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MULTIOBJECTIVE PROGRAMMING PROBLEMS WITH GENERALIZED V-TYPE-I UNIVEXITY AND RELATED n-SET FUNCTIONS

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We study optimality conditions and generalized Mond-Weir duality for multiobjective programming involving n-set functions which satisfy appropriate generalized V-type-I univexity conditions.

Key words: optimality, duality, multiobjective programming, n-set function, generalized V-type-I univexity.

1. PRELIMINARIES

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ its positive orthant, i.e.

$$\mathbb{R}^n_+ = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_j \geqq 0, j = 1, \ldots, n\right\}.$$

For any vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, we use the following notation: $x < y$ if $x_i < y_i$, $i = 1, 2, ..., n$; $x \leqq y$ iff $x_i \leqq y_i$, $i = 1, 2, ..., n$; $x \leq y$ if $x \leqq y$, but $x \neq y$; $x^\top y = \sum_{i=1}^n x_i y_i$.

For an arbitrary vector $x \in \mathbb{R}^n$ and a subset J of the index set $\{1, 2, ..., n\}$, we denote by x_J the vector with components x_j, $j \in J$.

Let (X, Γ, μ) be a finite atomless measure space and assume that $L_1(X, \Gamma, \mu)$ is a separable function space. For $h \in L_1(X, \Gamma, \mu)$ and $Z \in \Gamma$ with indicator function $I_Z \in L_\infty(X, \Gamma, \mu)$, the integral $\int_Z h \,\mathrm{d}\mu$ will be denoted by $\langle h, I_Z \rangle$.

Now, we shall define the notion of differentiability for *n*-set functions. Morris [7] introduced differentiability for set functions and Corley [4] defined this notion for n-set functions.

A function $\varphi : \Gamma \to \mathbb{R}$ is said to be differentiable at T if there exists $D\varphi_T \in L_1(X, \Gamma, \mu)$, called the derivative of φ at T, such that

$$\varphi(S) = \varphi(T) + \langle D\varphi_T, I_S - I_T \rangle + \psi(S, T)$$

for each $S \in \Gamma$, where $\psi : \Gamma \times \Gamma \to \mathbb{R}$ and has the property that $\psi(S, T)$ is $o(d(S, T))$, that is $\lim_{d(S,T) \to 0} \psi(S, T)/d(S, T) = 0$, and d is a pseudometric on Γ [7].

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A function $h:\Gamma^n \to \mathbb{R}$ is said to have a partial derivative at $S^0 = \left(S_1^0,\dots,S_n^0\right)$ with respect to its k-th argument, $1 \leqq k \leqq n$, if the function

$$\varphi(S_k) = h\left(S_1^0,\dots,S_{k-1}^0,S_k,S_{k+1}^0,\dots,S_n^0\right)$$

has derivative $D\varphi_{S_k^0}$, and we define $D_k h\left(S^0\right) = D\varphi_{S_k^0}$. If the $D_k h\left(S^0\right)$, $1 \leqq k \leqq n$, all exist, then we put $Dh\left(S^0\right) = \left(D_1 h\left(S^0\right),\dots,D_n h\left(S^0\right)\right)$. If $H:\Gamma^n \to \mathbb{R}^m$, $H = (H_1,\dots,H_m)$, we put $D_k H\left(S^0\right) = \left(D_k H_1\left(S^0\right)\right)$.

A function $h:\Gamma^n \to \mathbb{R}$ is said to be differentiable at $S^0 \in \Gamma^n$ if there exist $Dh\left(S^0\right)$ and $\psi:\Gamma^n \times \Gamma^n \to \mathbb{R}$ such that

$$h(S) = h\left(S^0\right) + \sum_{k=1}^{n}\left\langle D_k h\left(S^0\right), I_{S_k} - I_{S_k^0}\right\rangle + \psi\left(S,S^0\right)$$

where $\psi\left(S,S^0\right)$ is $o\left[d\left(S,S^0\right)\right]$ for all $S \in \Gamma^n$.

A vector set function $f = \left(f_1,\dots,f_p\right):\Gamma \to \mathbb{R}^p$ is differentiable on Γ if all its component functions f_i, $1 \leqq i \leqq p$, are differentiable on Γ.

In this paper we consider the n-set function multiobjective optimization problem

(VP) $\qquad$ $$minimize\left\{f(S) = \left(f_1(S),\dots,f_p(S)\right) \mid g(S) \leqq 0, S \in \Gamma^n\right\},$$

where $f:\Gamma^n \to \mathbb{R}^p$ and $g:\Gamma^n \to \mathbb{R}^m$ are differentiable n-set functions on Γ^n. The problem is to find the collection of (properly) efficient sets defined below.

Let $\mathcal{P} = \left\{S \in \Gamma^n \mid g(S) \leqq 0\right\}$ be the set of all feasible solutions for problem (VP).

Definition 1.1. *A feasible solution $S^0 \in \mathcal{P}$ is said to be an efficient solution (Pareto solution) for problem (VP) if there exists no other feasible solution $S \in \mathcal{P}$ such that $f(S) \leq f\left(S^0\right)$.*

Definition 1.2. *An efficient solution S^0 to (VP) is called properly efficient, if there exists a positive number M with the property that, if $f_i(S) < f_i\left(S^0\right)$, for each i and $S \in P$, then $\dfrac{f_i\left(S^0\right) - f_i(S)}{f_j(S) - f_j\left(S^0\right)} \leqq M$ for some j for which $f_j(S) > f_j\left(S^0\right)$.*

We shall consider a partition $\{J_0, J_1,\dots,J_k\}$ of the index set $\{1,2,\dots,m\}$, that is, $\bigcup_{s=0}^{k} J_s = \{1,2,\dots,m\}$, and $J_s \cap J_t = \varnothing$ for any $s \neq t$. Put

$$\psi_i\left(S,\lambda_{J_0}\right) = f_i(S) + \lambda_{J_0}^{\top} g_{J_0}(S)$$

for any i, $1 \leqq i \leqq p$, where $\lambda \in \mathbb{R}_+^m$ is a given vector. Moreover, we consider vectors $\rho = \left(\rho_1,\dots,\rho_p\right) \in \mathbb{R}^p$, $\rho' = \left(\rho_1',\dots,\rho_k'\right) \in \mathbb{R}^k$, and real numbers $\rho_0, \rho_0' \in \mathbb{R}$.

The following definitions extend similar concepts defined by Jeyakumar and Mond [5], Mishra et al. [6] and Bătătorescu [1],[2],[3].

Definition 1.3. *We say that problem (VP) is (ρ,ρ')-V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1,\dots,J_k\}$ if there exist positive real functions $\alpha_1,\dots,\alpha_p$ and $\beta_1,\dots,\beta_m$ defined on $\Gamma^n \times \Gamma^n$,* *nonnegative functions* b_0 *and* b_1, *also defined on* $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, *and a vector* $\lambda \in \mathbb{R}^m_+$ *such that*

$$b_0\left(S,S^0\right)\varphi_0\left[\psi_i\left(S,\lambda_{J_0}\right)-\psi_i\left(S^0,\lambda_{J_0}\right)\right] \geqq \alpha_i\left(S,S^0\right)\sum_{t=1}^{n}\left\langle D_t\psi_i\left(S^0,\lambda_{J_0}\right), I_{S_t}-I_{S_t^0}\right\rangle + \rho_i d^2\left(S,S^0\right) \tag{1}$$

and

$$-b_1\left(S,S^0\right)\varphi_1\left[\sum_{j\in J_s}\lambda_j g_j\left(S^0\right)\right] \geqq \beta_s\left(S,S^0\right)\sum_{j\in J_s}\lambda_j\sum_{t=1}^{n}\left\langle D_t g_j\left(S^0\right), I_{S_t}-I_{S_t^0}\right\rangle + \rho_s' d^2\left(S,S^0\right) \tag{2}$$

for any $S \in \mathcal{P}$, $i \in \{1,\ldots,p\}$, *and* $s \in \{1,\ldots,k\}$.

If (VP) is (ρ,ρ')-V-univex type I at all $S^0 \in \mathcal{P}$, according to the partition $\{J_0, J_1,\ldots, J_k\}$, then we say that (VP) is (ρ,ρ')-V-univex type I on $\mathcal{P}$, according to the partition $\{J_0, J_1,\ldots, J_k\}$.

If strict inequality holds in (1) (whenever $S \neq S^0$), then we say that (VP) is (ρ,ρ')- semi strictly V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1,\ldots, J_k\}$, depending on the case.

Definition 1.4. *We say that problem (VP) is* (ρ_0,ρ_0')*-quasi V-univex type I at* $S^0 \in \mathcal{P}$ *according to the partition* $\{J_0, J_1,\ldots, J_k\}$ *if there exist positive real functions* $\alpha_1,\ldots,\alpha_p$ *and* $\beta_1,\ldots,\beta_m$ *defined on* $\Gamma^n \times \Gamma^n$, *nonnegative functions* b_0 *and* b_1, *also defined on* $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, *such that for some vectors* $\tau \in \mathbb{R}^p_+$ *and* $\lambda \in \mathbb{R}^m_+$ *the implications*

$$\begin{aligned} & b_0\left(S,S^0\right)\varphi_0\left[\sum_{i=1}^{p}\tau_i\alpha_i\left(S,S^0\right)\left[\psi_i\left(S,\lambda_{J_0}\right)-\psi_i\left(S^0,\lambda_{J_0}\right)\right]\right] \leqq 0 \\ & \Rightarrow \sum_{i=1}^{p}\tau_i\sum_{t=1}^{n}\left\langle D_t\psi_i\left(S^0,\lambda_{J_0}\right), I_{S_t}-I_{S_t^0}\right\rangle \leqq -\rho_0 d^2\left(S,S^0\right), \forall S \in \mathcal{P}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} & b_1\left(S,S^0\right)\varphi_1\left[\sum_{s=1}^{k}\beta_s\left(S,S^0\right)\sum_{j\in J_s}\lambda_j g_j\left(S^0\right)\right] \leqq 0 \\ & \Rightarrow \sum_{j=1,\, j\notin J_0}^{m}\lambda_j\sum_{t=1}^{n}\left\langle D_t g_j\left(S^0\right), I_{S_t}-I_{S_t^0}\right\rangle \leqq -\rho_0' d^2\left(S,S^0\right), \forall S \in \mathcal{P}, \end{aligned} \tag{4}$$

both hold.

If (VP) is (ρ_0,ρ_0')-quasi V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1,\ldots, J_k\}$, then we say that (VP) is (ρ_0,ρ_0')-quasi V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1,\ldots, J_k\}$.

If the second (implied) inequality in (3) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0,ρ_0')- semi strictly quasi V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1,\ldots, J_k\}$, depending on the case.

Definition 1.5. *We say that problem* (VP) *is* (ρ_0,ρ_0')*-pseudo V-univex type I at* $S^0 \in \mathcal{P}$ *according to the partition* $\{J_0, J_1,\ldots, J_k\}$, *if there exist positive real functions* $\alpha_1,\ldots,\alpha_p$ *and* $\beta_1,\ldots,\beta_m$ *defined on* $\Gamma^n \times \Gamma^n$, *nonnegative functions* b_0 *and* b_1, *also defined on* $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, *such that for some vectors* $\tau \in \mathbb{R}^p_+$, $\lambda \in \mathbb{R}^m_+$ *and* $\forall S \in \mathcal{P}$, *the implications*

$$\sum_{i=1}^{p}\tau_i\sum_{t=1}^{n}\left\langle D_t\psi_i\left(S^0,\lambda_{J_0}\right),I_{S_t}-I_{S_t^0}\right\rangle \leqq -\rho_0 d^2\left(S,S^0\right)\Rightarrow$$
$$b_0\left(S,S^0\right)\varphi_0\left[\sum_{i=1}^{p}\tau_i\alpha_i\left(S,S^0\right)\left[\psi_i\left(S,\lambda_{J_0}\right)-\psi_i\left(S^0,\lambda_{J_0}\right)\right]\right]\geqq 0, \tag{5}$$

and

$$\sum_{j=1,\ j\notin J_0}^{m}\lambda_j\sum_{t=1}^{n}\left\langle D_t g_j\left(S^0\right),I_{S_t}-I_{S_t^0}\right\rangle \geqq -\rho_0' d^2\left(S,S^0\right)\Rightarrow$$
$$b_1\left(S,S^0\right)\varphi_1\left[\sum_{s=1}^{k}\beta_s\left(S,S^0\right)\sum_{j\in J_s}\lambda_j g_j\left(S^0\right)\right]\leqq 0, \tag{6}$$

both hold.

If (VP) is $\left(\rho_0,\rho_0'\right)$- pseudo V-univex type I at all $S^0\in\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$, then we say that (VP) is $\left(\rho_0,\rho_0'\right)$- pseudo V-univex type I on $\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$.

If the second (implied) inequality in (5) is strict ($S\neq S^0$), then we say that (VP) is $\left(\rho_0,\rho_0'\right)$- semi-strictly pseudo V-univex type I in f (in g) at S^0 or on $\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$, depending on the case.

If the second (implied) inequalities in (5) and (6) are both strict, then we say that (VP) is $\left(\rho_0,\rho_0'\right)$-strictly pseudo V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$, depending on the case.

Definition 1.6. *We say that problem (VP) is $\left(\rho_0,\rho_0'\right)$-quasi pseudo V-univex type I at $S^0\in\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$ if there exist positive real functions $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_m$ defined on $\Gamma^n\times\Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n\times\Gamma^n$, $\varphi_0:\mathbb{R}\to\mathbb{R}$, $\varphi_1:\mathbb{R}\to\mathbb{R}$, such that for some vectors $\tau\in\mathbb{R}_+^p$ and $\lambda\in\mathbb{R}_+^m$ the implications*

$$b_0\left(S,S^0\right)\varphi_0\left[\sum_{i=1}^{p}\tau_i\alpha_i\left(S,S^0\right)\left[\psi_i\left(S,\lambda_{J_0}\right)-\psi_i\left(S^0,\lambda_{J_0}\right)\right]\right]\leqq 0$$
$$\Rightarrow\sum_{i=1}^{p}\tau_i\sum_{t=1}^{n}\left\langle D_t\psi_i\left(S^0,\lambda_{J_0}\right),I_{S_t}-I_{S_t^0}\right\rangle \leqq -\rho_0 d^2\left(S,S^0\right),\ \forall\, S\in\mathcal{P} \tag{7}$$

and

$$\sum_{j=1,\ j\notin J_0}^{m}\lambda_j\sum_{t=1}^{n}\left\langle D_t g_j\left(S^0\right),I_{S_t}-I_{S_t^0}\right\rangle \geqq -\rho_0' d^2\left(S,S^0\right)\Rightarrow$$
$$b_1\left(S,S^0\right)\varphi_1\left[\sum_{s=1}^{k}\beta_s\left(S,S^0\right)\sum_{j\in J_s}\lambda_j g_j\left(S^0\right)\right]\leqq 0,\ \forall\, S\in\mathcal{P}, \tag{8}$$

do hold.

If (VP) is $\left(\rho_0,\rho_0'\right)$-quasi pseudo V-univex type I at all $S^0\in\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$, then we say that (VP) is $\left(\rho_0,\rho_0'\right)$-quasi pseudo V-univex type I on $\mathcal{P}$ according to the partition $\{J_0,J_1,\ldots,J_k\}$.

If the second (implied) inequality in (2) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ_0')-quasi strictly pseudo V-univex type I at S^0 or on $\mathcal{P}$, according to the partition $\{J_0, J_1, \ldots, J_k\}$, depending on the case.

Definition 1.7. *We say that problem (VP) is (ρ_0, ρ_0')-pseudo quasi V-univex type I at $S \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ if there exist positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_m$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that for some vectors $\tau \in \mathbb{R}_+^p$, $\lambda \in \mathbb{R}_+^m$ and $\forall\, S \in \mathcal{P}$, the implications*

$$\begin{gathered}\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \geqq -\rho_0 d^2\left(S, S^0\right) \Rightarrow \\ b_0\left(S, S^0\right) \varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i \left(S, S^0\right) \left[\psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right) \right] \right] \geqq 0\end{gathered} \tag{9}$$

and

$$\begin{gathered}b_1\left(S, S^0\right) \varphi_1 \left[\sum_{s=1}^{k} \beta_s \left(S, S^0\right) \sum_{j \in J_s} \lambda_j g_j \left(S^0\right) \right] \geqq 0 \Rightarrow \\ \sum_{j=1,\, j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j \left(S^0\right), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0' d^2\left(S, S^0\right)\end{gathered} \tag{10}$$

do hold.

If (VP) is (ρ_0, ρ_0')-pseudo quasi V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$, then we say that (VP) is (ρ_0, ρ_0')-pseudo quasi V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$.

If the second (implied) inequality in (9) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ_0')-strictly pseudo quasi V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$, depending on the case.

Remark 1.8. If we take in the above definitions $J_0 = \varnothing$, $k = m$ and $J_s = \{s\}$ for $s \in \{1, 2, \ldots, m\}$, then we retrieve the concepts in Bătătorescu [1].

Remark 1.9. If we take in the above definitions $n = 1$, $\rho_0 = \rho_0' = 0$, respectively $\rho_i = 0$ for any $i = 1, \ldots, p$, and $\rho_s' = 0$ for any $s = 1, \ldots, k$, then we retrieve the concepts in Bătătorescu [3].

2. SOME OPTIMALITY CONDITIONS

The following results give sufficient conditions for a set to be an efficient solution to problem (VP) under generalized type I conditions with respect to a partition of the constraints.

Theorem 2.1 (Sufficiency). *Assume that*

(a1) $S^0 \in \mathcal{P}$;

(a2) *there exist* $\tau^0 \in \mathbb{R}_+^p$, $\sum_{i=1}^{p} \tau_i^0 = 1$, *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that*

- *for any* $S \in \mathcal{P}$ *we have*

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i\left(S^0\right), I_{S_t} - I_{S_t^0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j\left(S^0\right), I_{S_t} - I_{S_t^0} \right\rangle \geqq 0,$$

- with respect to the partition $\{J_0, J_1, \ldots, J_k\}$, *we have*

$$\sum_{j\in J_s} \lambda_j^0 g_j\left(S^0\right) = 0 \text{ for any } s \in \{0,1,\ldots,k\};$$

(a3) *problem* (VP) *is* (ρ_0, ρ_0')*-quasi strictly pseudo* V*-univex type I at* S^0 *with* $\rho_0 + \rho_0' \geq 0$ *according to the partition* $\{J_0, J_1, \ldots, J_k\}$, *with respect to* τ^0, λ^0 *and for some positive functions* α_i, $i \in \{1,\ldots,p\}$ *and* β_j, $j \in \{1,\ldots,m\}$.

Further, suppose that for $r \in \mathbb{R}$, *we have,*

$$r \leqq 0 \Rightarrow \varphi_0(r) \leqq 0, \tag{11}$$

$$\varphi_1(r) < 0 \Rightarrow r < 0 \tag{12}$$

and

$$b_0\left(S, S^0\right) > 0,\ b_1\left(S, S^0\right) > 0,\ \forall\, S \in \mathcal{P}. \tag{13}$$

Then S^0 *is an efficient solution to* (VP).

Remark 2.2. For $n = 1$ and $\rho_0 = \rho_0' = 0$ we retrieve Theorem 1 of [3].

Remark 2.3. For $J_0 =, \varnothing$ $k = m$ and $J_s = \{s\}$, $s \in \{1,2,\ldots,m\}$, the above result reduces to Theorem 3.1 in [3]. We easily can state sufficient optimality theorems similar to those in [1].

The next result gives necessary condition for a properly efficient solution to (VP).

Theorem 2.4 (Necessity, Zalmai [9]). *Assume that*

(b1) S^0 *is a properly efficient solution to* (VP);

(b2) *there exists* $S^* \in \mathcal{P}$ *with* $g_{M_0}\left(S^*\right) < 0$, *where* $M_0 = \left\{ j \mid g_j\left(S^0\right) = 0 \right\}$, *such that*

$$g_j\left(S^0\right) + \sum_{t=1}^{n} \left\langle D_t g_j\left(S^0\right), I_{S_t^*} - I_{S_t^0} \right\rangle < 0, \forall j \in \{1,\ldots,m\},$$

Then there exist $\tau^0 \in \mathbb{R}^p$, $\tau^0 > 0$, *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that we have*

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i\left(S^0\right), I_{S_t} - I_{S_t^0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j\left(S^0\right), I_{S_t} - I_{S_t^0} \right\rangle \geqq 0,\ \mathrm{S} \in \mathcal{P},$$

and

$$\lambda_j^0 g_j\left(S^0\right) = 0,\ j \in \{1,\ldots,m\}.$$

3. GENERALIZED MOND-WEIR DUALITY

With respect to the partition $\{J_0, J_1, \ldots, J_k\}$ of its constraints, we associate with problem (VP) the following multiobjective dual problem:

(*GMWD*) maximize $\left(f_1(T)+\lambda_{J_0}^{\top} g_{J_0}(T),\ldots, f_p(T)+\lambda_{J_0}^{\top} g_{J_0}(T)\right)$

subject to $(T,\tau,\lambda)\in \mathsf{D}$

where

$$\mathsf{D}=\left\{(T,\tau,\lambda)\left|\begin{array}{c}\sum_{i=1}^{p}\tau_i\sum_{t=1}^{n}\left\langle D_t\left(f_i+\lambda_{J_0}^{\top}g_{J_0}\right)(T), I_{S_t}-I_{T_t}\right\rangle+ \\ +\sum_{j=1}^{m}\lambda_j\sum_{t=1}^{n}\left\langle D_t g_j(T), I_{S_t}-I_{T_t}\right\rangle \geqq 0, \forall S\in\Gamma^n \\ \lambda_{J_s}^{\top}g_{J_s}(T)\geqq 0,\ s=1,\ldots,k, \\ T\in\Gamma^n, \tau\in\mathbb{R}_+^p, e^{\top}\tau=1, \lambda\in\mathbb{R}_+^m\end{array}\right.\right\}$$

is the set of feasible solutions, with $e=(1,\ldots,1)^{\top}\in\mathbb{R}^p$.

Theorem 3.1. (Weak duality). *Assume that*

(i) $S\in P$;

(i2) $(T,\tau,\lambda)\in\mathsf{D}$ *and* $\tau>0$;

(i3) *problem* (VP) *is* (ρ_0,ρ_0')*- pseudo quasi* V*-univex type* I *at* T *with* $\rho_0+\rho_0'\geq 0$ *according to the partition* $\{J_0,J_1,\ldots,J_k\}$ *and some positive functions* α_i, $i\in\{1,\ldots,p\}$, β_s, $s\in\{1,\ldots,k\}$.

Further, assume that for $r\in\mathbb{R}$ *we have*

$$\varphi_0(r)\geqq 0\Rightarrow r\geqq 0, \tag{14}$$

$$r\geqq 0\Rightarrow\varphi_1(r)\geqq 0, \tag{15}$$

and

$$b_0(S,T)>0, b_1(S,T)\geqq 0. \tag{16}$$

Then $f(S)\not\leq f(T)+\lambda_{J_0}^{\top}g_{J_0}(T)e$.

Theorem 3.2.(Weak duality). *Assume that assumptions* (i1) *and* (i2) *of Theorem* 3.1 *hold. We also assume that*

(i.3′) *problem* (VP) *is* (ρ,ρ')*- semi strictly* V*-univex type* I *at* T *with* $\sum_{i=1}^{p}\frac{\tau_i\rho_i}{\alpha_i(S,T)}+\sum_{s=1}^{k}\frac{\rho_s'}{\beta_s(S,T)}\geqq 0$ *according to the partition* $\{J_0,J_1,\ldots,J_k\}$, *with respect to* λ *and some positive functions* α_i^*, $i\in\{1,\ldots,p\}$, β_s^*, $s\in\{1,\ldots,k\}$.

Further, assume that the functions φ_0 *and* φ_1 *have the properties*

$$\varphi_0(r)\geqq 0\Rightarrow r\geqq 0 \tag{17}$$

and

$$r\geqq 0\Rightarrow\varphi_1(r)\geqq 0, \tag{18}$$

with φ_0 *linear, and*

$$b_0(S,T)>0, b_1(S,T)\geqq 0. \tag{19}$$

Then $f(S)\, f(T)+\lambda_{J_0}^{\top}g_{J_0}(T)e$.

Theorem 3.3. (Strong duality). *Assume that* (j1) S^0 *is a properly efficient solution to* (VP);

(j2) *there exists* $S^* \in \mathrm{P}$ *with* $g_{M_0}\left(S^*\right) < 0,$ *where* $M_0 = \left\{ j \mid g_j\left(S^0\right) = 0 \right\},$ *such that*

$$g_j\left(S^0\right) + \sum_{t=1}^{n} \left\langle D_t g_j\left(S^0\right), I_{S_t^*} - I_{S_t^0} \right\rangle < 0, \forall j \in \{1,...,m\}.$$

Then there exist $\tau^0 \in \mathbb{R}^p,$ $\tau^0 > 0$ *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that* $\left(S^0, \tau^0, \lambda^0\right) \in D$ *and the objective functions of* (VP) *and* (GMWD) *have the same values at* S^0 *and* $\left(S^0, \tau^0, \lambda^0\right),$ *respectively. If problem* (VP) *is* $\left(\rho_0, \rho_0'\right)$*- pseudo quasi* V*-univex type* I *with* $\rho_0 + \rho_0' \geqq 0$ *at all feasible solutions of* (GMWD) *according to the partition* $\{J_0, J_1, \ldots, J_k\}$*,with respect to* $\tau^0,$ λ^0*, and conditions* (14)-(16) *of Theorem* 3.1 *are satisfied, then* $\left(S^0, \tau^0, \lambda^0\right) \in D$ *is an efficient solution for* (GMWD).

Theorem 3.4. (Strong duality). *Assume that* (j1) *and* (j2) *of Theorem* 3.3 *are satisfied. Then there exist* $\tau^0 \in \mathbb{R}^p,$ $\tau^0 > 0$ *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that* $\left(S^0, \tau^0, \lambda^0\right) \in D$ *and the objective functions of* (VP) *and* (GMWD) *have the same values at* S^0 *and* $\left(S^0, \tau^0, \lambda^0\right),$ *respectively. If problem* (VP) *is* $\left(\rho_0, \rho_0'\right)$*- semi strictly* V*-univex type* I *with* $\rho_0 + \rho_0' \geqq 0$ *at all feasible solutions of* (GMWD) *according to the partition* $\{J_0, J_1, \ldots, J_k\}$*, with respect to* λ^0*, and conditions* (17)-(19) *of Theorem* 3.2 *are satisfied, then* $\left(S^0, \tau^0, \lambda^0\right) \in \mathrm{D}$ *is an efficient solution for* (GMWD).

Theorem 3.5. (Converse duality). *Assume that*

(k1) $\left(T^0, \tau^0, \lambda^0\right) \in D$ *with* $\tau^0 > 0$;

(k2) $T^0 \in \mathrm{P}$;

(k3) *problem* (VP) *is* $\left(\rho, \rho'\right)$ *-*V*-univex type* I *at* $T^0,$ *with* $\sum_{i=1}^{p} \frac{\tau_i^0 \rho_i}{\alpha_i\left(S, T^0\right)} + \sum_{s=1}^{k} \frac{\rho_s'}{\beta_s\left(S, T^0\right)} \geqq 0,$ *according to the partition* $\{J_0, J_1, \ldots, J_k\}$*, with respect to* λ^0 *and some positive functions* $\alpha_i,$ $i \in \{1,..., p\},$ *and* $\beta_s,$ $s \in \{1,...,k\}.$

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$r < 0 \Rightarrow \varphi_0(r) < 0 \quad ; \quad \varphi_0(0) \leqq 0 \quad ; \quad r_1 \leqq r_2 \Rightarrow \varphi_0(r_1) \leqq \varphi_0(r_2), \tag{20}$$

$$r \geqq 0 \Rightarrow \varphi_1(r) \geqq 0 \tag{21}$$

and

$$b_0\left(S, T^0\right) > 0, b_1\left(S, T^0\right) \geqq 0, \ \forall S \in \mathrm{P}. \tag{22}$$

Then T^0 *is an efficient solution to* (VP).

If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i\left(S, T^0\right) < m_i$ *for all* $S \in \mathrm{P}$ *and* $i = 1,..., p,$ *then* T^0 *is properly efficient for* (VP).

Theorem 3.6. (Converse duality). *Assume that* (k1) *and* (k2) *of Theorem* 3.5 *are fulfilled and problem* (VP) *is* $\left(\rho_0, \rho_0'\right)$*- semi strictly pseudo V-univex type* I *in* $g,$ *at* $T^0,$ *with* $\rho_0 + \rho_0' \geq 0,$ *according to the partition* $\{J_0, J_1, \ldots, J_k\}$*, with respect to* $\tau^0,$ λ^0 *and some positive functions* $\alpha_i,$ $i \in \{1,..., p\},$ $\beta_s,$ $s \in \{1,...,k\}.$

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$\varphi_0(r) \geqq 0 \Rightarrow r \geqq 0, \tag{23}$$

$$r \geqq 0 \Rightarrow \varphi_1(r) \geqq 0 \tag{24}$$

and

$$b_0(S,T^0) > 0, b_1(S,T^0) \geqq 0, \ \forall S \in \mathsf{P}. \tag{25}$$

Then T^0 *is an efficient solution to* (VP).

If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i(S,T^0) < m_i$ *for all* $S \in P$ *and* $i \in \{1,..., p\}$, *then* T^0 *is properly efficient for* (VP).

Theorem 3.7. *Assume that* (k1) *and* (k2) *of Theorem* 3.5.*are fulfilled and*

(k3′) *problem* (VP) *is* (ρ_0, ρ_0')*-strictly pseudo quasi* V-*univex type* I *at* T^0, *with* $\rho_0 + \rho_0' \geq 0$, *according to the partition* $\{J_0, J_1,.., J_k\}$, *with respect to* τ^0, λ^0 *and some positive functions* α_i, $i \in \{1,..., p\}$, β_s, $s \in \{1,...,k\}$.

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$r < 0 \Rightarrow \varphi_0(r) \leqq 0 \quad ; \quad r_1 \leqq r_2 \Rightarrow \varphi_0(r_1) \leqq \varphi_0(r_2), \tag{26}$$

$$r \geqq 0 \Rightarrow \varphi_1(r) \geqq 0 \tag{27}$$

and

$$b_0(S,T^0) > 0, b_1(S,T^0) \geqq 0, \ \forall S \in \mathsf{P}. \tag{28}$$

Then T^0 *is an efficient solution for* (VP).

If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i(S,T^0) < m_i$ *for all* $S \in \mathsf{P}$ *and* $i = 1,..., p$, *then* T^0 *is properly efficient for* (VP).

The proofs will appear in [8].

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