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FACES IN CONVEX METRIC SPACES

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Abstract. We examine the role of the convex structure in a metric space on which it is defined. First, we introduce the notion of extreme point and face of a convex set. Second, we present the idea of core in a convex metric space. Several properties are proved and examples to support are given.

***Key words**:* convexity, face, core, extreme point, interior point, metric space
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1. INTRODUCTION

The fundamental idea of convexity in metric spaces was given by Takahashi in his seminal paper [16]. Later on geometric properties of the convex metric spaces were studied by Shimizu and Takahashi [15] and Beg [2]. Recently Berinde and Pacurar [5], Ghanifard et al. [6] and Kumar and Tas [12] studied existence of fixed points of multivalued mappings on convex metric spaces under different contractive conditions. Their main thrust was on obtaining fixed point. On the other hand, convex sets are an active area of research in functional analysis/linear algebra due to their significant applications in optimization theory/linear programming [1, 14]. There are so many challenging new key mathematical concepts involved, which continuously keep attracting the researchers. The notion and applications of extreme point, faces and core have been recently studied by several researchers under different names and slightly different definitions [7–10, 13]. In all these works the underlying structure is of a linear space. In the present work we wish to move away from linear structure. We study the geometric properties of convex set in spaces without linear structure. We introduce and study novel notions of extreme point, interior point and face of a convex set in a convex metric space. After this we broach the idea of core of a convex set.

This paper is set up as follows: In the Section 2, concept of convex metric spaces is revised and a review of related works that will be used in our proposal are presented. Section 3, introduces and study novel notions of extreme point, outer point, interior point and face of a convex set. Section 4 broach the idea of core of a convex set in convex metric spaces. Finally, Sect.5 concludes this paper.

2. PRELIMINARIES

This section is devoted to laying out the related notation pertaining to convex metric spaces and some related works that will be used afterward.

Definition 1 [16]. Let (X,d) be a metric space and $I=[0,1]$. A mapping $W : X\times X\times I \to X$ is said to be a convex structure on X if for each $(x,y,\alpha)\in X\times X\times I$ and $u\in X$,

$$d(u,W(x,y,\alpha)) \le \alpha d(u,x)+(1-\alpha)d(u,y). \tag{1}$$

A metric space X together with a convex structure W is said to be a convex metric space (X,d,W). A non empty subset M of X is called convex if $W(a,b,\lambda)\in M$ whenever $(a,b,\lambda)\in M\times M\times I$

Remark 1 [16]. The convex metric space X has the properties;
i. $W(a,b,1)=a,\ W(a,b,0)=b,\ W(a,a,\lambda)=a.$
ii. Open balls $B(a,r)=\{b\in X: d(a,b)<r\}$ and closed balls $B[a,r]=\{b\in X: d(a,b)\leq r\}$ are convex.
iii. If $\{K_\alpha:\alpha\in A\}$ is a family of convex subsets of X, then $\underset{\alpha\in A}{\cap}K_\alpha$ is convex.

Any normed space and a convex subset of a normed space is a convex metric space. There are several examples in the existing literature [2, 15, 16] of convex metric spaces which are not embedded in any normed space. Denote by $co(M)$ the convex hull of M. We also use the following notation;

$$\begin{aligned}
[x,y] &= \{z: z=W(x,y,\alpha) \text{ for some } \alpha\in I\},\\
(x,y] &= \{z: z=W(x,y,\alpha) \text{ for some } \alpha\in[0,1)\},\\
[x,y) &= \{z: z=W(x,y,\alpha) \text{ for some } \alpha\in(0,1]\},\\
(x,y) &= \{z: z=W(x,y,\alpha) \text{ for some } \alpha\in(0,1)\}.
\end{aligned}$$

Definition 2 [3]. A convex metric space (X,d,W) is said to have property $\mathscr{L}$ if for all $x,y,z\in X$ and α,β,γ in I, we have
i. $W(W(x,y,\alpha),W(x,y,\beta),\gamma)=W(x,y,\gamma\alpha+(1-\gamma)\beta)$
ii. $W(x,y,\alpha)=W(y,x,1-\alpha)$.

Remark 2. Taking $\beta=0$ in Definition 2(i), we obtain by Remark 1(i)

$$W(W(x,y,\alpha),y,\gamma)=W(x,y,\alpha\gamma). \tag{2}$$

Each normed space has property $\mathscr{L}$, if we define $W(x,y,t)=tx+(1-t)y$.

Definition 3 [4]. A function h from a convex metric space (X,d,W) to a real vector space is said to be convexity preserving (CP) if $h(W(x,y,\alpha))=W(h(x),h(y),\alpha)$.

The set of all real valued CP functions on (X,d,W) is denoted by $CP(X)$.

THEOREM 1 [4]. *Let (X,d,W) be a convex metric space having property $\mathscr{L}$. If*

$$W(W(x,y,\alpha),z,\gamma)=W(x,W(y,z,\frac{\gamma(1-\alpha)}{1-\alpha\gamma}),\alpha\gamma)\ \ for\ \ \alpha\gamma\neq 1, \tag{3}$$

and

$$W(x,y,\alpha)=W(x,z,\alpha)\Rightarrow y=z \tag{4}$$

then X is isomorphic to some convex subset Y of a real vector space Z.

3. FACES

In this section we discuss the notion of a face of a convex set in a convex metric space that can be define using the convexity structure of the space. Our main goal is to construct proper faces and discuss interesting properties with examples.

Definition 4. An extreme point of a convex subset M of a convex metric space X is a point x in M with the property that if $x=W(y,z,\alpha)$, where $y,z\in M$ and $\alpha\in I$, then $x=y$ and/or $x=z$.

The set of all extreme points of M is denoted by $\mathbf{E}(M)$.

Example 1. It is easy to see that the set $\{z : z = W(x,y,\alpha)\ \forall\ \alpha \in I, x \neq y\}$ in a convex metric space $(\mathbb{R}^2, d, W)$ has two extreme points. Here d is a Euclidean metric and $W(x,y,\alpha) = \alpha x + (1-\alpha)y$.

Example 2. Let $M = \{(0,y,z) : z^2 + y^2 = 1\} \cup \{(1,1,0), (-1,1,0)\} \subset (\mathbb{R}^3, d, W)$ with Euclidean metric d and $W(x,y,\alpha) = \alpha x + (1-\alpha)y$. Then

$$\mathbf{E}(co(M)) = \{(0,y,z) : z^2 + y^2 = 1, y \neq 1\} \cup \{(1,1,0), (-1,1,0)\}.$$

Definition 5. A convex subset M of X is W-closed if for every $(x,y) \subseteq M$ we have $[x,y] \subseteq M$. The W-closure of a subset S (denoted by $Wcl(S)$) is the smallest convex W-closed subset of X that contains S.

Definition 6. A point x in X is called an outer point of a subset M of X, if there exists two points $u \in M$ and $v \in X$ such that $W(x,u,\alpha) \in M$ and $W(x,v,\alpha) \notin M$ for all $\alpha \in (0,1)$. Set of all outer points of M is denoted by $\partial(M)$.

Obviously $\mathbf{E}(M) \subseteq \partial(M)$ and $\partial(M) = \partial(X \backslash M)$.

Definition 7. A point $z \in M$ is an interior point of M if it is not an outer point of M. Set of all interior points of M, is denoted by $\mathrm{int}(M)$.

Example 3. Consider the convex Euclidean metric space (R^2, d, W). Let $M = \{(x,y); y < -x+1 \text{ and } x,y \geq 0\}$. Now

$$\mathbf{E}(M) = \{(0,0), (0,1), (1,0)\},$$

$$\begin{aligned} \partial M &= \{(x,0) : 0 \leq x \leq 1\} \cup \{(0,y) : 0 \leq y \leq 1\} \cup \{(x,y) : x,y \geq 0 \text{ and} \\ y &= -x+1\}, \end{aligned}$$

and

$$\mathrm{int}(M) = \{(x,y) : x,y > 0 \text{ and } y < -x+1\}.$$

Definition 8. A face of a nonempty convex set M of a convex metric space X is a nonempty set $F \subset M$ with the property that if $x,y \in M$, $\alpha \in I$, and $W(x,y,\alpha) \in F$ then $x,y \in F$. A nonempty face $F \neq M$ is called a proper face.

Example 4. Any convex set M itself is its own face. The empty set ϕ is a face of any convex set M.

Example 5. In Example 3, $\{(x,0) : 0 \leq x \leq 1\}$ and $\{(0,y) : 0 \leq y \leq 1\}$ are two faces of of M.

Example 6. In Example 2, the faces of $co(M)$ are its extreme points; $\{(y,1,0) : |y| \leq 1\}$, $\{W((0,a,b),(1,1,0),\alpha)\}$, and $\{W((0,a,b),(-1,1,0),\alpha)\}$ where a,b are points satisfying $a^2 + b^2 = 1$ and $a \neq 1$.

Remark 3.
i. A face F is a subset of M so that any $[y,z] \subset M$, with interior points in F must lie in F.
ii. Extreme points are one point faces of M.

Throughout rest of this paper we assume that X is a convex metric space (X,d,W) satisfying hypothesis of Theorem 1 unless stated otherwise.

In our next theorem we present a novel way to construct proper faces via functions.

THEOREM 2. *Let $h : X \to Y \subset Z$ and $g : Y \to R$, where h be a CP function and g be a linear function. Assume that $g \circ h : X \to R$ is a non constant function satisfying*

$$\sup_{x \in X} g \circ h(x) = \gamma < \infty.$$

Then the set $\{x : g \circ h(x) = \gamma\} = F$ (if nonempty) is a proper face of X.

Proof. Assume without loss of generality that F is nonempty. First we show that F is a convex set. Let x, y be two points in F i.e., $g \circ h(x) = \gamma$ and $g \circ h(y) = \gamma$. Then for any $\alpha \in I$,

$$\begin{aligned} g \circ h(W(x,y,\alpha)) &= g(h(W(x,y,\alpha))) \\ &= g(\alpha h(x) + (1-\alpha)h(y)) \\ &= \alpha g(h(x)) + (1-\alpha)g(h(y)) \\ &= \alpha\, g \circ h(x) + (1-\alpha) g \circ h(y) = \gamma. \end{aligned}$$

Thus F is a convex set. Let x, y be any arbitrary points in X with $W(x,y,\alpha) \in F$. In case $\alpha = 0$, Remark 1(i) implies $y \in F$ and if $\alpha = 1$, Remark 1(i) implies $x \in F$. When $x, y \in X, \alpha \in (0,1)$ and $W(x,y,\alpha) \in F$, then $g \circ h(W(x,y,\alpha)) = \gamma$ and $g \circ h(x) \leq \gamma$, $g \circ h(y) \leq \gamma$ imply $g \circ h(x) = \gamma$, $g \circ h(y) = \gamma$. Thus $x, y \in F$. By hypothesis $g \circ h$ is not a constant function. Therefore, F is a proper subset of X. $\square$

The set $\{x : g \circ h(x) = \gamma\}$ is also called exposed set. In case, it is singleton then that point is called exposed point.

PROPOSITION 1. *Any proper face F of a convex subset M of X is a subset of outer points $\partial(M)$.*

Proof. Let x be any point in F. Choose $y \in X \backslash M$, the set $\Omega = \{\alpha : z = W(x,y,\alpha) \in F\} \subset I$. But if $1 \in \Omega$ then Remarks 1(i) & 3 imply that x is an interior point of $[x,y] \subset M$ with at least one end point y in $X \backslash M$. Therefore, $x \in Wcl(M) \cap Wcl(X \backslash M) = \partial(M)$. $\square$

THEOREM 3. *Let F be a proper face of a convex subset $M \subset X$. A subset P of F is a face of F if and only if it is a face of M.*

Proof. Suppose that P is a face of M, $x \in P$ and x is an interior point of $[y,z] \subset F$. Then x is an interior point of $[y,z] \subset M$. Therefore, $y, z \in M$. Hence $y, z \in P$. It further implies that P is a face of F.

Conversely, *if P is a face of F*, $x \in P$ and $x \in [y,z] \subset M$. As x is in F and F is a face of M, it imply that $y, z \in F$. Therefore, $[y,z] \subset F$. As P is a face of F and $y, z \in P$. Hence P is a face of M. $\square$

COROLLARY 1. *A point x in a proper face F of a convex subset M of X. is in $\mathbf{E}(F)$ if and only if $x \in \mathbf{E}(M)$.*

COROLLARY 2. *Let F be a proper face of a convex subset M of X. Then*

$$\mathbf{E}(F) = F \cap (\mathbf{E}(M)).$$

THEOREM 4. *Let (M) be a collection of faces F of a convex subset M of X then $\bigcap_{F \in (M)} F$ is also a face of M.*

Proof. Remark 1(iii) implies that $\bigcap_{F \in (M)} F$ is a convex subset of M. If for some points x, y in M, $(x,y) \cap (\bigcap_{F \in (M)} F) \neq \phi$. Then (x,y) intersects with each F in (M). Hence $[x,y] \subseteq \bigcap_{F \in (M)} F$. Therefore, $\bigcap_{F \in (M)} F$ is a face of M. $\square$

Let (M) be a collection of faces F of a convex subset M of X, orered by inclusion. Now for every chain $\mathscr{F} \subset (M)$, if $F_1, F_2 \in \mathscr{F}$ then either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$. Theorem 3 implies that in this case either F_1 is a proper face of F_2 or F_2 is a proper face of F_1.

THEOREM 5. *Let M be a convex subset of X. Let $\mathscr{F}$ be a chain of faces of M. Then $\bigcup_{F \in \mathscr{F}} F$ is a face of M.*

Proof. First we show that $\bigcup_{F \in \mathscr{F}} F$ is convex. To show this let $x \in \bigcup_{F \in \mathscr{F}} F$. Then this x is an element of some face F of M. Therefore $x \in M$. Now for any u, v in $\bigcup_{F \in \mathscr{F}} F$ there exists F_u, F_v in $\mathscr{F}$ such that $u \in F_u$ and $v \in F_v$. As $\mathscr{F}$ is a chain , we can take $F_u \subset F_v$.Thus, $u, v \in F_v$. Therefore, $[u,v] \subseteq F_v \subseteq \bigcup_{F \in \mathscr{F}} F$. Hence $\bigcup_{F \in \mathscr{F}} F$ is convex.

Next assume that $x \in \underset{F\in\mathscr{F}}{\cup} F$ and $y,z \in M$ be such that $x \in (y,z)$. Now it implies that there exists a face $F_x \in \mathscr{F}$ such that $x \in F_x$. Thus F_x is a face of M. Therefore, $[y,z] \subseteq F_x \subseteq \underset{F\in\mathscr{F}}{\cup} F$. Hence $\underset{F\in\mathscr{F}}{\cup} F$ is a face of M. $\square$

4. CORE

In functional analysis several equivalent definitions of the core are given. In this section, we give a definition of core similar to Klee [10] and then study its convexity properties.

Definition 9. Let M be a convex subset of a convex metric space (X,d,W). The core(M) is defined as

$$\operatorname{core}(M) = \{x \in M : \forall y \in M \;\exists\; z \in M \text{ such that } x \in (y,z)\}.$$

Note difference between Definitions 7 and 9. If M is singleton then $\operatorname{int}(M) = \phi \neq M = \operatorname{core}(M)$.

THEOREM 6. *Let M be a convex subset of X. Then* core(M) *is convex.*

Proof. Let x,y be two distinct points in core(M) and $z \in (x,y)$. Now for any $a \in M$ there exists $b,c \in M$ such that $x \in (a,b)$ and $y \in (a,c)$. Now for α,β,γ in $(0,1)$ we have

$$x = W(b,a,\alpha), \quad y = W(c,a,\beta), \quad z = W(y,x,\gamma).$$

Using Definition 2(ii) and equality (3) it further implies that

$$\begin{aligned} z &= W(W(b,a,\alpha),W(c,a,\beta),\gamma) \\ &= W(W(b,c,\frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\beta\gamma}),a,\alpha(1-\gamma)+\beta\gamma) \\ &= W(e,a,\alpha(1-\gamma)+\beta\gamma) \end{aligned}$$

where $e = W(b,c,\frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\beta\gamma}) \in M$ by convexity of M. Thus $z \in (a,e)$ with e in M. Therefore, $z \in \operatorname{core}(M)$. Hence core$(M)$ is convex. $\square$

We recall from Kreyszig [11] that the sequence space c_0 is defined as the space of all sequences converging to zero, with metric identical to l_∞. A subspace $c_{00} \subset c_0$, contains only eventually zero sequences (sequences with finite many nonzero terms).

Example 7. Let c_{00} be a convex metric space of eventually zero sequences with l_∞ metric and $W(x,y,\alpha) = \alpha x + (1-\alpha)y$. Let

$$M = \{x \in c_{00} : x_i \in [0,1]\}.$$

Obviously M is convex. Let $x = (x_1,x_2,x_3,...,x_\alpha,..) \in M$ then there exists $\beta \in \mathbb{N}$ such that $x_\alpha = 0$ for all $\alpha \geqslant \beta$. Next choose $y = (y_1,y_2,y_3,...,y_\alpha,..) \in M$ such that

$$d(x,y) = \begin{cases} x_\alpha & \alpha \in \mathbb{N}\backslash\{\beta\}, \\ 1 & \alpha = \beta. \end{cases} \tag{5}$$

Then the sequence y has same entries like sequence x, up till $\beta-1$ entry, has 1 in the $\beta-th$ entry and zero after $\beta-th$ entry. Let $z \in M$ be such that $x \in (y,z)$. Then there exists γ in $(0,1)$ such that $x = \gamma y + (1-\gamma)z$. It implies that $0 = x_\beta = \gamma y_\beta + (1-\gamma)z_\beta = \gamma + (1-\gamma)z_\beta$. It further implies that $z_\beta = \frac{-\gamma}{1-\gamma} < 0$. Thus a contradiction. Therefore, core$(M) = \phi$.

Remark 4. From example (7) we have following interesting observation; that two convex sets E and F with $E \subset F$ did not imply core$(E) \subset$ core(F).

LEMMA 1. *Let* $w,x,y,z \in X$*.If* $w \in (y,z)$*, then for all* $u \in (w,x)$ *there exists* $v \in (z,x)$ *such that* $u \in (y,v)$.

Proof. If $w \in (y,z)$, and $u \in (w,x)$ then there are $\alpha,\beta \in (0,1)$ such that $w = W(y,z,\alpha),\ u = W(w,x,\beta)$. Using equality (3) it implies that,

$$\begin{aligned} u &= W(w,x,\beta) = W(W(y,z,\alpha),x,\beta) \\ &= W(y,W(z,x,\frac{\beta(1-\alpha)}{1-\alpha\beta}),\alpha\beta) \\ &= W(y,v,\alpha\beta). \end{aligned} \tag{6}$$

Here $v = W(z,x,\frac{\beta(1-\alpha)}{1-\alpha\beta}) \in (z,x)$. Equality (6) implies that $u \in (y,v)$. □

PROPOSITION 2. *Let* M *be a convex subset of* X*. If* $x \in \mathrm{core}(M)$ *and* $y \in M$ *then* $[x,y) \subset \mathrm{core}(M)$

Proof. Let $x \in \mathrm{core}(M)$ and $y \in M$. Choose an arbitrary point $z \in (x,y)$.Using Definition 9 of core, for any $a \in M$ there exists $b \in M$ such that $x \in (a,b)$. Lemma 1 further implies that there exists $c \in (b,y)$ such that $z \in (a,c)$. As M is convex and $b,y \in M$. Therefore $c \in M$. Using Definition 9 it further implies that $z \in \mathrm{core}(M)$. Hence $[x,y) \subseteq \mathrm{core}(M)$. □

THEOREM 7. *Let* M *be a convex subset of* X*. If* $x \in \mathrm{core}(M)$ *and* $y \in M$ *then there exists* $e \in \mathrm{core}(M)$ *such that* $x \in (e,y) \subseteq \mathrm{core}(M)$.

Proof. Let $x \in \mathrm{core}(M)$ and $y \in M$ then there must be some $e \in M$ such that $x \in (e,y)$. Now using Proposition 2 we obtain $[x,e) \subset \mathrm{core}(M)$. Thus $x \in (e,y) \subseteq \mathrm{core}(M)$. □

THEOREM 8. *Let* M *be a convex subset of* X *then* $\mathrm{core}(\mathrm{core}(M)) = \mathrm{core}(M)$.

Proof. Let $x \in \mathrm{core}(M)$. Now for any $y \in \mathrm{core}(M)$ we also have $y \in M$. Using Theorem 7 there must be $e \in core(M)$ such that $x \in (e,y) \subseteq \mathrm{core}(M)$. Therefore $x \in \mathrm{core}(\mathrm{core}(M))$.Thus $\mathrm{core}(M) \subseteq \mathrm{core}(\mathrm{core}(M))$. Converse is obvious from Definition 9. Hence $\mathrm{core}(\mathrm{core}(M)) = \mathrm{core}(M)$. □

THEOREM 9. *Let* M *be a convex subset of* X *then* $\mathrm{core}(M) = M \backslash \eth(M)$.

Proof. Let $x \in M$. Now using Definition 6, $x \in \eth(M)$ if and only if there exists a $v \in X \backslash M$ with $W(x,v,\alpha) \notin M$. Hence $x \notin \mathrm{core}(M)$. □

5. CONCLUSION

In this article, we have introduced the definitions and some properties of face and core of a convex set in a convex metric space (without any linear structure). Core appears in the literature on functional analysis/linear algebra under different names, including the set of relatively absorbing points, the pseudo-relative interior and the set of inner points. The next challenging task in future research in this area is to obtain Krein–Milman type theorem or separation of convex sets.

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