THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 6, Number 3/2005, pp. 000-000

BAYESIAN PREMIUM AND ASYMPTOTIC APPROXIMATION FOR A CLASS OF LOSS FUNCTIONS

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We derive Bayesian premium for a class of loss functions. An exact calculation of Bayesian premium is possible under restrictive circumstances only, therefore we apply a theorem of Gatto [5] in order to derive an approximation for the premium. This approximation is based on Laplace's method, is simply computed and allows for analytical interpretations. Different forms of weighted quadratic loss functions are considered. We prove that the Laplace approximation of Bayesian premium does not depend on the loss function involved in exact evaluation of the Bayesian premium if this loss function is of weighted quadratic type or exponentially scaled.

Key words: Bayesian premium, Laplace approximation, weighted quadratic loss function, exponentially scaled loss function .

1. INTRODUCTION

The determination of the premium for an insured risk is one of the important problems of risk theory. By premium calculation it is meant the determination of the adequate premium for the risk assumed by an insured individual within a collectivity.

Moreover, if some claim experience for the given individual is available, we seek to determine the premium which exploits these actual claim amounts. Therefore, we compute Bayesian premium which is a premium based on claim experience and it is defined as the value which minimizes an expected loss with respect to a posterior distribution.

An exact calculation for the Bayesian premium is possible under restrictive circumstances only, regarding the prior distribution, the likelihood function and the loss function. That is why it would be helpful if we had a good approximation of the Bayesian premium simply to compute and which allows for analytical interpretations.

In order to derive an approximation for the Bayesian premium, we employ Laplace method by using a theorem of Gatto [5]. We do this considering a weighted quadratic loss function.

Important research concerning Bayesian analysis in actuarial science is due to Bailey [1], Buhlmann [4], Klugman [7], Makov [8], Schnieper [10].

In what concerns the use of Laplace approximation in actuarial science, we have to mention the work of Bleistein and Handelsman [3], Tierney and Kadane [11], Gatto [5].

2. BAYESIAN PREMIUM AND ITS LAPLACE APPROXIMATION

We consider random variables X_1, X_2, …, X_n on a probability field $\{\Omega, K, \Pr\}$, representing the size of claims in n consecutive periods for a given individual. X_i takes positive real values.

Reccomended by Marius IOSIFESCU, member of the Romanian Academy

Let $\{S,\Im,\rho\}$ be a measurable space (with measure ρ) and $\Theta:\Omega\to S$ a random variable taking the values $\theta\in S$.

We now consider a probability measure Π on $\{S,\Im\}$ which is absolutely continuous with respect to ρ and π is its corresponding density. We say that π is the prior density of Θ. The parameter θ (values of Θ) is the individual risk within a collectivity.

We assume that given $\Theta=\theta$ the random variables X_1, X_2, …, X_n are independently and identically distributed with common distribution function $F_{X|\Theta}(x|\theta)$ and density $f_{X|\Theta}(x|\theta)$ (for a fixed individual, the claim amounts of consecutive periods are independently and identically distributed).

Let $\boldsymbol{y}=(y_1,y_2,...,y_n)$ be an observed claim experience during the last n periods, i.e. a realization of the conditional random vector $\boldsymbol{X}=(X_1,X_2,...,X_n|\Theta=\theta)$. Denote by

$$l(y_1,y_2,...,y_n|\theta)=\prod_{i=1}^{n} f_{X|\Theta}(y_i|\theta)$$

the likelihood function. Then the posterior distribution is given by

$$\Pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})=\Pr(\Theta\le\theta|\boldsymbol{X}=\boldsymbol{y})$$

and the posterior density (from Bayes' theorem) by

$$\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})=\frac{l(y_1,y_2,...,y_n|\theta)\pi(\theta)}{\int_S l(y_1,y_2,...,y_n|\theta)\pi(\theta)\mathrm{d}\theta}$$

Next, define the loss function

$$L:\mathbf{R}_+^2\to\mathbf{R}_+$$

where $L(x,v)$ is the loss incurred by a decision maker taking the action v and facing the outcome x (v - the premium paid by the insured and x - the claim size).

Definition 2.1 The (individual) risk premium is a function $P_I:S\to\mathbf{R}_+$ which is measurable on $\{S,\Im\}$, where

$$P_I(\theta)=\arg\min_v E_{f_{X|\Theta}}\left(L(X,v)|\Theta=\theta\right),$$

or, equivalently,

$$P_I(\theta)=\arg\min_v\int_\Omega L(x,v)f_{X|\Theta}(x|\theta)\mathrm{d}x,$$

if the expectation involved in the above expression exists.

Similarly, we define Bayesian premium.

Definition 2.2 The Bayesian premium is the measurable function $P_B:\mathbf{R}_+^n\to\mathbf{R}_+$ given by

$$P_B(\boldsymbol{y})=\arg\min_v E_{\pi_{\Theta|\boldsymbol{X}}}\left(L(P_I(\Theta),v)|\boldsymbol{X}=\boldsymbol{y}\right)$$

or, equivalently,

$$P_B(\boldsymbol{y})=\arg\min_v\int_S L(P_I(\Theta),v)\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})\mathrm{d}\theta,$$

if the expectation involved in the above expression exists.

Note that the Bayesian premium is the real number $P_B(\boldsymbol{y})$ minimizing the posterior expected loss $E_{\pi_{\Theta|\boldsymbol{X}}}\left(L(P_I(\Theta),v)\middle|\boldsymbol{X}=\boldsymbol{y}\right)$ over v, where P_I is the individual risk premium.

Remark 2.1 In fact, the Bayesian premium is

$$P_B(\boldsymbol{y})=\arg\min_v E_{\pi_{\Theta|\boldsymbol{X}}}\left(L_2(P_I(\Theta),v)\middle|\boldsymbol{X}=\boldsymbol{y}\right),$$

where

$$P_I(\theta)=\arg\min_v E_{f_{X|\Theta}}\left(L_1(X,v)\middle|\Theta=\theta\right)$$

and $L_1, L_2 : \mathbf{R}_+^2 \to \mathbf{R}_+$ are loss functions not necessarily identical.

If nothing else is mentioned, we will consider that $L_1 \equiv L_2$.

In order to compute the Bayesian premium, we need to evaluate the integral $\int_S L(P_I(\theta),v)\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})d\theta$. This could be solved analitically under rather restrictive circumstances only, which concerns the choice of the loss function and the prior density, and the likelihood function giving the posterior density. Anyway, this choice should be based on objective criteria, possibly related to the real environment of the insurer, rather than on the feasibility of the ensuing calculations.

Of course, numerical integration (whenever it is possible) provides the most general solutions, but they do not yield an analytical result which would allow for interpretations.

In this context, the Laplace method gives an asymptotic approximation of the integral above, with a small asymptotic relativ error and this allows to obtain an accurate asymptotic approximation of the Bayesian premium (Gatto [5]).

The results obtained in this paper are based on the result below.

Theorem 2.1 (Gatto [5]) Let $\boldsymbol{y}=(y_1,y_2,...,y_n)$ denote a realization of $X_1,X_2,...,X_n$ given a fixed value of $\Theta=\theta$, $\boldsymbol{X}=(X_1,X_2,...,X_n|\Theta=\theta)$ and suppose that $\Pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})$ is an absolutely continuous distribution with density $\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})$ which is positive over an interval (a,b). If $\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})$ is differentiable over (a,b), has a unique maximum at $\tilde{\theta}\in(a,b)$, $\tilde{\theta}$ is the only point satisfying $\frac{\partial}{\partial\theta}\pi_{\Theta|\boldsymbol{X}}(\tilde{\theta}|\boldsymbol{y})=0$, and $\left.\frac{\partial^2}{\partial\theta^2}\pi_{\Theta|\boldsymbol{X}}(\theta|\boldsymbol{y})\right|_{\theta=\tilde{\theta}}$ exists and is strictly negative, then

$$P_B(\boldsymbol{y})\sim\arg\min_v L\left(P_I(\tilde{\theta}),v\right) \text{ as } n\to\infty .$$

(i.e. $\frac{P_B(\boldsymbol{y})}{\arg\min_v L\left(P_I(\tilde{\theta}),v\right)}\xrightarrow[n\to\infty]{}1$).

Theorem (Bleistein, Handelsman – 1986) Let $g:(a,b)\to\mathbf{R}_+$ be a differentiable function over (a,b) such that it has a unique minimum at an interior point $\tilde{\theta}$ which is the only point satisfying $\frac{\partial}{\partial\theta}g(\tilde{\theta})=0$ and $\left.\frac{\partial^2}{\partial\theta^2}g(\theta)\right|_{\theta=\tilde{\theta}}$ exists and is strictly positive. If $h:(a,b)\times\mathbf{R}_+\to\mathbf{R}_+$ is differentiable with respect to the first argument on (a,b), then

$$\arg\min_v\int h(\theta,v)e^{-ng(\theta)}d\theta \sim \arg\min_v h(\tilde{\theta},v)e^{-ng(\tilde{\theta})} \quad as \ n\to\infty .$$

In fact, these authors prove that $\arg\min_v \int h(\theta,v)e^{-ng(\theta)}\mathrm{d}\theta = \arg\min_v h(\tilde{\theta},v)e^{-ng(\tilde{\theta})} + O\left(\frac{1}{\sqrt{n}}\right)$. It should be mentioned that the error term contains v, otherwise the approximation would be an exact one.

The above result is based on Laplace's method of approximation. Briefly, this means that g and h are approximated by a Taylor polynomial of order 2 around $\tilde{\theta}$ and then their product is integrated on a small neighborhood of $\tilde{\theta}$.

Remark 2.3 If the loss function involved in the calculation of Bayesian premium is $L_2(x,v)$ different than the one involved in individual risk premium, $L_1(x,v)$, then, under the assumptions of Theorem 2.1 we get

$$P_B(\boldsymbol{y}) \sim \arg\min_v L_2\left(P_I\left(\tilde{\theta}\right),v\right) \text{ as } n\to\infty .$$

Remark 2.4 $\tilde{\theta}$ from Theorem 2.1 is the posterior mode of Θ given $\boldsymbol{X}=\boldsymbol{y}$.

Gatto [5] proposed an alternative approximation of Bayesian premium as

$$P_B(\boldsymbol{y}) \sim \arg\min_v L\left(P_I\left(\hat{\theta}\right),v\right) \text{ as } n\to\infty ,$$

where $\hat{\theta}$ is the maximum likelihood estimate of the risk parameter θ (i.e. the solution of the likelihood equation $\frac{\partial}{\partial\theta} l(y_1,y_2,...,y_n|\theta)=0$). The main disadvantage of this approximation is that the prior information is lost.

Denote

$$P_L^1(\boldsymbol{y}) = \arg\min_v L_2\left(P_I\left(\tilde{\theta}\right),v\right) \text{ or } P_L^1(\boldsymbol{y}) = \arg\min_v L\left(P_I\left(\tilde{\theta}\right),v\right) \text{ (if } L_1 \equiv L_2 \text{)}$$

and

$$P_L^2(\boldsymbol{y}) = \arg\min_v L_2\left(P_I\left(\hat{\theta}\right),v\right) \text{ or } P_L^2(\boldsymbol{y}) = \arg\min_v L\left(P_I\left(\hat{\theta}\right),v\right) \text{ (if } L_1 \equiv L_2 \text{)}.$$

In what follows we will consider a loss function of weighted quadratic type and suppose that **the conditions of Theorem 2.1 are fulfilled.**

Theorem 2.2 Let $L:\mathbf{R}_+^2 \to \mathbf{R}_+$ be the loss function given by $L(x,v)=w(x)(x-v)^2$, where $w:\mathbf{R}_+ \to \mathbf{R}_+$ is a measurable function. Then the individual risk premium is

$$P_I(\theta) = \frac{E_{f_{X|\Theta}}\left(Xw(X)|\Theta=\theta\right)}{E_{f_{X|\Theta}}\left(w(X)|\Theta=\theta\right)},$$

while the Bayesian premium is given by

$$P_B(\boldsymbol{y}) = \frac{E_{\pi_{\Theta|X}}\left(P_I(\Theta)w(P_I(\Theta))|\boldsymbol{X}=\boldsymbol{y}\right)}{E_{\pi_{\Theta|X}}\left(w(P_I(\Theta))|\boldsymbol{X}=\boldsymbol{y}\right)}.$$

Furthermore, an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y}) = P_I\left(\tilde{\theta}\right),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

***Proof*:** Apply Theorem 2.1 for the weighted quadratic loss function .

Proposition 2.1 Under the assumptions of Theorems 2.1. and 2.2 and assuming that w is differentiable of order $k \geq 1$ at 0, the individual risk premium is

$$P_I(\theta)=\frac{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{f_{X|\Theta}}\left(X^{k+1}\middle|\Theta=\theta\right)}{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{f_{X|\Theta}}\left(X^{k}\middle|\Theta=\theta\right)},$$

while the Bayesian premium is given by

$$P_B(\boldsymbol{y})=\frac{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{\pi_{\Theta|\boldsymbol{X}}}\left(P_I(\Theta)^{k+1}\middle|\boldsymbol{X}=\boldsymbol{y}\right)}{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{\pi_{\Theta|\boldsymbol{X}}}\left(P_I(\Theta)^{k}\middle|\boldsymbol{X}=\boldsymbol{y}\right)},$$

which could be approximated by

$$P_L^1(\boldsymbol{y})=\frac{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{f_{X|\Theta}}\left(X^{k+1}\middle|\Theta=\tilde{\theta}\right)}{\sum_{k=0}^{\infty}\frac{w^{(k)}(0)}{k!}E_{f_{X|\Theta}}\left(X^{k}\middle|\Theta=\tilde{\theta}\right)},$$

assuming that all expections involved exist.

Proof: Apply Theorem 2.2.

We will next analyze what happens if the loss functions involved in calculations of risk premium and Bayesian premium are both of weighted quadratic type, but they are not identical.

Theorem 2.3 Let $L_1, L_2 : \mathbf{R}_+^2 \to \mathbf{R}_+$ be the loss functions given by $L_i = w_i(x)(x-v)^2$, $i=1,2$ where $w_i : \mathbf{R}_+ \to \mathbf{R}_+$ are measurable functions. Suppose that the risk premium is computed from L_1 and the Bayesian premium is based on loss function L_2. Then the individual risk premium is

$$P_I(\theta)=\frac{E_{f_{X|\Theta}}\left(Xw_1(X)\middle|\Theta=\theta\right)}{E_{f_{X|\Theta}}\left(w_1(X)\middle|\Theta=\theta\right)},$$

the Bayesian premium is given by

$$P_B(\boldsymbol{y})=\frac{E_{\pi_{\Theta|\boldsymbol{X}}}\left(P_I(\Theta)w_2(P_I(\Theta))\middle|\boldsymbol{X}=\boldsymbol{y}\right)}{E_{\pi_{\Theta|\boldsymbol{X}}}\left(w_2(P_I(\Theta))\middle|\boldsymbol{X}=\boldsymbol{y}\right)},$$

and an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y})=\left.\frac{E_{f_{X|\Theta}}\left(Xw_1(X)\middle|\Theta=\theta\right)}{E_{f_{X|\Theta}}\left(w_1(X)\middle|\Theta=\theta\right)}\right|_{\theta=\tilde{\theta}}=P_I(\tilde{\theta}),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

Proof: Apply Theorem 2.1 and use the fact that $L_i = w_i(x)(x-v)^2$, for $i=1,2$.

Remark 2.5 If the loss function L_2 involved in the calculation of the Bayesian premium is of weighted quadratic type and different from the loss function L_1 involved in evaluating the risk premium, the approximation of Bayesian premium does not depend on L_2.

The weighted quadratic loss function involved in the calculation of Bayesian premium is not the only loss function that leads to an approximation of Bayesian premium that depends only on the loss function involved in evaluating the risk premium. In the next theorem we consider an exponentially scaled loss function.

Theorem 2.4 Let $L:\mathbf{R}_+^2 \to \mathbf{R}_+$ be the exponentially scaled loss function given by $L(x,v)=\left(e^{\alpha x}-e^{\alpha v}\right)^2$, where $\alpha>0$. Suppose that all conditions of Theorem 2.1 are fulfilled. Then the individual risk premium is

$$P_I(\theta)=\frac{1}{\alpha}\ln E_{f_{X|\Theta}}\left(e^{\alpha X}\middle|\Theta=\theta\right),$$

while the Bayesian premium is given by

$$P_B(\boldsymbol{y})=\frac{1}{\alpha}\ln E_{\pi_{\Theta|\boldsymbol{X}}}\left(e^{\alpha P_I(\Theta)}\middle|\boldsymbol{X}=\boldsymbol{y}\right).$$

Furthermore, an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y})=P_I\left(\tilde{\theta}\right),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

Proof: Apply Theorem 2.1. for $L(x,v)=\left(e^{\alpha x}-e^{\alpha v}\right)^2$.

The next propositions analyze the cases when the loss functions involved in evaluating risk premium and Bayesian premium are different and at least one of them is exponentially scaled.

Proposition 2.2 Let $L_i:\mathbf{R}_+^2 \to \mathbf{R}_+$, $i=1,2$, be loss functions given by $L_1(x,v)=w_1(x)(x-v)^2$, $L_2(x,v)=\left(e^{\alpha x}-e^{\alpha v}\right)^2$, where $w_1:\mathbf{R}_+ \to \mathbf{R}_+$ is a measurable function and $\alpha>0$. Suppose that the risk premium is computed from L_1 and the Bayesian premium is based on loss function L_2. Then the individual risk premium is

$$P_I(\theta)=\frac{E_{f_{X|\Theta}}\left(Xw_1(X)\middle|\Theta=\theta\right)}{E_{f_{X|\Theta}}\left(w_1(X)\middle|\Theta=\theta\right)},$$

the Bayesian premium is given by

$$P_B(\boldsymbol{y})=\frac{1}{\alpha}\ln E_{\pi_{\Theta|\boldsymbol{X}}}\left(e^{\alpha P_I(\Theta)}\middle|\boldsymbol{X}=\boldsymbol{y}\right),$$

and an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y})=P_I\left(\tilde{\theta}\right),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

Proof: Apply Theorem 2.1 for the two loss functions mentioned above.

Proposition 2.3 Let $L_i:\mathbf{R}_+^2 \to \mathbf{R}_+$, $i=1,2$, be loss functions given by $L_1(x,v)=\left(e^{\alpha x}-e^{\alpha v}\right)^2$, $L_2(x,v)=w_2(x)(x-v)^2$ where $w_2:\mathbf{R}_+ \to \mathbf{R}_+$ is a measurable function and $\alpha>0$. Suppose that the risk premium is computed from L_1 and the Bayesian premium is based on loss function L_2. Then the individual risk premium is

$$P_I(\theta)=\frac{1}{\alpha}\ln E_{f_{X|\Theta}}\left(e^{\alpha X}\middle|\Theta=\theta\right),$$

the Bayesian premium is given by

$$P_B(\boldsymbol{y})=\frac{E_{\pi_{\Theta|\boldsymbol{X}}}\left(P_I(\Theta)w_2(P_I(\Theta))\middle|\boldsymbol{X}=\boldsymbol{y}\right)}{E_{\pi_{\Theta|\boldsymbol{X}}}\left(w_2(P_I(\Theta))\middle|\boldsymbol{X}=\boldsymbol{y}\right)}$$

and an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y}) = P_I(\tilde{\theta}),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

Proof: Apply Theorem 2.1 for the two loss functions mentioned above.

Proposition 2.4 Let $L_i : \mathbf{R}_+^2 \to \mathbf{R}_+$, $i = 1,2$ be loss functions given by $L_1(x,v) = \left(e^{\alpha x} - e^{\alpha v}\right)^2$, $L_2(x,v) = \left(e^{\beta x} - e^{\beta v}\right)^2$ where $\alpha, \beta > 0$. Suppose that the risk premium is computed from L_1 and the Bayesian premium is based on loss function L_2. Then the individual risk premium is

$$P_I(\theta) = \frac{1}{\alpha} \ln E_{f_{X|\Theta}}\left(e^{\alpha X} \middle| \Theta = \theta\right),$$

the Bayesian premium is given by

$$P_B(\boldsymbol{y}) = \frac{1}{\beta} \ln E_{\pi_{\Theta|\boldsymbol{X}}}\left(e^{\beta P_I(\Theta)} \middle| \boldsymbol{X} = \boldsymbol{y}\right)$$

and an approximation of the Bayesian premium is

$$P_L^1(\boldsymbol{y}) = P_I(\tilde{\theta}),$$

where $\tilde{\theta}$ is the posterior mode of Θ, assuming that all expectations above exist.

Proof: Apply Theorem 2.1 for the two loss functions mentioned above.

Remark 2.6 In all the cases presented in Propositions 2.2 through 2.4 the approximation of the Bayesian premium does not depend on the loss function involved in the calculation of Bayesian premium.

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Received[h] September 19, 2005