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A BOUNDARY BETWEEN RELATIVISTIC AND NON-RELATIVISTIC MECHANICS

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The paper takes into account a particular case of movement, namely a body acted by a constant force, and establishes a bound speed between Newton's mechanics (non-relativistic) and relativistic mechanics. The procedure consists in applying a theorem of non-linear phenomena, resulting a speed equal to 1/√2 of light speed in vacuum.

1. INTRODUCTION

The mechanics makes, usually, a distinct separation between the non-relativistic domain of the small speeds ($v \ll c$) and the relativistic one, of the speeds near the light speed [1]. In the following, a new viewpoint is considered, that delimits the two domains without considering an intermediate domain of speed. The considered case is the movement of a body acted by a constant force. The procedure draws one's inspiration from electronics, dealing with non-linear electrical conduction, where a law and a theorem have been formulated [2], [3]. If the considered body is an electrically charged particle situated in a constant electrical field, the analogy mechanics - electronics used here becomes identity.

According to the law of the non-linear electrical conduction [3], any non-linear electrical process has the tendency to become linear when the voltages or currents extend to unlimited values. This asymptotic behaviour permits to apply the theorem of the non-linear electrical conduction that states the existence of a least one point where the third derivative of the conduction function gets null value. This theorem is a result of Roll's theorem applied for mathematical functions having asymptotes. The points that make zero the third derivative have the signification of the thresholds that separate the domains where different physical phenomena occur.

The mechanics offers a similar example for a body having an accelerate motion. Due to mass - speed dependence, the motion function is non-linear. For the relativistic domain, where the speed is limited to the light speed, the space - time function has an asymptotic behaviour. As consequence, it can apply the mathematical methodology of zeroing the third derivative to determine a bound speed that separates the two domains.

2. THE MOTION FUNCTION OF A BODY ACTED BY A CONSTANT FORCE

Considering a particular case, of a body acted by a constant force with a null initial velocity, diminishes the sphere of the problem. However, this case has the advantage of the simplest model for a motion that covers all spectrum of velocities $v \in [0, c]$.

Time dependence of the speed for a body having a rest mass m_0, acted by a constant force F is given by [4]:

$$v = c\frac{t}{\sqrt{t^2 + \frac{c^2}{a_0^2}}} \tag{1}$$

where a_0 is the initial acceleration:

$$a_0 = \frac{F}{m_0} \tag{2}$$

The motion function will be described with normalised coordinates, as follows:

- normalised speed (β):

$$\beta = \frac{v}{c} \quad (3)$$

and

- normalised time (τ):

$$\tau = \frac{t}{t_0} \quad (4)$$

where t_0 is given by:

$$t_0 = \frac{c}{a_0} \quad (5)$$

Using the normalised variables, the equation (1) becomes:

$$\beta = \frac{\tau}{\sqrt{1+\tau^2}} \quad (6)$$

The next step of the mathematical procedure consists in passing to logarithmic coordinates:

$$X = \ln \tau \quad (7a)$$

$$Y = \ln \beta \quad (7b)$$

The logarithmic description of the time has the advantage of transforming its definition domain from $\tau \in [0,\infty)$ to $X \in (-\infty,+\infty)$; in this way, the mathematical function can be analysed symmetrically in relation with asymptotic behaviour. Using the logarithm form of the time represents an usual procedure to define the *propre time*. Concerning the logarithm of the speed, this permits to enlarge the domain, especially for low values, from $\beta \in [0,1)$ to $Y \in (-\infty,0)$.

Considering the equations (7a) and (7b), the equation (6) becomes:

$$Y = \ln \frac{e^X}{\sqrt{1+e^{2X}}} \quad (8)$$

3. BOUNDARY SPEED

The equation (8) has asymptotes toward the both ends of the definition domain. For $X \to -\infty$ (corresponding to $t \to 0$), the asymptote is:

$$Y = X \quad (9)$$

and represents the non-relativistic mechanics approximation ($v = a_0 t$). For $X \to +\infty$ (corresponding to $t \to +\infty$), the asymptote is:

$$Y = 0 \quad (10)$$

and represents the relativistic mechanics approximation ($v = c$). The figure 1 shows the graph of the equation (8) and its asymptotes.

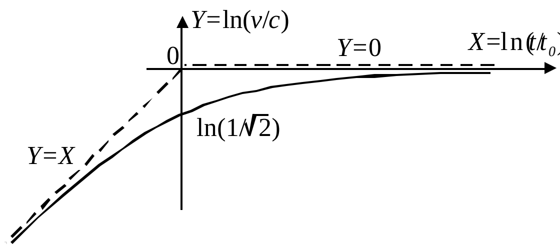

Fig. 1 - The speed - time dependence using normalised logarithmic coordinates.

The asymptotic behaviour of the function (8) determines the zeroing of the second derivative at the ends of the definition domain:

$$\frac{d^2Y}{dX^2} \to 0 \text{ for } X \to \pm\infty \quad (11)$$

Consequently, taking into account the continuity of the function and its derivatives (at least, up to 3-rd order), the Roll's theorem can be applied for the second derivative, resulting the statement:

«The function *Y(X)* has at least one point (X_c) where $\left.\frac{d^3Y}{dX^3}\right|_{X_c} = 0$. »

The third derivative of the function (8) has the form:

$$\frac{d^3Y}{dX^3} = -\frac{8e^{2X}(1-e^{2X})}{(1+e^{2X})^3} \quad (12)$$

The critical point (X_c) that zeroes the above equation is:

$$X_c \quad (13)$$

This point corresponds to the crossing point of the asymptotes, as is shown in Fig. 1, and it can be interpreted as a boundary (threshold) between the relativistic and non-relativistic domains. The corresponding value of the function for this point is:

$$Y_c = \ln\frac{1}{\sqrt{2}} \tag{14}$$

Passing to absolute coordinates, this point corresponds to:

$$t_c = t_0 = \frac{c}{a_0} \tag{15a}$$

$$v_c = \frac{1}{\sqrt{2}}c \tag{15b}$$

The equation (15b) establishes, in conclusion, a speed equal to $1/\sqrt{2}$ of light speed in vacuum as a bound between relativistic and non-relativistic mechanics. In term of mass, this boundary corresponds to a $\sqrt{2}$ times multiplication of rest mass ($m_c = \sqrt{2}\, m_0$).

4. OBSERVATION

The mathematical procedure of zeroing the third derivative can be applied for the function (1), also, namely for the absolute variables (non-logarithmic). Indeed, this function has null values for the second derivative at the both ends of its definition domain $t \in [0,\infty)$. For $t \to +\infty$ the second derivative has a zero limit due to horizontal asymptotes $v = c$. For $t = 0$ the second derivative is null, also; this value corresponds to the "asymptotic" type behaviour $v = a_0 t$ at small velocities, but a real asymptote can not be defined. Applying the Roll's theorem in this case, the following critical point results:

$$t'_c = \frac{t_0}{2} = \frac{c}{2a_0} \tag{16a}$$

that represents half of previous obtained value (15b). The corresponding bound speed is:

$$v'_c = \frac{1}{\sqrt{5}}c \tag{16b}$$

Comparing this result ($v'_c \cong 0{,}447\ c$) with the previous one ($v_c \cong 0{,}707\ c$), these can be considered closely, taking into account the very large domain of speed. However, using the logarithmic coordinates makes a net and symmetrical separation of the relativistic domain ($X > 0$) and the non-relativistic domain ($X < 0$) and gives a concrete meaning of the critical point.

5. AN ANALOGY WITH ELECTRICAL PHENOMENA

Electronics knows functions similar to function (6) for the case of a linear system having a single pole and a single zero [5], [6]. The transfer function A has a frequency dependence given by:

$$A(\omega) = \frac{\dfrac{\omega}{\omega_j}}{\sqrt{1+\left(\dfrac{\omega}{\omega_j}\right)^2}} \tag{17}$$

where ω_j is the critical point (frequency or pulsation) that separates the low domain and the medium domain of frequencies. The practice of logarithmic coordinates and asymptotes is well known for such function, as Bode diagrams. It is well known, also, the particular value of the function (17) for the ω_j value:

$$A(\omega_j) = \frac{1}{\sqrt{2}} \tag{18}$$

similar with β bound value (15b).

The analogy electric - mechanic goes deeply, remembering that equation (17) represents the modulus of the complex function $\underline{A}(i\omega)$:

$$\underline{A}(i\omega) = \frac{i\omega}{i\omega + \omega_j} \tag{19}$$

The same procedure can be applied for the mechanic case considered here, using Minkowski space ($x_1 = x$, 0, 0, $x_4 = ict$). The equation (1) can be derived from the equation:

$$\underline{v} = c\frac{it}{it + t_0} \tag{20}$$

where $\underline{v}$ is the complex speed and t_0 is a physical constant whose sense will be determined further. Considering the normalised complex speed $\underline{\beta}$, the equation (20) can be written:

$$\underline{\beta} = \frac{\underline{v}}{c} = \frac{ict}{ict + ct_0} \tag{21}$$

Considering the coordinates of the Minkowski space, the equation (21) becomes:

$$\underline{\beta} = \frac{x_4}{x_4 + x_0} \tag{22}$$

where x_0 is a notation for:

$$x_0 = ct_0 \tag{23}$$

Indeed, the modulus of this function is:

$$\beta = |\underline{\beta}| = \frac{|x_4|}{|x_4 + x_0|} = \frac{ct}{\sqrt{c^2t^2 + x_0^2}} = \\ = \frac{\frac{ct}{x_0}}{\sqrt{1+\left(\frac{ct}{x_0}\right)^2}} = -\frac{\tau}{\sqrt{1+\tau^2}} \tag{24}$$

that is the equation (6). Consequently, knowing the equation (15a) the sense of x_0 is the modulus of x_4 coordinate in the critical point between relativistic and non-relativistic domains.

6. CONCLUSIONS

The paper considered the case of a body motion, acted by a constant force, starting with null speed and continuing to move up to the relativistic domain. Expressing the time and the speed in logarithmic coordinates, the two domains, relativistic and non-relativistic, were described by two corresponding asymptotes. The boundary that separates the two domains resulted in the crossing point of the asymptotes and is equal to $1/\sqrt{2}$ of light speed in vacuum. An analogy with electrical phenomena permitted a presentation of the mechanics problem using the Minkowski space.

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