nu\right) v = 0, \tag{43}$$

from which we obtain the following two values

$$v_{(1)} = 0, \qquad v_{(2)} = \sqrt{\frac{\mathscr{K} + D_\nu}{\tau^q + \tau^\nu}}, \quad \text{being} \quad \frac{\mathscr{K} + D_\nu}{\tau^q + \tau^\nu} > 0. \tag{44}$$

From $(44)_2$ the wave propagation velocity $v_{(2)}$ *is always real*.

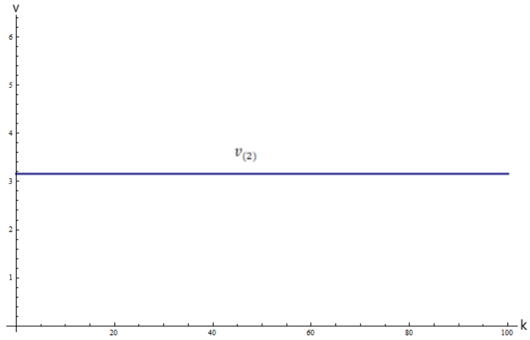

Fig. 1 – Representation of the wave propagation speed $v_{(2)}$ as function of k.

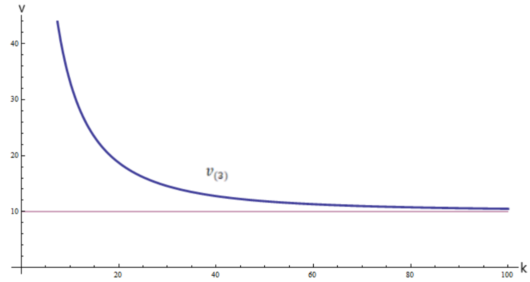

Fig. 2 – Representation of the wave propagation speed $v_{(3)}$ as function of k. The fuchsia horizontal line is its horizontal asymptote.

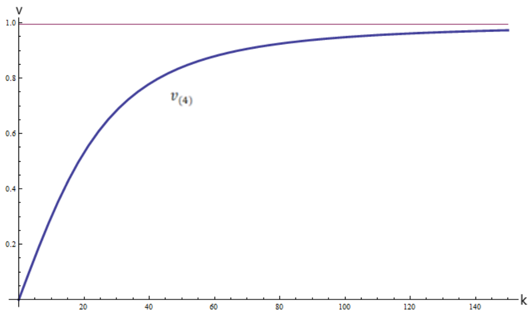

Fig. 3 – Representation of the wave propagation speed $v_{(4)}$ as function of k.

From the real part of the dispersion relation (42), we obtain

$$\tau^q \tau^\nu k^2 v^4 - \left[\left(\mathscr{K} \tau^\nu + D_\nu \tau^q \right) k^2 + 1 \right] v^2 + k^2 \left(\mathscr{K} D_\nu - \alpha_T \beta_\nu \right) = 0, \tag{45}$$

from which we have two possible modes

$$v_{(3)} = \sqrt{\mathscr{H}_1 + \sqrt{\mathscr{H}_1^2 - \mathscr{H}_2}}, \quad v_{(4)} = \sqrt{\mathscr{H}_1 - \sqrt{\mathscr{H}_1^2 - \mathscr{H}_2}}, \tag{46}$$

$$\text{where} \quad \mathscr{H}_1 = \frac{\mathscr{K} \tau^\nu + D_\nu \tau^q}{2 \tau^q \tau^\nu} + \frac{1}{2 \tau^q \tau^\nu k^2} \quad (\text{with} \quad \mathscr{H}_1 > 0), \quad \mathscr{H}_2 = \frac{\mathscr{K} D_\nu - \alpha_T \beta_\nu}{\tau^q \tau^\nu}. \tag{47}$$

The velocity $v_{(3)}$ is real when $\mathscr{H}_1^2 - \mathscr{H}_2 \geq 0$, namely when

$$\left[(\mathscr{K}\tau^{v} - D_v\tau^{q})^2 + 4\tau^{q}\tau^{v}\alpha_T\beta_v\right]k^4 + 2\,(\mathscr{K}\tau^{v} + D_v\tau^{q})\,k^2 + 1 \geq 0, \tag{48}$$

that is *always true* because sum of positive quantities, and then also the velocity $v_{(3)}$ *is always real*. From $(46)_2$ the velocity $v_{(4)}$ is real when

$$\mathscr{H}_1 - \sqrt{\mathscr{H}_1^2 - \mathscr{H}_2} \geq 0, \tag{49}$$

from which we obtain

$$\mathscr{H}_2 \geq 0, \quad \text{i.e.} \quad \mathscr{K}D_v \geq \alpha_T\beta_v. \tag{50}$$

Therefore, *in the assumption that* $(50)_1$ *(or* $(50)_2$*) holds* $v_{(4)}$ *is real*. In Figs. 1–3 the propagation speeds as functions of k are represented for a given numerical set of the several coefficients present in the equations of the developed model: $\mathscr{K} = 10^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $D_v = 10^{-1}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\tau^q = 10^{-2}\,\mathrm{s}$, $\tau^v = 10^{-3}\,\mathrm{s}$, $\beta_v = 10^{-3}\,\mathrm{s}^{-1}$ and $\alpha_T = 10^{-2}\,\mathrm{m}^4\,\mathrm{s}^{-1}$. In this assumption the condition $(50)_2$ is satisfied and then the velocity $v_{(2)}$ is real.

The results presented in Figs. 1–3 show that for increasing values of k (for decreasing wave lengths λ) the propagation velocity $v_{(2)}$ remains constant, the propagation velocity $v_{(3)}$ decreases, while the propagation velocity $v_{(4)}$ increases.

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