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ESTIMATING PROB{Y<X} IN THE CASE OF THE POWER DISTRIBUTION

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We consider the problem of estimating the probability $\text{Prob}\{Y < X\}$, where X and Y are two independent random variables having power distributions. We obtain a parametric estimator $\hat{R}_n$ and a non-parametric estimator $\overline{R}_n$ for the quantity $R = \text{Prob}\{Y < X\}$. We compare the performances of these two estimators using Monte Carlo techniques and we find that the procedure used for the estimates is satisfactory.
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1. INTRODUCTION

The problem of estimating $R = \text{Prob}\{Y < X\}$ where X and Y are independent random variables has been studied for the exponential distribution by Kelley et al. [3], for the double exponential distribution in Awad and Fayoumi [2], and for the Luceño distribution [5].

We shall consider the power distribution $\pi_{b,\delta}$ with parameters $b, \delta > 0$, that is the distribution on IR with the probability density function:

$$\rho(x; b, \delta) = \delta b^{-\delta} x^{\delta-1} \mathbf{1}_{(0,b)}(x), \; x \in \text{IR}$$

Let X and Y be two independent random variables such that $X \sim \pi_{b_1,\delta_1}$, $Y \sim \pi_{b_1,\delta_1}$. We can think of X as the strength of a mechanical system being subjected to a stress Y. The purpose of this paper is to give a measure of the mechanical reliability of the system, that is estimating $\text{Prob}\{Y < X\}$ with δ_1, δ_2 known and b_1, b_2 unknown.

2. ESTIMATING THE RELIABILITY

Since

$$\text{Prob}(Y < X \mid X) = \begin{cases} \left(\dfrac{X}{b_2}\right)^{\delta_2}, & \text{when } 0 < X < b_2, \\ 1, & \text{when } X \geq b_2 \end{cases}$$

and

$$\text{Prob}(Y < X) = E(\text{Prob}(Y < X \mid X))$$

we obtain

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$$\text{Prob}(Y<X)=\int_0^\infty \text{Prob}(Y<X\mid X=x)\rho(x;b_1,\delta_1)dx==\begin{cases}\int_0^{b_1}\left(\frac{x}{b_2}\right)^{\delta_2}\delta_1 b_1^{-\delta_1}x^{\delta_1-1}dx, & \text{if } b_1\le b_2,\\ \int_0^{b_2}\left(\frac{x}{b_2}\right)^{\delta_2}\delta_1 b_1^{-\delta_1}x^{\delta_1-1}dx+\int_{b_2}^{b_1}\delta_1 b_1^{-\delta_1}x^{\delta_1-1}dx, & \text{if } b_1>b_2\end{cases}=$$

$$=\begin{cases}\left(\frac{b_1}{b_2}\right)^{\delta_2}\frac{\delta_1}{\delta_1+\delta_2}, & \text{if } b_1\le b_2,\\ 1-\left(\frac{b_1}{b_2}\right)^{-\delta_1}\frac{\delta_2}{\delta_1+\delta_2}, & \text{if } b_1>b_2\end{cases}$$

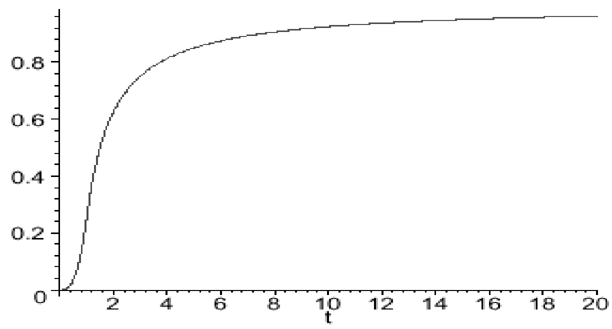
Let $t=b_1/b_2$. Then we define

$$f(t)=\text{Prob}(Y<X)=\begin{cases}t^{\delta_2}\frac{\delta_1}{\delta_1+\delta_2}, & \text{if } 0<t\le 1,\\ 1-t^{-\delta_1}\frac{\delta_2}{\delta_1+\delta_2}, & \text{if t}>1\end{cases} \tag{1}$$

The function $f(\cdot)$ is obviously continuous and differentiable on $(0,\infty)$, increasing and convex on $(0,1)$ and non-convex on $(1,\infty)$, as we can see in the following:

$$f'(t)=\begin{cases}\frac{\delta_1\delta_2}{\delta_1+\delta_2}t^{\delta_2-1}, & \text{if } 0<t\le 1,\\ \frac{\delta_1\delta_2}{\delta_1+\delta_2}t^{-\delta_1-1}, & \text{if t}>1\end{cases};\quad f''(t)=\begin{cases}\frac{\delta_1\delta_2(\delta_2-1)}{\delta_1+\delta_2}t^{\delta_2-2}, & \text{if } 0<t\le 1,\\ \frac{-\delta_1\delta_2(\delta_1+1)}{\delta_1+\delta_2}t^{-\delta_1-2}, & \text{if t}>1\end{cases}$$

The graph of f has the following appearance:



For fixed values of the parameters δ_1,δ_2 we can obtain the value of t necessary to achieve a reliability value of 0.95, or 0.05, specifically $t_{0.95}=2.9240,\ t_{0.5}=0.0444$, for $\delta_1=1.5,\delta_2=0.5$. In the case $\delta_1=\delta_2$ we have $t_{1-\alpha}=\frac{1}{t_\alpha}$.

3. ESTIMATION OF $R=\text{Prob}\{Y<X\}$

Remark. Let $Z\sim\pi_{b,\delta}$ and $z_1,...,z_n$ be a random sample from Z. Since $E(Z)=\frac{b\delta}{\delta+1}$, for $\delta>0$ known, with the method of moments we find that

$$\hat{b}_n = \frac{1+\delta}{\delta}\bar{z}_n$$

is an estimate for the parameter b.

Obviously, $\hat{b}_n = \varphi(\bar{z}_n)$, where $\varphi:(0,\infty)\to(0,\infty), \varphi(t)=\frac{1+\delta}{\delta}t$, is a differentiable function. Then, (see [4], pg. 119), it follows that

$$\hat{b}_n \xrightarrow{a.s.} b, \text{ as } n\to\infty \tag{2}$$

Now let $x_1,...,x_n$ be a random sample from X and $y_1,...,y_n$ a random sample from Y. We assume that the samples are independent. Taking into account the fact that

$$E(X)=\frac{b_1\delta_1}{1+\delta_1},\quad E(Y)=\frac{b_2\delta_2}{1+\delta_2},$$

and using the method of moments, we obtain that

$$\hat{b}_{1,n}=\frac{1+\delta_1}{\delta_1}\bar{x}_n \quad \text{and} \quad \hat{b}_{2,n}=\frac{1+\delta_2}{\delta_2}\bar{y}_n$$

are estimates for the parameters b_1 and b_2.
From (2) it follows that

$$\hat{b}_{1,n}\xrightarrow{a.s.} b_1 \text{ and } \hat{b}_n \xrightarrow{a.s.} b_2 \text{ as } n\to\infty.$$

Then,

$$\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}} \xrightarrow{a.s.} \frac{b_1}{b_2}, \text{ as } n\to\infty \tag{3}$$

Since f is continuous, from (3) we get

$$\hat{R}_n \overset{def}{=} f\left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right) \xrightarrow{a.s.} f\left(\frac{b_1}{b_2}\right), \text{ as } n\to\infty. \tag{4}$$

Therefore, for n large enough,

$$f\left(\frac{b_1}{b_2}\right)\approx \hat{R}_n,$$

that is

$$\hat{R}_n = f\left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right)=\begin{cases}\dfrac{\delta_1}{\delta_1+\delta_2}\left(\dfrac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right)^{\delta_2}, & \text{if } \hat{b}_{1,n}\le\hat{b}_{2,n}\\ 1-\dfrac{\delta_2}{\delta_1+\delta_2}\left(\dfrac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right)^{-\delta_1}, & \text{if } \hat{b}_{1,n}>\hat{b}_{2,n}\end{cases} = \begin{cases}\dfrac{\delta_1}{\delta_1+\delta_2}\hat{r}_n^{\,\delta_2}, & \text{if } \hat{r}_n\le 1\\ 1-\dfrac{\delta_2}{\delta_1+\delta_2}\hat{r}_n^{\,-\delta_1}, & \text{if } \hat{r}_n>1\end{cases}$$

where $\hat{r}_n=\frac{b_1}{b_2}$, is an estimate for $R=f\left(\frac{b_1}{b_2}\right)=\text{Prob}(Y<X)$.

4. SIMULATION STUDY

In this section we will consider, besides the parametric estimator $\hat{R}_n$, defined in the preceding section, a non-parametric estimator (see [5]), defined as follows:

$$\overline{R}_n = \frac{card\{(X_i, Y_j) \mid Y_j < X_i,\ 1 \le i, j \le n\}}{n^2}$$

We will compare the mean bias (*MB*) and mean square error (*MSE*) for the two estimators. For an estimator $\mathbf{R}$ we define the two above quantities by means of the following formulas:

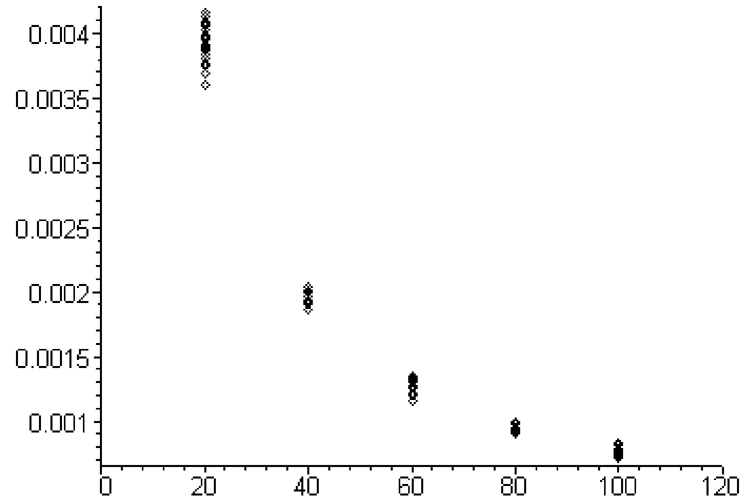
$$MB(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}\left(R(r;\delta_1,\delta_2) - \mathbf{R}(r_i;\delta_1,\delta_2)\right)$$

$$MSE(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}\left(R(r;\delta_1,\delta_2) - \mathbf{R}(r_i;\delta_1,\delta_2)\right)^2$$

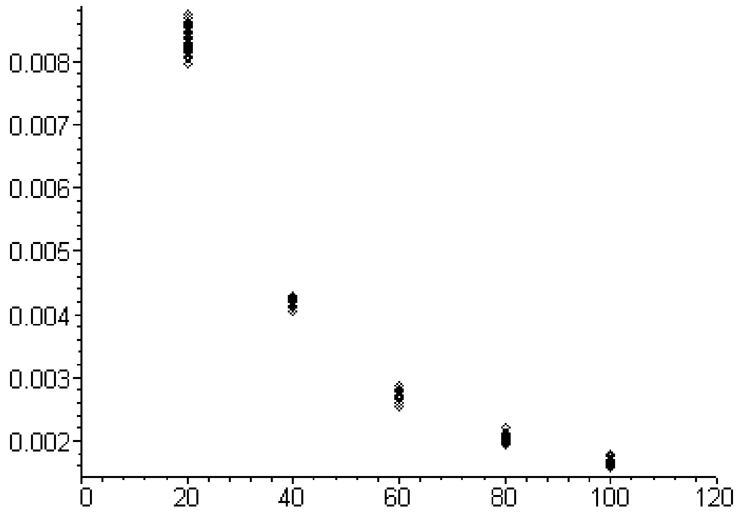
where N represents the number of experiments, in our case estimating R.

Random samples from $X \sim \pi_{b\ ,\delta_1}$, $Y \sim \pi_{b_2,\delta_2}$ were generated, with $(b_1, b_2) \in \{(3,4), (3,6), (3,10), (3,10)\}$ and $(\delta_1, \delta_2) \in \{(0.5,0.5), (0.5,1.5), (0.5,3), (0.5,10)\}$. In order to obtain the *MB* and *MSE* the experiment was repeated $N = 1000$ times. The results can be obtained, on request, from the authors. The simulation showed that:

1) $\hat{R}_n$ and $\overline{R}_n$ estimate R with errors of the 10^{-2} order in the worse case, that is when $b_1 = b_2$. According to the values obtained for the *MSE*, $\hat{R}_n$ is superior. Also we noticed that *MSE* appears to decrease exponentially when the sample size increases, as in the following plots(*MSE* versus sample size):

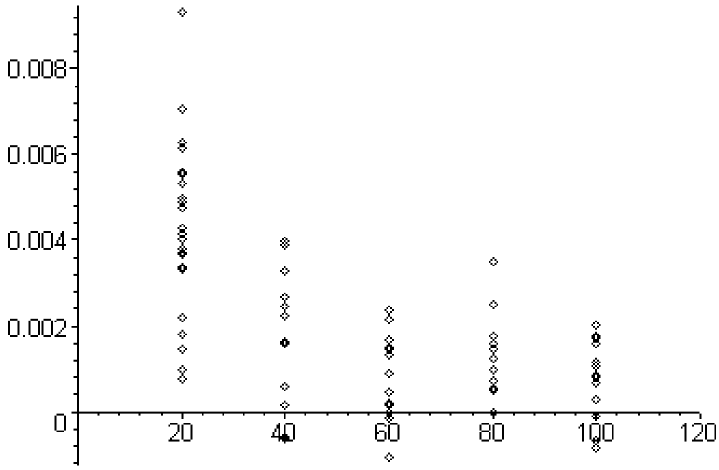


$MSE(\hat{R}_n)$

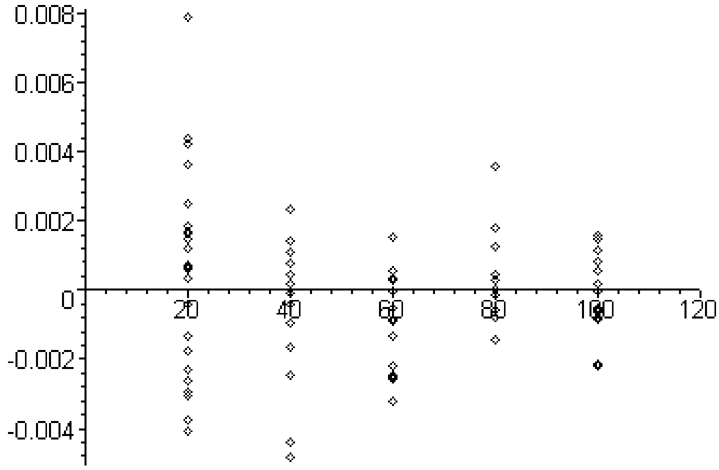


$MSE(\bar{R}_n)$

2) Generally, $\hat{R}_n$ underestimates R, as in the next plots (*MB* versus sample size):

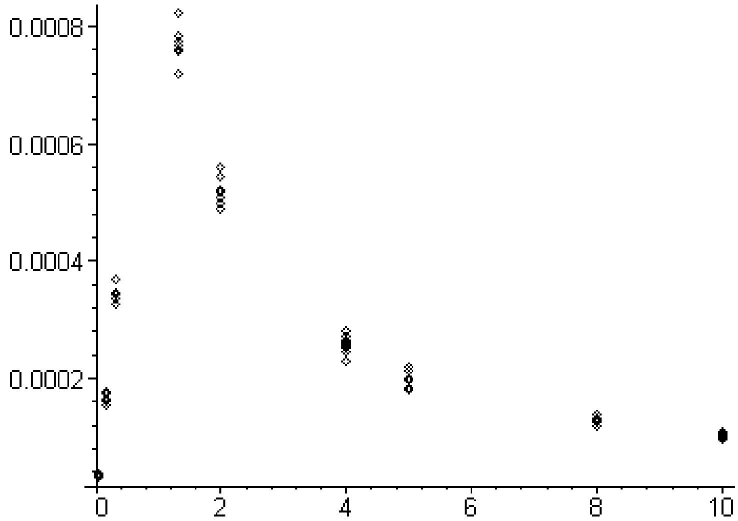


$MB(\hat{R}_n)$



$MB(\bar{R}_n)$

3) $MSE(\hat{R}_n)$ increases when $r = \frac{b_1}{b_2}$ is approaching 1, as it can be observed in the next plot (*MSE* versus r):



We conclude that both estimators appear to work well, with an advantage for the parametric estimator $\hat{R}_n$.

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