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DETERMINATION OF THE MOVING EQUATIONS FOR A MANY DEGREES OF FREEDOM SYSTEM USING KANE'S EQUATIONS

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Our aim is to determine the moving equation for a many degrees of freedom system using Kane's equations. It is well known that Lagrange's equations form a set of second order differential equations, but Kane's equations form a set of first order differential equations, which are very useful in numerical studies. Kane's equations are known as Lagrange type equations for d'Alembert principle.

Key words: Lagrange's equations, Kane's equations, mechanical systems.

1. INTRODUCTION

Let us consider the system from figure 1, formed by the body of mass m_1 and the bars of masses m_2 and m_3, all of them placed in a vertical plane. The bar AB is jointed in the point A with the sieve of mass m_1, and the bar BC is jointed in the point B with the bar AB. The lengths of the bars are $AB = l_2$ and $BC = l_3$, respectively (fig. 1). The mass centres of the two bars are in the points C_2 and C_3, such that

$$AC_2 = C_2B = \frac{l_2}{2};\ BC_3 = C_3C = \frac{l_3}{2}. \tag{1.1}$$

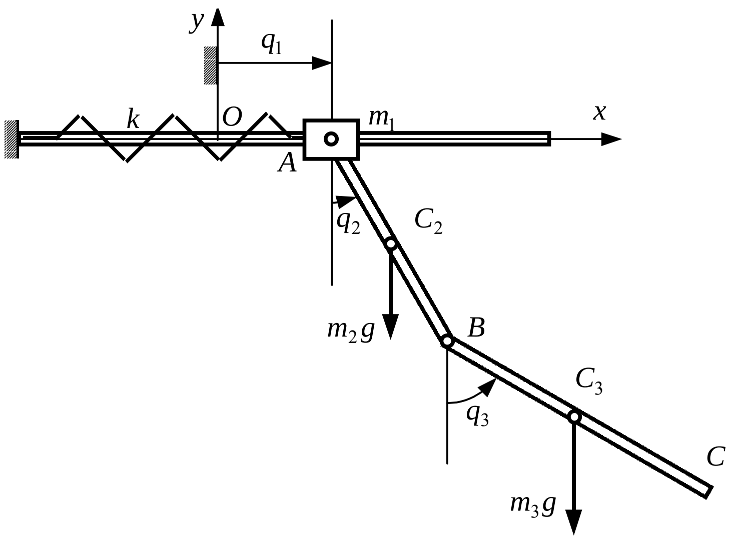

Fig. 1. Many degrees of freedom studied system.

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The body of mass m_1 is connected by the spring of stiffness k. We have selected the generalised co-ordinates q_1 (the horizontal displacement of the sieve relative to the non-tensioned position of the spring), q_2 and q_3 (defining the position relative to vertical axis of the two bars).

If we should use the Lagrange's equations, we should obtain the system of differential equations

$$A_{11}\ddot{q}_1 + A_{12}\ddot{q}_2 + A_{13}\ddot{q}_3 = B_1;\ A_{21}\ddot{q}_1 + A_{22}\ddot{q}_2 + A_{23}\ddot{q}_3 = B_2;\ A_{31}\ddot{q}_1 + A_{32}\ddot{q}_2 + A_{33}\ddot{q}_3 = B_3, \tag{1.2}$$

where A_{ij}, $i = 1, 2, 3$, $j = 1, 2, 3$ are functions of the generalised co-ordinates q_i, $i = 1, 2, 3$, and B_i, $i = 1, 2, 3$ are functions of the generalised co-ordinates and of the generalised speeds $\dot{q}_i$, $i = 1, 2, 3$.

The system (1-2) is a linear system of 3 equations in the unknowns $\ddot{q}_1$, $\ddot{q}_2$ and $\ddot{q}_3$ which can be determined with the Cramer's rule. One obtains:

$$\ddot{q}_i = E_i\left(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3\right) \tag{1.3}$$

with $i = 1, 2, 3$.

Noting:

$$\xi_i = q_i;\ \xi_{i+3} = \dot{q}_i, \tag{1.4}$$

where $i = 1, 2, 3$, we obtain a system of 6 non-linear differential equations of first order,

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \xi_{i+3};\ \frac{\mathrm{d}\xi_{i+3}}{\mathrm{d}t} = E_i\left(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3\right). \tag{1.5}$$

Again $i = 1, 2, 3$.

2. KINEMATICS OF THE MOTION

We shall note:

$$u_i = \dot{q}_i;\ i = 1, 2, 3, \tag{2.1}$$

the generalised speeds of the system.

We have:

$$v_1 = \dot{q}_1 = u_1;\ \omega_2 = \dot{q}_2 = u_2;\ \omega_3 = \dot{q}_3 = u_3. \tag{2.2}$$

The positions of the mass centres with respect to the frame xOy are:

$$\begin{gathered}\mathbf{r}_{\mathbf{C_1}} = q_1\mathbf{i};\ \mathbf{r}_{\mathbf{C_2}} = \left(q_1 + \frac{l_2}{2}\sin q_2\right)\mathbf{i} - \frac{l_2}{2}\cos q_2\mathbf{j};\\ \mathbf{r}_{\mathbf{C_3}} = \left(q_1 + l_2\sin q_2 + \frac{l_3}{2}\sin q_3\right)\mathbf{i} - \left(l_2\cos q_2 + \frac{l_3}{2}\cos q_3\right)\mathbf{j}.\end{gathered} \tag{2.3}$$

The speeds of the mass centres read:

$$\begin{gathered}\mathbf{v}_{\mathbf{C_1}} = \dot{q}_1\mathbf{i} = u_1\mathbf{i};\ \bar{v}_{C_2} = \left(\dot{q}_1 + \frac{l_2}{2}\dot{q}_2\cos q_2\right)\bar{i} + \frac{l_2}{2}\dot{q}_2\sin q_2\,\bar{j};\\ \mathbf{v}_{\mathbf{C_3}} = \left(\dot{q}_1 + l_2\dot{q}_2\cos q_2 + \frac{l_3}{2}\dot{q}_3\cos q_3\right)\mathbf{i} + \left(l_2\dot{q}_2\sin q_2 + \frac{l_3}{2}\dot{q}_3\sin q_3\right)\mathbf{j},\end{gathered} \tag{2.4}$$

and the accelerations of the mass centres are:

$$\mathbf{a}_{C_1} = \ddot{q}_1\mathbf{i}\,;\; \mathbf{a}_{C_2} = \left(\ddot{q}_1 + \frac{l_2}{2}\ddot{q}_2 - \frac{l_2}{2}\dot{q}_2^2 \sin q_2\right)\mathbf{i} + \left(\frac{l_2}{2}\ddot{q}_2 \sin q_2 + \frac{l_2}{2}\dot{q}_2^2 \cos q_2\right)\mathbf{j}\,;$$
$$\bar{a}_{C_3} = \left(\ddot{q}_1 + l_2\ddot{q}_2 \cos q_2 - l_2\dot{q}_2^2 \sin q_2 + \frac{l_3}{2}\ddot{q}_3 \cos q_3 - \frac{l_3}{2}\dot{q}_3^2 \sin q_3\right)\bar{i} + \left(l_2\ddot{q}_2 \sin q_2 + l_2\dot{q}_2^2 \cos q_2 + \frac{l_3}{2}\ddot{q}_3 \sin q_3 + \frac{l_3}{2}\dot{q}_3^2 \cos q_3\right)\bar{j}. \tag{2.5}$$

3. GENERALIZED INERTIAL FORCES

For a rigid body in an arbitrary motion, the generalised inertial force is given by

$$F_k^* = \frac{\partial \mathbf{v}_{\mathbf{CG}}}{\partial u_k} \cdot \mathbf{F}_{\mathbf{in}} + \frac{\partial \boldsymbol{\omega}}{\partial u_k} \cdot \mathbf{M}_{\mathbf{in}}\,, \tag{3.1}$$

where $\mathbf{v}_{\mathbf{CG}}$ is the speed of the mass centre, $\boldsymbol{\omega}$ is the angular speed, $\mathbf{F}_{\mathbf{in}}$ is the inertial force, and $\mathbf{M}_{\mathbf{in}}$ is the inertial moment.

For our system, we can write:

$$\mathbf{F}_{\mathbf{in_1}} = m_1\dot{u}_1\mathbf{i}\,;\; \mathbf{F}_{\mathbf{in_2}} = -m_2\left[\left(\dot{u}_1 + \frac{l_2}{2}\dot{u}_2 \cos q_2 - \frac{l_2}{2}u_2^2 \sin q_2\right)\mathbf{i} + \left(\frac{l_2}{2}\dot{u}_2 \sin q_2 + \frac{l_2}{2}u_2^2 \cos q_2\right)\mathbf{j}\right];$$
$$\mathbf{F}_{\mathbf{in_3}} = -m_3\left[\left(\ddot{q}_1 + l_2\dot{u}_2 \cos q_2 - l_2u_2^2 \sin q_2 + \frac{l_3}{2}\dot{u}_3 \cos q_3 - \frac{l_3}{2}u_3^2 \sin q_3\right)\mathbf{i} + \left(l_2\dot{u}_2 \sin q_2 + l_2u_2^2 \cos q_2 + \frac{l_3}{2}\dot{u}_3 \sin q_3 + \frac{l_3}{2}u_3^2 \cos q_3\right)\mathbf{j}\right], \tag{3.2}$$

$$\mathbf{M}_{\mathbf{in_1}} = 0\,;\; \mathbf{M}_{\mathbf{in_2}} = -J_{C_2}\dot{u}_2\mathbf{k}\,;\; \mathbf{M}_{\mathbf{in_3}} = -J_{C_3}\dot{u}_3\mathbf{k}\,, \tag{3.3}$$

$$\frac{\partial \bar{v}_{C_1}}{\partial u_1} = \frac{\partial \bar{v}_{C_2}}{\partial u_1} = \frac{\partial \bar{v}_{C_3}}{\partial u_1} = \bar{i}\,;\; \frac{\partial \mathbf{v}_{\mathbf{C_1}}}{\partial u_2} = \frac{\partial \mathbf{v}_{\mathbf{C_1}}}{\partial u_3} = \frac{\partial \mathbf{v}_{\mathbf{C_2}}}{\partial u_3} = 0\,;\; \frac{\partial \mathbf{v}_{\mathbf{C_2}}}{\partial u_2} = \frac{l_2}{2}\cos q_2\mathbf{i} + \frac{l_2}{2}\sin q_2\mathbf{j}\,;$$
$$\frac{\partial \mathbf{v}_{\mathbf{C_3}}}{\partial u_2} = l_2 \cos q_2\mathbf{i} + l_2 \sin q_2\mathbf{j}\,;\; \frac{\partial \mathbf{v}_{\mathbf{C_3}}}{\partial u_3} = \frac{l_3}{2}\cos q_3\mathbf{i} + \frac{l_3}{2}\sin q_3\mathbf{j}\,, \tag{3.4}$$

$$\frac{\partial \boldsymbol{\omega}_1}{\partial u_1} = \frac{\partial \boldsymbol{\omega}_1}{\partial u_2} = \frac{\partial \boldsymbol{\omega}_1}{\partial u_3} = \frac{\partial \boldsymbol{\omega}_2}{\partial u_1} = \frac{\partial \boldsymbol{\omega}_2}{\partial u_3} = \frac{\partial \boldsymbol{\omega}_3}{\partial u_1} = \frac{\partial \boldsymbol{\omega}_3}{\partial u_2} = 0\,;\; \frac{\partial \boldsymbol{\omega}_2}{\partial u_2} = \frac{\partial \boldsymbol{\omega}_3}{\partial u_3} = -\mathbf{k}\,. \tag{3.5}$$

We find the generalised inertial forces:

$$F_1^* = -\dot{u}_1(m_1 + m_2 + m_3) - \dot{u}_2\left(\frac{m_2}{2} + m_3\right)l_2 \cos q_2 - \dot{u}_3m_3\frac{l_3}{2}\cos q_3 + u_2^2\left(\frac{m_2}{2} + m_3\right)l_2 \sin q_2 + u_3^2m_3\frac{l_3}{2}\sin q_3\,;$$
$$F_2^* = -\dot{u}_1\left(\frac{m_2}{2} + m_3\right)l_2 \cos q_2 - \dot{u}_2\left(m_2l_2^2 + J_{C_2} + m_3l_2^2\right) - \dot{u}_3m_3l_2\frac{l_3}{2}\cos(q_3 - q_2)\,;$$
$$F_3^* = -\dot{u}_1m_3\frac{l_3}{2}\cos q_3 - \dot{u}_2m_3l_2\frac{l_3}{2}\cos(q_3 - q_2) - \dot{u}_3\left(m_3\frac{l_3^2}{4} + J_{C_3}\right) - u_2^2m_3l_2\frac{l_3}{2}\sin(q_3 - q_2)\,. \tag{3.6}$$

4. GENERALIZED ACTIVE FORCES

The active forces in the system are:

$$\mathbf{F_e} = -kq_1\mathbf{i}\,;\ \mathbf{G_2} = -m_2 g\mathbf{j}\,;\ \mathbf{G_3} = -m_3 g\mathbf{j}\,. \tag{4.1}$$

The generalised active forces are defined by the following expressions:

$$F_1 = \frac{\partial \mathbf{v_{C_1}}}{\partial u_1}\cdot\mathbf{F_e} + \frac{\partial \mathbf{v_{C_2}}}{\partial u_1}\cdot\mathbf{G_2} + \frac{\partial \mathbf{v_{C_3}}}{\partial u_1}\cdot\mathbf{G_3}\,;\ F_2 = \frac{\partial \mathbf{v_{C_1}}}{\partial u_2}\cdot\mathbf{F_e} + \frac{\partial \mathbf{v_{C_2}}}{\partial u_2}\cdot\mathbf{G_2} + \frac{\partial \mathbf{v_{C_3}}}{\partial u_2}\cdot\mathbf{G_3}\,;$$
$$F_3 = \frac{\partial \mathbf{v_{C_1}}}{\partial u_3}\cdot\mathbf{F_e} + \frac{\partial \mathbf{v_{C_2}}}{\partial u_3}\cdot\mathbf{G_2} + \frac{\partial \mathbf{v_{C_3}}}{\partial u_3}\cdot\mathbf{G_3} \tag{4.2}$$

and they lead to

$$F_1 = -kq_1\,;\ F_2 = -m_2 g\frac{l_2}{2}\sin q_2 - m_3 g l_2 \sin q_3\,;\ F_3 = -m_3 g\frac{l_3}{2}\sin q_3\,. \tag{4.3}$$

5. KANE'S DYNAMICAL EQUATIONS

These equations read:

$$F_k^* + F_k = 0\,;\ k = 1,2,3\,. \tag{5.1}$$

They offer us:

$$\dot u_1(m_1 + m_2 + m_3) + \dot u_2\left(\frac{m_2}{2} + m_3\right)l_2\cos q_2 + \dot u_3 m_3\frac{l_3}{2}\cos q_3 -$$
$$- u_2^2\left(\frac{m_2}{2} + m_3\right)l_2\sin q_2 - u_3^2 m_3\frac{l_3}{2}\sin q_3 + kq_1 = 0\,;$$
$$\dot u_1\left(\frac{m_2}{2} + m_3\right)l_2\cos q_2 + \dot u_2\left(m_2 l_2^2 + J_{C_2} + m_3 l_2^2\right) + \dot u_3 m_3 l_2\frac{l_3}{2}\cos(q_3 - q_2) +$$
$$+ m_2 g\frac{l_2}{2}\sin q_2 + m_3 g l_2 \sin q_3 = 0\,;$$
$$\dot u_1 m_3\frac{l_3}{2}\cos q_3 + \dot u_2 m_3 l_2\frac{l_3}{2}\cos(q_3 - q_2) + \dot u_3\left(m_3\frac{l_3^2}{4} + J_{C_3}\right) + u_2^2 m_3 l_2\frac{l_3}{2}\sin(q_3 - q_2) +$$
$$+ m_3 g\frac{l_3}{2}\sin q_3 = 0\,. \tag{5.2}$$

The system (5-2) with the relations (2-2) forms a system of 6 non-linear differential equations.

6. NUMERICAL RESULTS

For the numerical simulation we selected the working parameters as follows:

$$k = 1000\left[{}^{\mathrm{N}}\!/\!_{\mathrm{m}}\right];\ l_2 = 0.2[\mathrm{m}];\ l_3 = 0.3[\mathrm{m}];\ m_1 = 1[\mathrm{kg}];\ m_1 = 1[\mathrm{kg}];\ m_2 = 1.2[\mathrm{kg}];\ m_3 = 1.4[\mathrm{kg}];$$
$$J_{C_2} = \frac{m_2 l_2^2}{12} = 0.004\left[\mathrm{kgm}^2\right];\ J_{C_2} = \frac{m_3 l_3^2}{12} = 0.0105\left[\mathrm{kgm}^2\right]. \tag{6.1}$$

The initial values are:

$$q_1^0 = q_1(0) = 0.01[\text{m}];\ q_2^0 = q_2(0) = \frac{\pi}{6}[\text{rad}];\ q_3^0 = q_3(0) = \frac{\pi}{3}[\text{rad}];\ u_1^0 = u_1(0) = 0.01\left[\text{m}/\text{s}\right];$$
$$u_2^0 = u_2(0) = \left[\text{rad}/\text{s}\right];\ u_3^0 = u_3(0) = \left[\text{rad}/\text{s}\right]. \quad (62)$$

In figure 2 we have captured the variation of q_1 as a function of time t for $0 \le t \le 3[\text{s}]$. In the same conditions, in figure 3 we have captured the variation of u_2 as a function of time t for $0 \le t \le 3[\text{s}]$.

The reader can easy observe that the motion is not a periodical one, but the allure of the diagrams suggests an original periodical one. In we should select the initial values closer to the vertical equilibrium point, the motion could consider to be periodic.

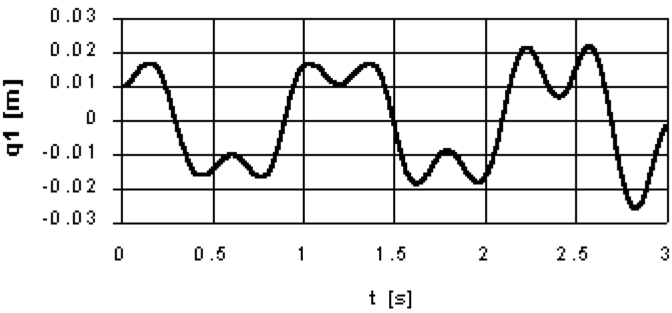

Fig. 2. The variation $q_1 = q_1(t)$ for $0 \le t \le 3[\text{s}]$.

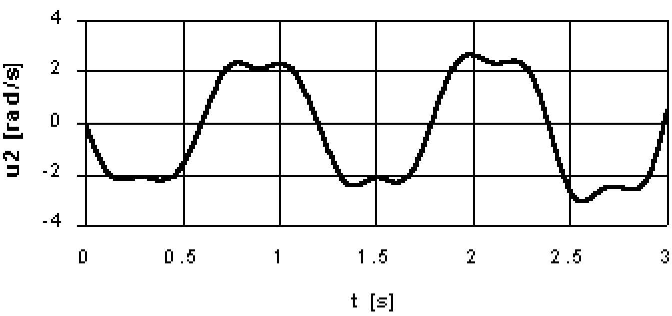

Fig. 3. The variation $u_2 = u_2(t)$ for $0 \le t \le 3[\text{s}]$.

7. CONCLUSIONS

Kane's equations are, in our example, similar to Lagrange's equations.

The great advantage is that Kane's equations are first order differential equations, but Lagrange's are second order. Another advantage, probably the most important one, is that in the case of Kane's equations the generalised speeds may be not only the derivatives of the generalised co-ordinates, but also combinations of them.

Lesser proved that in some cases there exists a selection of the generalised speed such that the system (5-2) becomes a diagonal one.

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