THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 24, Number 4/2023, pp. 297–306
DOI: 10.59277/PRA-SER.A.24.4.01

A REMARK ON $*$-RICCI PARALLELISM ON ALMOST COKÄHLER 3-MANIFOLDS

Wenjie WANG

Zhengzhou University of Aeronautics, School of Mathematics
Zhengzhou 450046, Henan, P.R. China
Corresponding author: Wenjie WANG, E-mail: wangwj072@163.com

Abstract. In this paper, we give a local classification theorem of almost coKähler 3-manifolds whose $*$-Ricci operators are parallel under a weak restriction.

Key words: almost coKähler 3-manifold; $*$-Ricci tensor; parallelism.
Mathematics Subject Classification (MSC2020): 53D15, 53C25.

1. INTRODUCTION

The Ricci tensor S of a semi-Riemannian manifold (M,g) is defined by

$$S(X,Y) = \text{trace}\{Z \to R(Z,X)Y\},$$

where R is the curvature tensor of M which is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and ∇ denotes the Levi-Civita connection of the metric g; and X,Y denote arbitrary tangent vector fields of the tangent bundle of the manifold. The Ricci tensor S is said to be parallel (with respect to the Levi-Civita connection) if

$$\nabla S = 0. \tag{1}$$

The parallelism of the Ricci tensor S has been studied by many researchers for a long time in differential geometry (see some earlier literature [13, 20]). When the manifold admits some additional structures, new parallelism of the Ricci tensor appeared. Next, let M^{2n+1} be an almost contact metric manifold (see its detailed definition in Section two) endowed with an almost contact metric structure (ϕ,ξ,η,g). The Ricci tensor S of M^{2n+1} is said to be η-parallel (see [11]) if

$$(\nabla_{\phi X} S)(\phi Y, \phi Z) = 0 \tag{2}$$

for any vector fields X,Y and Z. By definition, the following relationship is valid:

$$(1) \Rightarrow (2).$$

In general, the converse of the above relationship is not necessarily valid on an almost contact metric manifold. For example see some results for almost coKähler cases of dimension three in Section three. Here, we have to point out that (1) is a very strong condition for an almost contact metric manifold, while (2) is more adapted to the associated almost contact metric structure.

The so called $*$-Ricci tensor was first introduced on an almost Hermitian manifold by Tachibana in [21]. Later, such a notion was defined on real hypersurfaces of nonflat complex space forms by Kaimakamis and Panagiotidou in [10]. In recent time, $*$-Ricci tensor on an almost contact metric manifold (M,ϕ,ξ,η,g) was considered in [7, 14, 27] as the following

$$S^*(X,Y) = \frac{1}{2}\mathrm{trace}\{Z \to R(X,\phi Y)\phi Z\} \tag{3}$$

for any vector fields X,Y. The $*$-Ricci operator Q^* of $*$-Ricci tensor S^* with respect to g is expressed by $g(Q^*X,Y) = S^*(X,Y)$. Note that Q^* is not a symmetric operator in general. In analogy with the usual Ricci tensor, the $*$-Ricci tensor S^* is said to be parallel (with respect to the Levi-Civita connection) if

$$\nabla S^* = 0. \tag{4}$$

Generalizing condition (4), on an almost contact metric manifold one may consider

$$(\nabla_X Q^*)Y = (\nabla_Y Q^*)X \tag{5}$$

and

$$(\nabla_{\phi X} S^*)(\phi Y, \phi Z) = 0 \tag{6}$$

for any vector fields X,Y,Z. In general, if (5) and (6) are true, then we say that the $*$-Ricci tensor is of Codazzi type and η-parallel, respectively.

Very recently, Venkatesha et al. in [22] considered (5) on a non-coKähler almost coKähler 3-manifold. However, their result (see [22, Theorem 3.6]) needs some strong restrictions (namely the Reeb vector field ξ is strongly normal and $\|\nabla_\xi h\|$ is invariant along the Reeb flow). In this paper, we shall prove a complete classification theorem for an almost coKähler 3-manifold whose $*$-Ricci tensor is parallel (or vanishing) under a more weaker restriction. This makes main results in [22] being some special cases of our main theorem. According to our theorem, some non-homogeneous almost coKähler 3-manifolds with $*$-Ricci parallelism can be found.

2. ALMOST COKÄHLER MANIFOLDS

By an almost contact metric manifold, we refer to a Riemannian manifold M^{2n+1} of dimension $2n+1$, $n \geq 1$, on which there exists an almost contact structure (ϕ,ξ,η) satisfying

$$\phi^2 = -\mathrm{id} + \eta\otimes\xi,\ \eta\circ\phi = 0, \tag{7}$$

and a Riemannian metric g such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{8}$$

for any vector fields X,Y, where ϕ is a $(1,1)$-type tensor field; ξ is a vector field called the Reeb vector field and η is a global one-form called the almost contact form ([2]). On the product $M^{2n+1}\times\mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, there is an almost complex structure J defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathscr{C}^\infty$-function on $M^{2n+1}\times\mathbb{R}$. The almost contact metric manifold is said to be normal if J is integrable, or equivalently,

$$[\phi,\phi] = -2\mathrm{d}\eta\otimes\xi,$$

where $[\phi,\phi]$ is the the Nijenhuis tensor of ϕ.

An almost coKähler manifold is defined as an almost contact metric manifold on which there hold $d\eta = 0$ and $d\Phi = 0$, where Φ is the fundamental two-form defined by $\Phi(X,Y) = g(X,\phi Y)$. A normal almost coKähler manifold is said to be a coKähler manifold (see [1,2]). An almost coKähler manifold is coKähler if and only if (see [1])

$$\nabla\phi = 0. \tag{9}$$

Let $2h$ be the Lie derivative of the structure tensor field ϕ along the Reeb vector field. On an almost coKähler manifold we put $l = R(\cdot,\xi)\xi$ and $h' = h\circ\phi$. The three $(1,1)$-type tensor fields l, h' and h are symmetric and satisfy

$$\nabla\xi = h', \tag{10}$$

$$\nabla_\xi h = -h^2\phi + \phi l, \tag{11}$$

and

$$h\xi = 0,\ l\xi = 0,\ \mathrm{tr}h = 0,\ \mathrm{tr}(h') = 0,\ h\phi + \phi h = 0. \tag{12}$$

We remark that (almost) coKähler manifolds are also known as (almost) cosymplectic manifolds (see [3]). The above all fundamental results on an almost coKähler manifold can be seen in [2,3,16].

3. RICCI PARALLELISM

In this section, first let us recall some known results regarding Ricci parallelism on an almost coKähler 3-manifold. On any Riemannian manifold (M,g) of dimension three, the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ &- g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{13}$$

for any vector fields X,Y and Z. Taking the covariant derivative of the above equality gives

$$\begin{aligned} &(\nabla_X R)(Y,Z)W \\ =& g(Z,W)(\nabla_X Q)Y - g(Y,W)(\nabla_X Q)Z + g((\nabla_X Q)Z,W)Y \\ &- g((\nabla_X Q)Y,W)Z - \frac{1}{2}X(r)(g(Z,W)Y - g(Y,W)Z) \end{aligned}$$

for any vector fields X,Y,Z and W. If the Ricci tensor is parallel, the scalar curvature is a constant and hence the above equality gives $\nabla S = 0 \Rightarrow \nabla R = 0$. In a word, the manifold is locally symmetric if and only if the Ricci tensor is parallel for Riemannian three-manifold. Therefore, the following result follows immediately from Perrone [17, Proposition 3.1].

THEOREM 1 ([17]). *The Ricci tensor of an almost coKähler 3-manifold is parallel if and only if the manifold is locally isometric to a product of a one-dimensional manifold and a Kähler surface of constant curvature.*

We remark that the above product space admits a coKähler structure. This shows that a locally symmetric almost coKähler 3-manifold must be coKähler. When relaxing Ricci parallelism to Ricci η-parallelism on an almost contact metric 3-manifold, the complete classification problem is hard to solve. The author [24] employed a restriction (namely $\nabla_\xi h = ah'$ and it is said to be *h-a* condition) on a strictly almost coKähler 3-manifold to give a local classification result (see [5] for contact metric case and [25] for almost Kenmotsu case). The following result was proved in [8,24].

THEOREM 2. *The Ricci tensor of an almost coKähler 3-h-a-manifold is η-parallel if and only if the manifold is locally isometric to a product manifold $\mathbb{R}\times N$ with N being of constant curvature of dimension two or a Lie group $E(1,1)$, $\tilde{E}(2)$ or the Heisenberg group Nil^3 equipped with a left invariant almost coKähler structure.*

Lie group $E(1,1)$ is the rigid motions group of the Minkowski two-plane and $\tilde{E}(2)$ is the universal covering $E(2)$ of the rigid motions group of the Euclidean two-plane.

4. $*$-RICCI PARALLELISM

Let M^3 be an almost coKähler 3-manifold. Such a manifold has been studied by many authors (see some recent references [4, 6, 12, 18, 26]) in recent time. It is known that M^3 is coKähler if and only if $h = 0$. On a coKähler 3-manifold, now we compute the $*$-Ricci tensor. Applying $h = 0$ in (10), we obtain $Q\xi = 0$. Using this and putting $Y = Z = \xi$ in (13) we obtain the Ricci operator

$$Q = \frac{r}{2}I - \frac{r}{2}\eta \otimes \xi. \tag{14}$$

Using (14), by definition (3) and a direct calculation, the $*$-Ricci tensor is given by

$$S^* = S. \tag{15}$$

In view of this, next we consider the $*$-Ricci tensor on an almost coKähler manifold.

Let $\mathscr{U}_1$ be the open subset of M^3 such that $h \neq 0$ and $\mathscr{U}_2$ the open subset of M^3 defined by $\mathscr{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathscr{U}_1 \cup \mathscr{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathscr{U}_1 \cup \mathscr{U}_2$. On $\mathscr{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where λ is a positive function on $\mathscr{U}_1$. Notice that the eigenvalue function λ is continuous on M^3 and smooth on $\mathscr{U}_1 \cup \mathscr{U}_2$. For simplicity, we write $e_1 := e$, $e_2 := \phi e$ and $e_3 := \xi$. The Levi-Civita connection of the metric on $\mathscr{U}_1$ can be seen in [18, Lemma 2.1]. In fact, on $\mathscr{U}_1$, we have

$$\nabla_{e_i} e_j = \begin{pmatrix} \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_2 & -\frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_1 + \lambda e_3 & -\lambda e_2 \\ -\frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_2 + \lambda e_3 & \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_1 & -\lambda e_1 \\ ae_2 & -ae_1 & 0 \end{pmatrix} \tag{16}$$

for any $i, j \in \{1,2,3\}$, where $\sigma(e_k) = g(Qe_3, e_k)$ for any $k \in \{1,2\}$ and a is a smooth function. Moreover, on $\mathscr{U}_1$, the Ricci tensor can be written as

$$S = \begin{pmatrix} \frac{1}{2}r + \lambda^2 - 2a\lambda & e_3(\lambda) & \sigma(e_1) \\ e_3(\lambda) & \frac{1}{2}r + \lambda^2 + 2a\lambda & \sigma(e_2) \\ \sigma(e_1) & \sigma(e_2) & -2\lambda^2 \end{pmatrix} \tag{17}$$

with respect to the local ϕ-basis $\{e_1, e_2, e_3\}$, where r is the scalar curvature. According to (3), with the help of (16) and (17), on $\mathscr{U}_1$, the $*$-Ricci tensor S^* is given by (see also [22, Lemma 3.1]):

$$S^* = \begin{pmatrix} \frac{1}{2}r + 2\lambda^2 & 0 & 0 \\ 0 & \frac{1}{2}r + 2\lambda^2 & 0 \\ \sigma(e_1) & \sigma(e_2) & 0 \end{pmatrix} \tag{18}$$

with respect to the local ϕ-basis $\{e_1, e_2, e_3\}$.

LEMMA 1. *If the $*$-Ricci tensor on $\mathscr{U}_1$ is parallel, then λ is invariant along the Reeb flow and* (32), (33) *are valid.*

Proof. For simplicity, we write $\nabla_i S^*_{jk} := (\nabla_{e_i} S^*)(e_j, e_k)$ for any $i, j \in \{1,2,3\}$. If the $*$-Ricci tensor is parallel, from $\nabla_1 S^*_{23} = 0$, we get

$$r = -4\lambda^2, \tag{19}$$

where notice that λ on $\mathscr{U}_1$ is supposed to be the positive eigenfunction of h. According to $\nabla_1 S^*_{21} = 0$, we get

$$\sigma(e_1) = 0. \tag{20}$$

According to $\nabla_2 S^*_{12} = 0$, we get

$$\sigma(e_2) = 0. \tag{21}$$

In this case, using the above three equalities, from (18), we observe that the $*$-Ricci tensor vanishes identically. With the help of (19), (20) and (21), the Ricci tensor (17) becomes

$$S = \begin{pmatrix} -\lambda^2 - 2a\lambda & e_3(\lambda) & 0 \\ e_3(\lambda) & -\lambda^2 + 2a\lambda & 0 \\ 0 & 0 & -2\lambda^2 \end{pmatrix} \tag{22}$$

with respect to the local basis $\{e_1, e_2, e_3\}$. Note that on any Riemannian manifold the following equality is valid:

$$\frac{1}{2}\mathrm{grad}(r) = \mathrm{div}Q.$$

From (22), the scalar curvature is given by $r = -4\lambda^2$. Therefore, the above formula becomes

$$-4\lambda X(\lambda) = \sum_{i=1}^{3} (\nabla_{e_i} S)(e_i, X) \tag{23}$$

for any vector field X. By a direct calculation, from (22) and (16), we have:

$$\nabla_1 S_{13} = \lambda e_3(\lambda).$$

$$\nabla_2 S_{23} = \lambda e_3(\lambda).$$

$$\nabla_3 S_{33} = -4\lambda e_3(\lambda).$$

Putting the above three equalities into (23) for $X = e_3$, we obtain

$$e_3(\lambda) = 0. \tag{24}$$

Similarly, by a direct calculation, with the help of (16), (22) and (24), we have:

$$\nabla_1 S_{11} = -2(a+\lambda)e_1(\lambda) - 2\lambda e_1(a).$$

$$\nabla_2 S_{21} = 2ae_1(\lambda).$$

$$\nabla_3 S_{31} = 0.$$

Putting the above three equalities into (23) for $X = e_1$, we obtain

$$e_1(a) = e_1(\lambda). \tag{25}$$

Similarly, by a direct calculation, with the help of (16), (22) and (24), we have:

$$\nabla_1 S_{12} = -2ae_2(\lambda).$$

$$\nabla_2 S_{22} = 2(a-\lambda)e_2(\lambda) + 2\lambda e_2(a).$$

$$\nabla_3 S_{32} = 0.$$

Putting the above three equalities into (23) for $X = e_2$, we obtain

$$e_2(a) = -e_2(\lambda). \tag{26}$$

According to (16), with the help of (20), (21), the Lie bracket of the Lie algebra containing all tangent vector fields is given by

$$[e_1,e_2]=-\frac{1}{2\lambda}e_2(\lambda)e_1+\frac{1}{2\lambda}e_1(\lambda)e_2,\ [e_2,e_3]=(a-\lambda)e_1,\ [e_3,e_1]=(a+\lambda)e_2. \tag{27}$$

Putting these three equalities into the following well-known Jacobi identity

$$[[e_1,e_2],e_3]+[[e_2,e_3],e_1]+[[e_3,e_1],e_2]=0$$

yields that

$$e_1(\lambda-a)+\frac{1}{2\lambda}e_3(e_2(\lambda))+\frac{a-\lambda}{2\lambda}e_1(\lambda)=0 \tag{28}$$

and

$$e_2(\lambda+a)+\frac{1}{2\lambda}e_3(e_1(\lambda))-\frac{a+\lambda}{2\lambda}e_2(\lambda)=0, \tag{29}$$

respectively, where we used (19), (20) and (21). With the help of (25) and (26), (28) and (29) become

$$e_3(e_2(\lambda))=(\lambda-a)e_1(\lambda) \tag{30}$$

and

$$e_3(e_1(\lambda))=(\lambda+a)e_2(\lambda), \tag{31}$$

respectively. Taking the derivative of a along $[e_2,e_3]$, with the aid of the second term of (27), we get

$$e_2(e_3(a))=(a-\lambda)e_1(a)+e_3(e_2(a)).$$

Putting (25), (26) and (30) into the above equality gives

$$e_2(e_3(a))=2(a-\lambda)e_1(\lambda). \tag{32}$$

Similarly, taking the derivative of a along $[e_3,e_1]$, with the aid of the third term of (27), we get

$$e_1(e_3(a))=-(a+\lambda)e_2(a)+e_3(e_1(a)).$$

Putting (25), (26) and (31) into the above equality gives

$$e_1(e_3(a))=2(a+\lambda)e_2(\lambda). \tag{33}$$

This completes the proof. □

As seen before, the $*$-Ricci tensor and usual Ricci tensor are the same on a coKähler 3-manifold. So, in what follows we consider only non-coKähler case. Here we say that an almost coKähler manifold is strictly if $h\neq 0$ everywhere. By a direct calculation, on any strictly almost coKähler manifold, from (11) we have (see also [18]):

$$\nabla_\xi h=\frac{\xi(\lambda)}{\lambda}h-2ah'. \tag{34}$$

Notice that $\xi(\lambda)=0$ when $*$-Ricci tensor is parallel. So, in this paper we consider the following condition for an almost coKähler 3-manifold with $*$-Ricci parallelism:

$$\|\nabla_\xi h\|/\|h\| \text{ is invariant along the Reeb flow.} \tag{$\star$}$$

Before stating our main theorem in this paper, we construct a concrete example of strictly almost coKähler 3-manifold satisfying condition ($\star$). Such an example is a special case in [12, Example 3] or [9, Section 5.5].

Example 3. Let G be a three-dimensional non-unimodular Lie group endowed with a left invariant metric g whose Lie algebra is given by

$$[e_1,e_2]=e_2,\ [e_2,e_3]=0,\ [e_1,e_3]=e_2,$$

where $\{e_1,e_2,e_3\}$ is an orthonomal basis with respect to the metric g.

We define a vector field $\xi=e_3$ and its dual 1-form by $\eta=g(\xi,\cdot)$, and a $(1,1)$-type tensor field ϕ by $\phi\xi=0$, $\phi e_1=e_2$ and $\phi e_2=-e_1$. One can check that (G,ϕ,ξ,η,g) defines a three-dimensional left invariant non-coKähler almost coKähler manifold (for more details see [9,17]). By using the Koszul formula we have

$$(\nabla_{e_i}e_j)=\begin{pmatrix} 0 & -\frac{1}{2}\xi & \frac{1}{2}e_2 \\ -e_2-\frac{1}{2}\xi & e_1 & \frac{1}{2}e_1 \\ -\frac{1}{2}e_2 & \frac{1}{2}e_1 & 0 \end{pmatrix}$$

for any $i,j\in\{1,2,3\}$. The Ricci operator is given by

$$Q\xi=-\frac{1}{2}\xi-e_2, Qe_1=-\frac{3}{2}e_1,\ Qe_2=-\frac{1}{2}e_2-\xi.$$

The tensor field h is given by (see also [9, pp. 15]):

$$h\xi=0,\ he_1=-\frac{1}{2}e_1,\ he_2=\frac{1}{2}e_2.$$

Moreover, we have $\lambda=-\frac{1}{2}$ and $a=-\frac{1}{2}$, and then according to (34) we get $\nabla_\xi h=h'$. Because in this case both $\|\nabla_\xi h\|$ and $\|h\|$ are constant, then the condition $(\star)$ is valid. As introduced before, the authors in [22, Theorem 3.6] need that ξ is a strongly normal unit vector field and also $\|\nabla_\xi h\|$ is invariant along ξ. Perrone in [18, Proposition 4.3] proved that ξ on an almost coKähler 3-manifold is a strongly normal unit vector field if and only if ξ is minimal (or equivalently, by [18, Theorem 3.1], ξ is an eigenvector field of the Ricci operator) and $\|h\|^2$ is invariant along $\{\xi\}^\perp$. However, according to the above equalities, we observe that in our example ξ is not an eigenvector field of the Ricci operator and hence ξ is not a strongly normal unit vector field.

THEOREM 4. *If the $*$-Ricci tensor of a strictly almost coKähler 3-manifold is parallel and $(\star)$ is valid, then one of the following statements is valid:*

- *The manifold is locally isometric to a Lie group $E(1,1)$, $\widetilde{E}(2)$ or Heisenberg group Nil^3 equipped with a left invariant non-coKähler almost coKähler structure.*
- *There exists a chart $(U,(x,y,z))$ on an open subset of the manifold such that*

$$e_2=\frac{\partial}{\partial x},\ e_3=\frac{\partial}{\partial y},\ e_1=f_1\frac{\partial}{\partial x}+f_2\frac{\partial}{\partial y}+f_3\frac{\partial}{\partial z},$$

  where $f_1=\alpha(z)x+2a(z)y+\beta(z)$ with $\alpha(z)$, $\beta(z)$, $f_2(z)$ and $f_3(z)$ being three functions which vary only along z.
- *There exists a chart $(U,(x,y,z))$ on an open subset of the manifold such that*

$$e_1=\frac{\partial}{\partial x},\ e_3=\frac{\partial}{\partial y},\ e_2=f_1\frac{\partial}{\partial x}+f_2\frac{\partial}{\partial y}+f_3\frac{\partial}{\partial z},$$

  where $\bar{f}_1=\bar{\alpha}(z)x-2a(z)y+\bar{\beta}(z)$ with $\bar{\alpha}(z)$, $\bar{\beta}(z)$, $\bar{f}_2(z)$ and $\bar{f}_3(z)$ being three functions which vary only along z.

Proof. In the $*$-Ricci tensor is parallel and $(\star)$ is valid, according to (34) and Lemma 1, we obtain $e_3(a)=0$. Using this in (32) and (33), we get

$$(a-\lambda)e_1(\lambda)=0 \tag{35}$$

and

$$(a+\lambda)e_2(\lambda) = 0, \tag{36}$$

respectively. If λ is a constant, (25) and (26), together with $e_3(a) = 0$, shows that a is also a constant. Now (27) becomes

$$[e_1, e_2] = 0,\ [e_2, e_3] = (a-\lambda)e_1,\ [e_3, e_1] = (a+\lambda)e_2. \tag{37}$$

According to Milnor [15], the manifold is locally isometric to one of Lie groups $E(1,1)$, $\widetilde{E}(2)$ or Heisenberg group Nil^3. For almost coKähler structures defined on these Lie groups we refer the reader to [17, 19]. If λ is not a constant, in view of (24), there exists an open subset of the manifold such that either $e_1(\lambda) \neq 0$ or $e_2(\lambda) \neq 0$. Next, we consider these two cases.

If $e_1(\lambda) \neq 0$ on some open subset, say Ω_1, we work on this set. From (35) we get $\lambda = a$. Moreover, from (26) we get $e_2(\lambda) = 0$. Now (27) becomes

$$[e_1, e_2] = \frac{1}{2a}e_1(a)e_2,\ [e_2, e_3] = 0,\ [e_3, e_1] = 2ae_2, \tag{38}$$

where a is not a constant which is invariant only along the distribution span$\{e_2, e_3\}$. According to the second equality of (38), the distribution span$\{e_2, e_3\}$ is integrable. Then there exists a chart $(U, (x, y, z))$ for every point in Ω_1 such that

$$e_2 = \frac{\partial}{\partial x},\ e_3 = \frac{\partial}{\partial y}.$$

We set $e_1 = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}$ with f_i for $i \in \{1,2,3\}$ being three smooth functions. The first equality of (38) transforms into the following PDEs:

$$\frac{\partial f_1}{\partial x} = -\frac{1}{2a}e_1(a),\ \frac{\partial f_2}{\partial x} = 0,\ \frac{\partial f_3}{\partial x} = 0.$$

Similarly, the third equality of (38) transforms into the following PDEs:

$$\frac{\partial f_1}{\partial y} = 2a,\ \frac{\partial f_2}{\partial y} = 0,\ \frac{\partial f_3}{\partial y} = 0. \tag{39}$$

If $e_2(\lambda) \neq 0$ on some open subset, say Ω_2, we work on this set. From (36) we get $\lambda = -a$. Moreover, from (25) we get $e_1(\lambda) = 0$. Now (27) becomes

$$[e_1, e_2] = -\frac{1}{2a}e_2(a)e_1,\ [e_2, e_3] = 2ae_1,\ [e_3, e_1] = 0, \tag{40}$$

where a is not a constant which is invariant only along the distribution span$\{e_1, e_3\}$. According to the third equality of (40), the distribution span$\{e_1, e_3\}$ is integrable. Then there exists a chart $(U, (x, y, z))$ for every point in Ω_2 such that

$$e_1 = \frac{\partial}{\partial x},\ e_3 = \frac{\partial}{\partial y}.$$

We set $e_2 = \bar{f}_1\frac{\partial}{\partial x} + \bar{f}_2\frac{\partial}{\partial y} + \bar{f}_3\frac{\partial}{\partial z}$ with $\bar{f}_i$ for $i \in \{1,2,3\}$ being three smooth functions. The first equality of (40) transforms into the following PDEs:

$$\frac{\partial \bar{f}_1}{\partial x} = -\frac{1}{2a}e_2(a),\ \frac{\partial \bar{f}_2}{\partial x} = 0,\ \frac{\partial \bar{f}_3}{\partial x} = 0.$$

Similarly, the second equality of (40) transforms into the following PDEs:

$$\frac{\partial \bar{f}_1}{\partial y} = -2a,\ \frac{\partial \bar{f}_2}{\partial y} = 0,\ \frac{\partial \bar{f}_3}{\partial y} = 0. \tag{41}$$

Solving these PDEs (39) and (41) completes the proof. □

Remark 1. Venkatesha et al.'s result (see [22, Theorem 3.6]) becomes a special case of Theorem 4 (corresponding to the case $a, \lambda \in \mathbb{R}$).

Remark 2. According to Theorem 4, one finds non-homogeneous almost coKähler 3-manifolds whose $*$-Ricci tensor is parallel (or vanishing).

If one computes the derivative of the $*$-Ricci tensor (see also [22, Lemma 3.4]), the following theorem is true.

THEOREM 5. *On an almost coKähler* 3*-manifold the following conditions are equivalent.*

- *The $*$-Ricci tensor is vanishing.*
- *The $*$-Ricci tensor is parallel.*
- *The $*$-Ricci tensor is of Codazzi type, i.e.,* $(\nabla_X Q^*)Y - (\nabla_Y Q^*)X = 0$.
- *The $*$-Ricci tensor is of Killing type, i.e.,* $(\nabla_X Q^*)Y + (\nabla_Y Q^*)X = 0$.
- *The $*$-Ricci tensor is cyclic parallel, i.e.,* $\sum_{X,Y,Z} (\nabla_X S^*)(Y,Z) = 0$,

where X, Y, Z denote arbitrary vector fields.

According to the above two theorems and results in Section three, we observe that the properties of the $*$-Ricci tensors are much different from that of the usual Ricci tensors.

The set of all almost coKähler 3-manifolds is much huge, this makes many authors to consider such manifolds under some other restrictions in which the function defined in ($\star$) was discussed in many literature (see [12, 18, 22–24]).

ACKNOWLEDGEMENTS

The author was supported by the Nature Science Foundation of Henan Province (grant No. 232300420359) and the Youngth Scientific Research Program in Zhengzhou University of Aeronautics (grant No. 23ZHQN01010). The author would like to thank the referee for his or her careful reading.

REFERENCES

1. D.E. BLAIR, *The theory of quasi-Sasakian structures*, Journal of Differential Geometry, **1**, pp. 331–345, 1967.
2. D.E. BLAIR, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Volume 203, Birkhäuser, Boston, 2010.
3. B. CAPPELLETTI-MONTANO, A. DE NICOLA, I. YUDIN, *A survey on cosymplectic geometry*, Reviews in Mathematical Physics, **25**, *10*, p. 1343002, 2013.
4. J.T. CHO, *Reeb flow symmetry on almost cosymplectic three-manifolds*, Bulletin of the Korean Mathematical Society, **53**, *4*, pp. 1249–1257, 2016.
5. J.T. CHO, J.E. Lee, *η-parallel contact 3-manifolds*, Bulletin of the Korean Mathematical Society, **46**, *3*, pp. 577–589, 2009.
6. U.C. DE, P. MAJHI, Y.J. SUH, *Semisymmetric properties of almost coKähler* 3*-manifolds*, Bulletin of the Korean Mathematical Society, **56**, *1*, pp. 219–228, 2019.
7. A. GHOSH, D.S. PATRA, *$*$-Ricci soliton within the frame-work of Sasakian and (κ,μ)-contact manifold*, International Journal of Geometric Methods in Modern Physics, **15**, *7*, p. 1850120, 2018.
8. J. INOGUCHI, *A note on almost contact Riemannian* 3*-manifolds II*, Bulletin of the Korean Mathematical Society, **54**, *1*, pp. 85–97, 2017.
9. J. INOGUCHI, J.E. LEE, *Pseudo-symmmetric almost cosymplectic* 3*-manifolds*, International Journal of Geometric Methods in Modern Physics, **20**, *10*, p. 2350175, 2023.

10. G. KAIMAKAMIS, K. PANAGIOTIDOU, *$*$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, Journal of Geometry and Physics, **86**, pp. 408–413, 2014.
11. M. KIMURA, S. MAEDA, *On real hypersurfaces of a complex projective space*, Mathematische Zeitschrift, **202**, *3*, pp. 299–311, 1989.
12. X. LIU, W. WANG, *Locally ϕ-symmetric almost coKähler* 3*-manifolds*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, Nouvelle Série, **62**, *4*, pp. 427–438, 2019.
13. T. NAGANO, *The conformal transformation on a space with parallel Ricci tensor*, Journal of the Mathematical Society of Japan, **11**, pp. 10–14, 1959.
14. P. MAJHI, U.C. DE, Y.J. SUH, *$*$-Ricci solitons on Sasakian* 3*-manifolds*, Publicationes Mathematicae Debrecen, **93**, *1–2*, pp. 241–252, 2018.
15. J. MILNOR, *Curvature of left invariant metrics on Lie groups*, Advances in Mathematics, **21**, *3*, pp. 293–329, 1976.
16. Z. OLSZAK, *On almost cosymplectic manifolds*, Kodai Mathematical Journal, **4**, *2*, pp. 239–250, 1981.
17. D. PERRONE, *Classification of homogeneous almost cosymplectic three-manifolds*, Differential Geometry and its Applications, **30**, *1*, pp. 49–58, 2012.
18. D. PERRONE, *Minimal Reeb vector fields on almost cosymplectic* 3*-manifolds*, Kodai Mathematical Journal, **36**, *2*, pp. 258–274, 2013.
19. D. PERRONE, *Classification of homogeneous almost α-coKähler three-manifolds*, Differential Geometry and its Applications, **59**, pp. 66–90, 2018.
20. P.J. RYAN, *Hypersurfaces with parallel Ricci tensor*, Osaka Journal of Mathematics, **8**, pp. 251–259, 1971.
21. S. TACHIBANA, *On almost-analytic vectors in almost Kählerian manifolds*, Tôhoku Mathematical Journal, **11**, pp. 247–265, 1959.
22. V. VENKATESHA, U.C. DE, H.A. KUMARA, D.M. NAIK, *$*$-Ricci tensor on three dimensional almost coKähler manifolds*, Filomat, **37**, *6*, pp. 1793–1802, 2023.
23. W. WANG, X. LIU, *Three-dimensional almost co-Kähler manifolds with harmonic Reeb vector fields*, Revista de la Unión Matemática Argentina, **58**, *2*, pp. 307–317, 2017.
24. Y. WANG, *Ricci tensors on three dimensional almost coKähler manifolds*, Kodai Mathematical Journal, **39**, *3*, pp. 469–483, 2016.
25. Y. WANG, *Three-dimensional almost Kenmotsu manifolds with η-parallel Ricci tensor*, Journal of the Korean Mathematical Society, **54**, *3*, pp. 793–805, 2017.
26. Y. WANG, *Curvature homogeneity and ball-homogeneity on almost coKähler* 3*-manifolds*, Bulletin of the Korean Mathematical Society, **56**, *1*, pp. 253–263, 2019.
27. Y. WANG, *Contact* 3*-manifolds and $*$-Ricci soliton*, Kodai Mathematical Journal, **43**, *2*, pp. 256–267, 2020.

Received March 20, 2023