



## THE ACCURATE ELEMENT METHOD: A NEW PARADIGM FOR NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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This paper presents the development of a new methodology for numerically solving ordinary differential equations. This methodology, which we call the accurate element method (AEM), breaks the connection between the solution accuracy and the number of unknowns. The differential equations are discretized by using finite elements, like in the finite element method (FEM). A key feature of the AEM is the methodology developed for eliminating unknowns inside the element by using the relations provided by the governing equations. The results of the AEM applied to a second-order ordinary differential equation (ODE) show that the order of the local approximation function does not affect the number of unknowns used to discretize the differential equation. This is a breakthrough in the paradigm used to approach the numerical solution of differential equations. For a second-order ODE it is shown that the discretized solution has the same number of unknowns, whether the local approximation (or interpolation) function is a third-degree or a nineteen-degree polynomial. The implication of this result is that higher-degree interpolation functions can be used without increasing the number of unknowns. For a specified accuracy, the usage of elements with high-degree interpolation functions leads to a reduction in the necessary number of elements. It is shown herein that for the solution of a second-order ODE, two AEM elements with seventh-degree polynomial interpolation functions generate a more accurate solution than 100 FEM elements with a linear interpolation function. Since in this case the number of unknowns per element in the AEM is two times larger than in the FEM, the utilisation of AEM leads 25-fold reduction in the number of unknowns compared to FEM. This reduction of the number of unknowns results in a reduction of the computational time.

*Key words:* Ordinary differential equations; Finite element method.

### 1. INTRODUCTION

In recent years, various numerical methods have been developed for solving ordinary differential equations and partial differential equations. The most general and widely used methods include the finite difference method, finite volume method and finite element method (FEM). The common goal of these methods is to approximate as well as possible the solution of the governing equations describing various physical phenomena.

Several options are currently used to improve the accuracy of these methods: (1) increasing the degree of the polynomials that locally approximate the solution, that is,  $p$ -type refinement; (2) decreasing the size of the elements which discretize the computational domain in order to reduce the variation of the unknowns over the element, that is,  $h$ -type refinement; (3) using anisotropic elements, that is, having different orders of expansion for each spatial direction in an element; (4) using non-conforming elements with either  $h$ -type or  $p$ -type non-conformities; (5) using adaptive refinement strategies.

For all the methods currently used to solve differential equations, the outcome of increasing solution accuracy is an increase in the number of unknowns. To minimize the number of unknowns, each method tries to optimize the relation between the order of the approximation function and the size of the element, such as in the  $h$ - $p$  adaptive finite element or spectral element methods. This paper presents the development of a new methodology for the numerical solution of differential equations. This methodology, which we call

the accurate element method (AEM), breaks the connection between the solution accuracy and the number of unknowns. Consequently, the number of unknowns corresponding to high-accuracy solutions obtained using high-order approximation polynomials is equal to the number of unknowns corresponding to low-order approximation polynomials.

The AEM is described in the next section. The general description of the AEM is followed by the presentation of the methodology. A second-order, linear ODE is used to illustrate the implementation of the AEM.

## 2. GENERAL DESCRIPTION OF THE ACCURATE ELEMENT METHOD

The AEM will be presented by comparing it to the FEM. The strategies of both the AEM and the FEM can be summarized by the following three steps: (1) discretization of the domain  $D$  on which the governing equations must be integrated; (2) local approximation of the solution of the governing equations, in which the information given by each element is concentrated at the nodes of the element; and (3) reconstruction of the domain  $D$ , obtained by writing nodal equations that bring together the information given by the elements adjacent to each node. The solution of the system of equations obtained in this way represents the nodal unknowns.

The first and third steps of the AEM are identical to those of the FEM. The two methods differ only in how the second step of their common strategy is accomplished. Note that the second step is crucial in obtaining an accurate solution of the governing equations. As will be shown herein, the AEM uses a much smaller number of elements than the FEM to obtain a similar accuracy.

In step two, the local approximation of the solution in the FEM is done by using interpolation functions (IF). Usually these interpolation functions are low degree polynomials that lead to poor approximations. Consequently, the number of elements used in the FEM must be large in order to obtain a result with reasonable accuracy. The local approximation in the AEM will be done by high-degree polynomial functions. The high-degree polynomial functions will be called concordant functions (CF). Because the functions used for local approximation are high-degree polynomials, only a small number of elements is needed to obtain an accurate solution in the AEM. The example presented below shows that the accuracy obtained in the AEM by using two elements with a seventh-degree polynomial CF is better than the accuracy obtained in the FEM by using 100 elements with a linear IF.

The degree of the polynomials used in the AEM is usually higher than seven, since, as will be shown herein, the computational effort in the AEM is not greatly affected by the degree of the polynomial. This is an important difference between the AEM and the FEM. High-degree polynomials may be used for the IF in the FEM, but this increases the number of unknowns of the discretized system of equations. In the AEM, the degree of the polynomial used in the local approximation does not affect the number of unknowns of the discretized system of equations. This is an important advantage of the AEM compared to the finite element, finite volume or finite difference methods.

## 3. METHODOLOGY AND EXAMPLE

To introduce the methodology of the AEM, let us start by presenting a simple but yet meaningful case, the solution of a linear, second-order ordinary differential equation [1]:

$$\phi'' + 4\phi - 4x^3 - 8x^2 - 18x - 8 = 0, \quad (1)$$

with the boundary conditions :

$$\phi_1 := \phi(x=0) = 0 \quad ; \quad \phi_2 := \phi(x=1) = 0 \quad (2)$$

The analytical solution of this equation with the specified boundary conditions is:

$$\phi(x) = C_1 \sin(2x) + C_2 \cos(2x) + x^3 + 2x^2 + 3x + 1 \quad (3)$$

Let us consider a CF that is a seventh-degree polynomial:

$$\phi(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7, \quad (4)$$

and let us write  $\phi$  using a natural coordinate  $\eta = x/L$ , where  $L$  is the length of the element. Equation (4) can be written in vectorial form using the natural coordinate  $\eta$ :

$$\phi(\eta) = [\bar{\eta}][CL], \quad (5)$$

where:

$$\begin{aligned} [\bar{\eta}] &= [1 \quad \eta \quad \eta^2 \quad \eta^3 \quad \eta^4 \quad \eta^5 \quad \eta^6 \quad \eta^7] \\ [CL] &= [C_0 \quad C_1L \quad C_2L^2 \quad C_3L^3 \quad C_4L^4 \quad C_5L^5 \quad C_6L^6 \quad C_7L^7]^T \end{aligned}$$

The values of the function  $\phi$  and its derivatives at both ends of the element can be written as:

$$\begin{bmatrix} \phi_1^{(0)} \\ L\phi_1^{(1)} \\ L^2\phi_1^{(2)} \\ L^3\phi_1^{(3)} \\ \phi_2^{(0)} \\ L\phi_2^{(1)} \\ L^2\phi_2^{(2)} \\ L^3\phi_2^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \end{bmatrix} [CL], \quad (6)$$

or :

$$[B] = [A][CL]. \quad (7)$$

The notation  $\phi_i^{(0)} := \phi_i$  and  $\phi_i^{(k)} := \frac{d^k \phi_i}{dx^k}$  with  $k \geq 1$  has been used in equation (6). The inverse of matrix [A] is:

$$[A]^{-1} = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -210 & -120 & -30 & -4 & 210 & -90 & 15 & -1 \\ 504 & 270 & 60 & 6 & -504 & 234 & -42 & 3 \\ -420 & -216 & -45 & -4 & 420 & -204 & 39 & -3 \\ 120 & 60 & 12 & 1 & -120 & 60 & -12 & 1 \end{bmatrix}, \quad (8)$$

and [B] can be rearranged as:

$$[B] = \begin{bmatrix} \phi_1^{(0)} \\ L\phi_1^{(1)} \\ L^2\phi_1^{(2)} \\ L^3\phi_1^{(3)} \\ \phi_2^{(0)} \\ L\phi_2^{(1)} \\ L^2\phi_2^{(2)} \\ L^3\phi_2^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + L \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + L^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{bmatrix} + L^3 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{bmatrix}. \quad (9)$$

The vector of coefficients [CL] can be written using equations (7), (8) and (9) as:

$$[CL] = [A]^{-1}[B] = [F80] \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + L[F81] \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + L^2[F82] \begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{bmatrix} + L^3[F83] \begin{bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{bmatrix}, \quad (10)$$

where

$$[F80] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -35 & 35 \\ 84 & -84 \\ -70 & 70 \\ 20 & -20 \end{bmatrix} \quad [F81] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ -20 & -15 \\ 45 & 39 \\ -36 & -34 \\ 10 & 10 \end{bmatrix} \quad [F82] = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -10 & 5 \\ 20 & -14 \\ -15 & 13 \\ 4 & -4 \end{bmatrix} \quad [F83] = \frac{1}{6} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -4 & -1 \\ 6 & 3 \\ -4 & -3 \\ 1 & 1 \end{bmatrix}.$$

Let us derive a relation similar to but more general than the stiffness matrix relation. This relation should provide a link between the variables  $\phi_i^{(0)}$  and the variable derivatives  $\phi_i^{(k)}$ ,  $k \geq 1$ . This relation, which is a generalization of the stiffness matrix relation, will be called the complete transfer relation (CTR) [1]. The derivation of the CTR can be done based on the physical phenomenon that is described by the governing equations or by using the Galerkin method. The latter approach will be used herein, because of its generality.

Let us use the following weighting function:

$$w(x) = \frac{L-x}{L}\phi_1 + \frac{x}{L}\phi_2 = [\bar{N}] \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (11)$$

in the Galerkin method, and let us rewrite the governing equation (1) in a more general form:

$$\frac{d^2\phi}{dx^2} + c\phi(x) + d(x) = 0.$$

Using the Galerkin projection one obtains:

$$\int_0^L [\bar{N}]^T \frac{d^2\phi}{dx^2} dx + \int_0^L [\bar{N}]^T \phi(x) dx + \int_0^L [\bar{N}]^T d(x) dx = [0] \quad (12)$$

After integrating by parts the first term, equation (12) becomes:

$$\frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} - c \int_0^L \frac{1}{L} \begin{bmatrix} L-x \\ x \end{bmatrix} \phi(x) dx = \int_0^L \frac{1}{L} \begin{bmatrix} L-x \\ x \end{bmatrix} d(x) dx \quad (13)$$

Let us now replace the function  $\phi$  of the third term of the left-hand side by the expression given in equation (5). Equation (13) becomes:

$$\begin{aligned} & \frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} - cL[GOF0]_8 \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} - cL^2[GOF1]_8 \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + \\ & \left. - cL^3[GOF2]_8 \begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{bmatrix} - cL^4[GOF3]_8 \begin{bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{bmatrix} \right] = \int_0^L \frac{1}{L} \begin{bmatrix} L-x \\ x \end{bmatrix} d(x) dx \end{aligned} \quad (14)$$

where:

$$[GOF0]_8 = \frac{1}{36} \begin{bmatrix} 13 & 5 \\ 5 & 13 \end{bmatrix} \quad [GOF1]_8 = \frac{1}{252} \begin{bmatrix} 17 & -10 \\ 10 & -17 \end{bmatrix} \quad [GOF2]_8 = \frac{1}{1008} \begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix} \quad [GOF3]_8 = \frac{1}{15120} \begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix}.$$

Note that often the third term of equation (13) is not accounted for while deriving the stiffness matrix in the FEM. Even when this term is accounted for, the function  $\phi$  in the integral is approximated by a low degree polynomial. This term is responsible for introducing errors in the FEM. Unlike the FEM, the AEM uses a high order polynomial to approximate  $f(x)$ . The result of this approach is that equation (14) has 8 unknowns. There are, however, only 4 available relations: 2 equations (14) and 2 boundary conditions (2). Consequently, 4 additional relations are necessary.

Usually, when more information is needed, this information is obtained either from some relations with other elements or by accepting a reasonable approximation. Neither of these options is used in the AEM. Instead, the necessary information is obtained accurately from inside the element, that is, from the governing equation itself. Consequently, the second and third derivatives are:

$$\begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{bmatrix} = -c \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} - \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix}, \quad (15)$$

and :

$$\begin{bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{bmatrix} = -c \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} - \begin{bmatrix} d_1^{(1)} \\ d_2^{(1)} \end{bmatrix}, \quad (16)$$

where :

$$d_1^{(0)} = d(x=0) ; d_2^{(0)} = d(x=L) ; d_1^{(1)} = d'(x=0) ; d_2^{(1)} = d'(x=L).$$

After substituting equations (15) and (16) in equation (14) one obtains:

$$\begin{aligned} & \frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} - cL[GOF0]_8 \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} - cL^2[GOF1]_8 \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + \\ & + c^2 \bar{E} [GOF2]_8 \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + cL^3 [GOF2]_8 \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} + c^2 \bar{E} [GOF3]_8 \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + cL^4 [GOF3]_8 \begin{bmatrix} d_1^{(1)} \\ d_2^{(1)} \end{bmatrix} = \\ & = \int_0^L \frac{1}{L} \begin{bmatrix} L-x \\ x \end{bmatrix} d(x) dx \end{aligned}$$

The unknowns  $\phi_1^{(2)}, \phi_2^{(2)}, \phi_1^{(3)}$  and  $\phi_2^{(3)}$ , which we call the ‘‘apparent’’ unknowns, have been eliminated by using the governing equations.

Similar equations can be obtained by using higher order CFs, as shown below:

- seventh-degree concordant function, CF8:

$$\begin{bmatrix} -1/3 & -1.476190476190476 \\ -1.476190476190476 & -1/3 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + \begin{bmatrix} 0.7354497354497356 & 0.1544973544973545 \\ -0.1544973544973545 & -0.7354497354497356 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} -6.915343915343915 \\ -11.62751322751323 \end{bmatrix},$$

- nineteen-degree concordant function, CF20:

$$\begin{bmatrix} -0.310606670618465 & -1.417281845711744 \\ -1.417281845711744 & -0.310606670618465 \end{bmatrix} \begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{bmatrix} + \begin{bmatrix} 0.7082522309668606 & 0.153520002220184 \\ -0.153520002220184 & -0.7082522309668606 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} -6.571622875498251 \\ -11.13461085637016 \end{bmatrix}.$$

For comparison, let us show the FEM solution obtained using a linear interpolation function (IF2):

$$\begin{bmatrix} -1/3 & -5/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^{(0)} \\ \mathbf{f}_2^{(0)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^{(1)} \\ \mathbf{f}_2^{(1)} \end{bmatrix} = \begin{bmatrix} -23.6/3 \\ -12.8 \end{bmatrix}.$$

Table 1 shows the variation of  $\phi_1^{(1)}$  as a function of the degree of the polynomial in the CF. This solution is based on one element. For comparison, the value of  $\phi_1^{(1)}$  calculated using the FEM with an IF2 is also included. The exact value is obtained from the solution (3) as  $\phi_1^{(1)} = -13.31181749284520$ . The variation of the error shows that an accurate solution can be obtained with only one AEM element with CF12.

Table 1: Solution based on one element.

|      | $\phi_1^{(1)}$     | Error [%] |
|------|--------------------|-----------|
| IF2  | -7.86666           | -40.90    |
| CF4  | -12.85714285714286 | -3.41     |
| CF8  | -13.31157287382312 | -1.83e-3  |
| CF12 | -13.31181747484904 | -1.35e-7  |
| CF16 | -13.31181749284484 | -2.72e-12 |
| CF20 | -13.31181749284521 | 4.00e-14  |

The variation of  $\phi_1^{(1)}$  as a function of the number of elements is shown in Table 2. Results are shown for the FEM with IF2 and the AEM with CF4 and CF8. The advantage of using the AEM and high-degree polynomials is obvious. Two AEM elements with CF8 produce more accurate results than 100 FEM elements with IF2. Four AEM elements with CF8 produce a solution with an error of the order of  $10^{-8}\%$ .

Table 2: Variation of the value of  $\phi_1^{(1)}$  as a function of the number of elements and degree of the CF/IF.

| Exact value: $\phi_1^{(1)} = \phi_1' = \phi'(x=0) = -13.31181749284520$ |                          |           |                          |           |           |
|---|--------------------------|-----------|--------------------------|-----------|-----------|
| NR.   | IF                       |           | CFI                      |           | CFI       |
| elem.   | $\phi_1^{(1)} = \phi_1'$ | Error %   | $\phi_1^{(1)} = \phi_1'$ | Error %   | Error %   |
| 1   | -7.86666                 | -40.90    | -12.85714285714286       | -3.41     | -1.83 e-3 |
| 2   | -11.632291666            | -12.62    | -13.28681646362417       | -1.87 e-1 | -4.74 e-6 |
| 4   | -12.85655835183100       | -3.42     | -13.31028065459640       | -1.15 e-2 | -1.84 e-8 |
| 6   | -13.10604588668801       | -1.55     | -13.31151472702216       | -2.27 e-3 | *         |
| 10  | -13.23707977753565       | -0.56     | -13.31177830616024       | -2.94 e-4 | *         |
| 20  | -13.29306202953536       | -0.14     | -13.31181504502042       | -1.83 e-5 | *         |
| 50  | -13.30881341149338       | -2.26 e-2 | -13.31181743019060       | -4.70 e-7 | *         |
| 100   | -13.31106635773660       | -5.60 e-3 | *                        | *         | *         |

#### 4. CONCLUSIONS

A new method, which we call the accurate element method, has been presented herein. The AEM has been applied to solve ordinary differential equations. A key feature of the AEM was the methodology developed for eliminating unknowns inside the element by using the relations provided by the governing equations. The results of the AEM applied to a second-order ODE showed that the order of the local approximation function did not affect the number of unknowns used to discretize the differential equation. This is a breakthrough in the paradigm used to approach the numerical solution of differential equations.

For a second-order ODE it was shown that the discretized solution has the same number of unknowns, whether the local approximation (or interpolation) function is a third-degree or a nineteen-degree polynomial. The implication of this result is that higher-degree interpolation functions can be used without increasing the number of unknowns. For a specified accuracy, the usage of elements with high-degree interpolation functions leads to a reduction in the necessary number of elements. It was shown herein that for the solution of a second-order ODE, two AEM elements with seventh-degree polynomial interpolation functions generate a more accurate solution than 100 FEM elements with a linear interpolation function. Since in this case the number of unknowns per element in the AEM is two times larger than in the FEM, the utilisation of AEM leads 25-fold reduction in the number of unknowns compared to FEM.

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