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CONSERVATIVE NEWTONIAN FORCES

Radu P.VOINEA*

Romanian Academy, Section of Technical Sciences
Calea Victoriei 125, 010071 Bucharest 1, Romania
E-mail: bratosin@acad.ro

The most general expression of conservative Newtonian generalized forces is determined.

1. NEWTONIAN FORCES

A force **F** acting on a particle is a Newtonian one, if and only if, their components *X, Y, Z* on the axes of a right-handed rectangular Cartesian coordinate system Oxyz are scalar functions of *x, y, z* (particle coordinates), $\dot{x},\dot{y},\dot{z}$ (components of the particle velocity) and of the instant *t* only, [4]i.e.

$$\begin{aligned} X &= X\left(x,y,z,\dot{x},\dot{y},\dot{z},t\right) \\ Y &= Y\left(x,y,z,\dot{x},\dot{y},\dot{z},t\right) \\ Z &= Z\left(x,y,z,\dot{x},\dot{y},\dot{z},t\right) \end{aligned} \tag{1}$$

In this case, Newton's second law leads to three scalar second-order differential equations.

2. NEWTONIAN GENERALIZED FORCES

Generalized forces $Q_1,Q_2.....Q_n$ are Newtonian, if and only if, they are scalar functions of $q_1,q_2.....q_n$ (generalized coordinates), $\dot{q}_1,\dot{q}_2.....\dot{q}_n$ (generalized velocities) and time *t*, i.e.

$$Q_k = Q_k\left(q_1.....q_n,\dot{q}_1.....\dot{q}_n,t\right) \qquad k=1,2.....n \tag{2}$$

In this case the Lagrange equations are second-order differential equations.

3. CONSERVATIVE NEWTONIAN FORCES

A force $\mathbf{F}(\mathbf{r})$ is a conservative one, if and only if [2]:

$$\oint_C \mathbf{F}(\mathbf{r})\cdot \mathrm{d}\mathbf{r} = 0 \tag{3}$$

where $\mathbf{F}(\mathbf{r})$ is a single-valued vector function and differentiable with continuous partial derivatives throughout a finite region *V* and *C* is a regular closed curve contained in this region.

The line integral (3) equals the flux of $\operatorname{curl}\mathbf{F}$ through a simply connected regular (one-side) surface segment *S* situated in *V* and bounded by *C* (Stoke's theorem). It follows:

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* Member of the Romanian Academy

$$\int_S \operatorname{curl}\mathbf{F}(\mathbf{r})\cdot\mathbf{n}\,\mathrm{d}A=0 \tag{4}$$

and because S is an arbitrary surface segment situated in V, $\operatorname{curl}\mathbf{F}(\mathbf{r})=0$, and:

$$\mathbf{F}=\operatorname{grad}\varphi \tag{5}$$

i.e.

$$X_1=\frac{\partial\varphi}{\partial x}\quad ;\quad Y_1=\frac{\partial\varphi}{\partial y}\quad ;\quad Z=\frac{\partial\varphi}{\partial z} \tag{6}$$

where $\varphi=\varphi(x,y,z)$.

The expressions (5) and (6) are not the most general expressions for a conservative Newtonian force in E_3.

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The line integral (3) is valid for any regular closed curve C situated in V. It is necessary to consider also the case $\mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}=0$, respective $\mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t=0$. From $\mathbf{F}\cdot\mathbf{v}=0$ it follows:

$$\mathbf{F}=\mathbf{A}\times\mathbf{v} \tag{7}$$

where $\mathbf{A}(A_x,A_y,A_z)$ is an arbitrary vector function of $\mathbf{r},\dot{\mathbf{r}}$ and t. The components of $\mathbf{F}$ are:

$$\begin{aligned} X_2&=A_y\dot z-A_z\dot y\\ Y_2&=A_z\dot y-A_x\dot z\\ Z_2&=A_x\dot y-A_y\dot x \end{aligned} \tag{8}$$

The most general expressions of the components X, Y, Z of a conservative-Newtonian force $\mathbf{F}$ in E_3 is:

$$\begin{aligned} X&=X_1+X_2=\frac{\partial\varphi(x,y,z)}{\partial x}+A_y(x,y,z,\dot x,\dot y,\dot z,t)\dot z-A_z(x,y,z,\dot x,\dot y,\dot z,t)\dot y\\ Y&=Y_1+Y_2=\frac{\partial\varphi(x,y,z)}{\partial y}+A_z(x,y,z,\dot x,\dot y,\dot z,t)\dot x-A_x(x,y,z,\dot x,\dot y,\dot z,t)\dot z\\ Z&=Z_1+Z_2=\frac{\partial\varphi(x,y,z)}{\partial z}+A_x(x,y,z,\dot x,\dot y,\dot z,t)\dot y-A_y(x,y,z,\dot x,\dot y,\dot z,t)\dot x \end{aligned} \tag{9}$$

and, in matrix form:

$$\begin{Bmatrix} X\\ Y\\ Z\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\varphi}{\partial x}\\ \dfrac{\partial\varphi}{\partial y}\\ \dfrac{\partial\varphi}{\partial z}\end{Bmatrix}+\begin{bmatrix}0 & -A_z & A_y\\ A_z & 0 & -A_x\\ -A_y & A_x & 0\end{bmatrix}\cdot\begin{Bmatrix}\dot x\\ \dot y\\ \dot z\end{Bmatrix} \tag{10}$$

An important particular case is:

$$\mathbf{A}=\operatorname{curl}\mathbf{B} \tag{11}$$

In this case:

$$A_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \quad ; \quad A_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \quad ; \quad A_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \tag{12}$$

The expression (10) becomes:

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{Bmatrix} + \begin{bmatrix} 0 & \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} & 0 & \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \\ \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} & \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} & 0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} \tag{13}$$

It is easy to recognize the electromagnetic force acting on a particle having an electromagnetic charge $e = 1$ [1], [3].

4. CONSERVATIVE NEWTONIAN GENERALISED FORCES

A system of generalized forces $Q_1, Q_2 .....Q_n$ is a conservative-Newtonian one if the expression:

$$Q_1 \,\mathrm{d}\, q_1 + Q_2 \,\mathrm{d}\, q_2 + ... + Q_n \,\mathrm{d}\, q_n \tag{14}$$

is an exact differential $\mathrm{d}\varphi$, or if

$$Q_1 \dot{q}_1 + Q_2 \dot{q}_2 + ... + Q_n \dot{q}_n \tag{15}$$

is identical null.

In the first case the generalized forces have the expressions:

$$Q'_k = \frac{\partial \varphi}{\partial q_k} \qquad k = 1, 2.....n \tag{16}$$

and in the second case the generalized forces have the expressions:

$$Q''_k = \sum_{i=1}^{n} A_{ki} \dot{q}_i \qquad \text{with} \qquad A_{ik} = -A_{ki} \tag{17}$$

From (16) and (17) there follows the most general expression of conservative-Newtonian generalized forces in matrix form:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ . \\ . \\ . \\ Q_n \end{Bmatrix} = \begin{Bmatrix} Q'_1 \\ Q'_2 \\ . \\ . \\ . \\ Q'_n \end{Bmatrix} + \begin{Bmatrix} Q''_1 \\ Q''_2 \\ . \\ . \\ . \\ Q''_n \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \varphi}{\partial q_1} \\ \frac{\partial \varphi}{\partial q_2} \\ . \\ . \\ . \\ \frac{\partial \varphi}{\partial q_n} \end{Bmatrix} + \begin{bmatrix} 0 & A_{12} & A_{13} & . & . & A_{1n} \\ A_{21} & 0 & A_{23} & . & . & A_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{n1} & A_{n2} & A_{n3} & . & . & 0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ . \\ . \\ . \\ \dot{q}_n \end{Bmatrix} \tag{18}$$

where:

$$\varphi=\varphi(q_1...q_n) \quad ; \quad A_{ki}=A_{ki}(q_1...q_n,\dot{q}_1...\dot{q}_n,t) \quad ; \quad A_{ik}=-A_{ki} \tag{19}$$

NOTES

1. The expression (13) corresponding to the electromagnetic force can be generalized in configuration space. Let $\mathbf{B}(B_1,B_2...B_n)$ be a vector in this space. It follows:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ Q_n \end{Bmatrix} = \begin{Bmatrix} \frac{\partial\varphi}{\partial q_1} \\ \frac{\partial\varphi}{\partial q_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial\varphi}{\partial q_n} \end{Bmatrix} + \begin{bmatrix} 0 & \frac{\partial B_1}{\partial q_2}-\frac{\partial B_2}{\partial q_1} & \frac{\partial B_1}{\partial q_3}-\frac{\partial B_3}{\partial q_1} & \cdot\ \cdot & \frac{\partial B_1}{\partial q_n}-\frac{\partial B_n}{\partial q_1} \\ \frac{\partial B_2}{\partial q_1}-\frac{\partial B_1}{\partial q_2} & 0 & \frac{\partial B_2}{\partial q_3}-\frac{\partial B_3}{\partial q_2} & \cdot\ \cdot & \frac{\partial B_2}{\partial q_n}-\frac{\partial B_n}{\partial q_2} \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \frac{\partial B_n}{\partial q_1}-\frac{\partial B_1}{\partial q_n} & \frac{\partial B_n}{\partial q_2}-\frac{\partial B_2}{\partial q_n} & \frac{\partial B_n}{\partial q_3}-\frac{\partial B_3}{\partial q_n} & \cdot\ \cdot & 0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n \end{Bmatrix} \tag{18}$$

2. The antisymetric $n\times n$ matrix (20) generalize the motion of curl, in matrix form, in configuration space.

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